Open access • Journal Article • DOI:10.1080/00927879408824934

Restricted injectivity, transfer property and decompositions of separative positively ordered monoids. — Source link

Friedrich Wehrung

Institutions: University of Caen Lower Normandy

Published on: 01 Jan 1994 - Communications in Algebra (Marcel Dekker, Inc.)

Topics: Monoid, Commutative property, Invariant (mathematics), Countable set and Bounded function

RESTRICTED INJECTIVITY, TRANSFER PROPERTY AND DECOMPOSITIONS OF SEPARATIVE POSITIVELY ORDERED MONOIDS.

Friedrich WEHRUNG
Université de Caen
Département de Mathématiques
14032 CAEN CEDEX
FRANCE

ABSTRACT.

We introduce a notion of separativeness for positively ordered monoids (P.O.M.'s), similar in definition to the notion of separativeness for commutative semigroups but which has a simple categorical equivalent, weaker that injectivity, the transfer property. We show that existence in a separative extension of the ground P.O.M. of a solution of a given linear system is equivalent to the satisfaction by the ground P.O.M. of a certain set of equations and inequations, the resolvent. We deduce in particular a characterization of the P.O.M.'s which are injective relatively to the class of embeddings of countable P.O.M.'s; those include in particular divisible weak cardinal algebras. We also deduce that finitely additive positive non-standard measures invariant relatively to a given exponentially bounded group separate equidecomposability types modulo this group.

1991 mathematics subject classification: 06F05, 20M14, 08B25, 08B30.

§0. INTRODUCTION.

Let A be a commutative monoid. Then A can be equipped with a preordering, defined by $x \leq y \Leftrightarrow (\exists z)(x + z = y)$. The corresponding structure $(A, +, 0, \leq)$ is an example of what we call a *positively ordered monoid*. By definition, a positively ordered monoid (from now on a P.O.M.) is a structure consisting on a commutative monoid, together with a preordering which is compatible with the addition and for which every element is positive; when a P.O.M. is obtained from a commutative monoid as above, then we will say that it is *minimal*. There are a lot of P.O.M.'s which are not minimal. One of the main reasons for which we work in the context of P.O.M.'s rather than in the context of commutative monoids is that if A is a submonoid of a commutative monoid B, then it is not necessary that the minimal preordering of A is the restriction to A of the minimal preordering of B, which brings unnecessary trouble in Hahn-Banach-type proofs and makes a lot of categorical statements rather cumbersome. One of the results of [17], which shows that injective objects in the class of P.O.M.'s equipped with their natural notion of embedding are exactly the retracts of the powers of $\overline{\mathbb{P}} = ([0, \infty], +, 0, \leq)$, shows that this difficulty is unavoidable: indeed, it is well-known that the only injective commutative monoid is $\{0\}$.

However, our methods tend to study essentially *minimal* P.O.M.'s. One of their important properties which is preserved under substructures is the property of *preminimality* (definition 1.2). But it finally turns out that this concept is slightly too general, so that instead, we will have to consider a weakened form of additive cancellation called *separativeness* (definition 1.2). This definition is strongly connected to the definition of separativeness of commutative semigroups (*see* [4], Vol. 1) — in particular, the underlying monoid of any separative P.O.M. is separative. But unlike the case of commutative semigroups, separative P.O.M.'s enjoy a special categorical property: they are exactly those which have the *transfer property* (theorem 2.9), where E has the transfer property if and only if for every sub-P.O.M. A of a P.O.M. B, every P.O.M.-homomorphism from A to E extends to a P.O.M.-homomorphism from B to some P.O.M. containing E. The case where one can always take $F = E$ (*i.e.* E is injective) has already been discussed in [16]; the case where one can always take $F = E$ with certain restrictions on B (on size or axioms) — we will speak about *restricted injectivity* — will be discussed in chapters 3 and 4, see *e.g.* 3.2, 3.6, 3.10, 3.13, 4.3, 4.4, 4.10. Note again that in the case of commutative semigroups, these notions trivialize more or less — totally in the case of injectivity, and partially in the case of the transfer property (see proposition 2.11) where it turns out that the transfer property is equivalent to additive cancellation. In theorem 3.13, we prove that a P.O.M. E is injective relatively to the class of all inclusion maps from A to B where B is preminimal of size at most a given infinite cardinal κ (we will say that E is $PREM_\kappa$-injective) if and only if it is injective relatively to both classes obtained respectively by restricting B to be idem-multiple generated over A, or B cancellative and minimal.

One of the essential tools of the proof of theorem 3.13 is the notion of *linear system*. We consider linear systems with coefficients in $\mathbb{N}$ and with coefficients in a given P.O.M.. To each such system, say (S), one can associate in an effective way a second linear system this time without unknowns, the *resolvent* of (S). Then one of our main results is that if E is separative, then existence of a solution of (S) in some separative extension of E is equivalent to the satisfaction by E of the resolvent of (S) (see theorems 3.14 and 3.16). This algorithm generalizes to cases without cancellation and with an ordering the one which is known to solve equation systems over abelian groups.

Chapter 1 is essentially devoted to introduce the terminology (preminimal or separative P.O.M.'s, cones...) and the technology (D.P.O.M.'s...) used in this work. This is not our first encounter with D.P.O.M.'s, *see* [16], chapter 2; futhermore, the latter structures seem in many cases to be the relevant alternative to abelian groups (or to cancellative P.O.M.'s) in the absence of additive cancellation.

In chapter 2, we study sufficient conditions of transferability of P.O.M.-embeddings; we deduce the characterization of the transfer property (theorem 2.9), but also *e.g.* how it can be used to embed 'painlessly' arbitrary P.O.M.'s into richer structures (here, for example, P.O.M.'s satisfying the finite refinement property — see corollary 2.7). We show examples, including weak cardinal algebras and equidecomposability types P.O.M.'s modulo exponentially bounded groups. In chapter 3, we characterize restricted forms of injectivity (theorems 3.2, 3.6, 3.10, 3.13) and we prove the correctness of the aforementioned resolvent test (theorem 3.14). In chapter 4, we characterize completely $PREM_\kappa$-injectivity in antisymmetric P.O.M.'s satisfying the multiplicative $\leq$-cancellation property (theorem 4.3), and then injectivity relatively to the class of all inclusion maps from A to B where B is an *arbitrary* P.O.M. of size at most κ (theorem 4.4). Strangely, the difficult part of this theorem (modulo the results of chapter 3) is to prove that such P.O.M.'s always embed into powers of $\overline{\mathbb{P}}$. They include divisible weak cardinal algebras. From the results of chapters 1, 2 and 3, we finally deduce a decomposition theorem for arbitrary separative P.O.M.'s (corollary 4.12), which is in some sense optimal (theorem 4.17); from a particular case with antisymmetry and multiplicative $\leq$-cancellation (theorem 4.14), we deduce in particular that if G is an exponentially bounded group operating on a set X, then finitely additive G-invariant measures on $\mathcal{P}(X)$ with positive non-standard values separate G-equidecomposability types (it is false for just $\overline{\mathbb{P}}$-valued measures), this is corollary 4.16. It is to be noticed that a lot of proofs of converses of injectivity / or transfer property statements showed in this work are not unlike arguments of 'reverse mathematics'; an important exception is for the essential lemmas 3.5 and 3.9.

We use standard terminology and notation. If X and Y are two sets, then we will denote by X^Y the set of all maps from Y to X. If $(A_i)_{i\in I}$ is a family of P.O.M.'s, then we will denote its direct sum (coproduct) by $\coprod_{i\in I} A_i$. When $(\forall i \in I)(A_i = A)$, we will write $A^{(I)}$. We will denote the set of all natural numbers by ω when it is considered as an ordinal, $\mathbb{N}$ otherwise. If $(\phi_i)_{i\in I}$ is a family of formulas, we will sometimes denote their conjunction (resp. disjunction) by $\bigwedge\!\!\!\bigwedge_{i\in I} \phi_i$ (resp. $\bigvee\!\!\!\bigvee_{i\in I} \phi_i$).

Before the statement of each theorem, we indicate the references of the relevant definitions introduced in this paper.

§1. PREMINIMAL P.O.M.'s, SEPARATIVE P.O.M.'s; DIFFERENCE P.O.M.'s.

We first recall the *context* in which this work is done; it is expressed by the following definition, taken from [16], and which we recall here:

1.1. Definition. *A* positively ordered monoid *(from now on P.O.M.) is a structure* $(A, +, 0, \leq)$ *such that* $(A, +, 0)$ *is a commutative monoid and* $\leq$ *is a preordering on* A *satisfying both following conditions:*

(i) $(\forall a, b, c)(a \leq b \Rightarrow a + c \leq b + c)$,

(ii) $(\forall a)(a \geq 0)$.

Our notations and terminology will be widely borrowed from [16]. For example, a P.O.M. is said *minimal* when it satisfies the statement

$$(\forall a, b)\big(a \leq b \;\Leftrightarrow\; (\exists x)(a + x = b)\big),$$

antisymmetric when its preordering is antisymmetric. If m is in $\mathbb{N} \setminus \{0\}$, then the *multiplicative* $\leq$*-cancellation property* is (*see* [16], chapter 1) the following statement:

$$(\forall x, y)(mx \leq my \Rightarrow x \leq y).$$

If X and Y are two subsets of a given P.O.M., we will write $X \leq Y$ instead of $(\forall (x, y) \in X \times Y)(x \leq y)$, $a \leq Y$ (resp. $X \leq a$, resp. $a_1, \ldots, a_m \leq b_1, \ldots, b_n$) when

$X = \{a\}$ (resp. $Y = \{a\}$, resp. $X = \{a_1, \ldots, a_m\}$ and $Y = \{b_1, \ldots, b_n\}$). In any P.O.M., we define binary relations $\equiv$, $\preceq$, $\asymp$ and $\ll$ by

$$\begin{aligned} x \equiv y &\Leftrightarrow x \leq y \textbf{ and } y \leq x, \\ x \preceq y &\Leftrightarrow (\exists n \in \mathbb{N})(x \leq ny), \\ x \asymp y &\Leftrightarrow x \preceq y \textbf{ and } y \preceq x, \\ x \ll y &\Leftrightarrow x + y = y. \end{aligned}$$

If the context does not make it clear, we will add an index A to the symbols $\leq$, $\equiv$, $\preceq$, $\asymp$ and $\ll$. An element a of a given P.O.M. A is *idem-multiple* when $a + a = a$. The set of idem-multiple elements of A will be denoted, for reasons which will appear clearly in the middle of this chapter, by $\frac{1}{\infty}A$. We recall here the statement of the *pseudo-cancellation property*, already studied in [16], chapter 1:

$$(\forall a, b, c)\big(a + c \leq b + c \Rightarrow (\exists d \ll c)(a \leq b + d)\big).$$

We introduce two more notations, already used in [16]: if A is a P.O.M. and a is an element of A, then we denote by $A|a$ the sub-P.O.M. of A of all *a-bounded* elements of A, *i.e.*

$$A|a = \{x \in A : \ x \preceq a\}.$$

On the other hand, we denote by $\dfrac{A}{a}$ the *quotient* P.O.M. of A by a, *i.e.* the P.O.M. of all equivalence classes of elements of A modulo the relation

$$x \equiv_a y \ \Leftrightarrow \ x + a = y + a,$$

equipped with the preordering defined by

$$[x]_a \leq [y]_a \ \Leftrightarrow \ x + a \leq y + a$$

(where $[x]_a$ denotes the equivalence class of x modulo $\equiv_a$).

Finally, for every P.O.M. A, we denote by $A \cup \{\infty\}$ the P.O.M. obtained by adjunction to A of a unique largest element (which is idem-multiple), which we will denote by ∞.

Now, we shall introduce a definition which will be very important throughout this work.

1.2. Definition. *Let A be a P.O.M.. Then we say that A is*

— cancellative *when it satisfies both following statements:*

$$\begin{aligned} &(\forall a, b, c)\big(a + c \leq b + c \Rightarrow a \leq b\big), \\ &(\forall a, b, c)\big(a + c = b + c \Rightarrow a = b\big); \end{aligned}$$

— preminimal *when it satisfies both following statements:*

$$(\forall a,b,c,d)\big((a+c\leq b+c \textbf{ and } c\leq d)\Rightarrow a+d\leq b+d\big),$$
$$(\forall a,b,c,d)\big((a+c=b+c \textbf{ and } c\leq d)\Rightarrow a+d=b+d\big);$$

— separative *when it satisfies both following statements:*

$$(\forall a,b,c)\big((a+c\leq b+c \textbf{ and } c\preceq b)\Rightarrow a\leq b\big),$$
$$(\forall a,b,c)\big((a+c=b+c \textbf{ and } c\preceq a,b)\Rightarrow a=b\big).$$

The term 'preminimal' comes from the fact that for all c, d in a P.O.M. A such that $c\leq d$, the conjunction of both following conditions

$$(\forall a,b)\big(a+c\leq b+c\Rightarrow a+d\leq b+d\big)$$

and

$$(\forall a,b)\big(a+c=b+c\Rightarrow a+d=b+d\big)$$

is equivalent to the fact that there are a P.O.M. B containing A and an element x of B such that $c+x=d$. We will neither use nor prove this fact here.

It is easy to see that every minimal or separative P.O.M. is preminimal. Both converses are false: $\mathbb{N}\setminus\{1\}$ is a sub-P.O.M. of the minimal P.O.M. $\mathbb{N}$, thus it is preminimal; but it is not minimal. And the free P.O.M. with the two generators a and b with both relations $a+b=2a=2b$ is easily seen to be an antisymmetric, minimal P.O.M. but non separative. Note also that if A is a separative P.O.M., then its underlying commutative semigroup is by definition separative in the sense of [4], Vol. 1. Finally, an example of non-preminimal P.O.M. is given by the lexicographical product $\mathbb{N}\times_{lex}\overline{\mathbb{N}}$, where $\overline{\mathbb{N}}=\mathbb{N}\cup\{\infty\}$: indeed, we have $(0,2)+(0,\infty)=(0,1)+(0,\infty)$ and $(0,\infty)<(1,0)$, but $(0,2)+(1,0)\not\leq(0,1)+(1,0)$.
The following definition provides us with a fundamental example of separative P.O.M.:

1.3. Definition. *A* cone *is a minimal, cancellative P.O.M., that is, the positive cone of a preordered commutative group. A* cone with infinity *is a P.O.M. of the form $A\cup\{\infty\}$ where A is a cone.*

With this definition, we can give several characterizations of separative P.O.M.'s:

1.4. Theorem. [definitions 1.2, 1.3] *Let A be a P.O.M.. Then the following are equivalent:*

(a) *A is separative;*

(b) *A satisfies the following three statements:*
 (i) *Preminimality;*
 (ii) $(\forall a,b)(a+b\leq 2b\Rightarrow a\leq b)$;
 (iii) $(\forall a,b)(a+b=2a=2b\Rightarrow a=b)$.

(c) *A embeds into a product of cones with infinity.*

Proof. (a)$\Rightarrow$(b) is trivial. Now assume (b), we prove (a) in a sequence of claims.

Claim 1. Let a, b, c in A such that $a+c \leq b+c$ and $c \leq b$; then $a \leq b$. Moreover, if $a+c = b+c$ and $c \leq a, b$, then $a = b$.

Proof of claim 1. Since A is preminimal and $c \leq b$, we also have $a + b \leq 2b$, thus $a \leq b$ by hypothesis. Furthermore, if $a+c = b+c$ and $c \leq a, b$, then, since $c \leq b$ and A is preminimal, $a + b = 2b$; similarly, $a + b = 2a$. By hypothesis, $a = b$. ∎ Claim 1.

Then, an easy induction argument yields easily the following

Claim 2. Let a, b in A, m in $\mathbb{N}$ such that $a + mb \leq (m+1)b$; then $a \leq b$. Moreover, if $a + mb = (m+1)b$ and $b \leq a$, then $a = b$. ∎ Claim 2.

Now we can complete the proof of separativeness of A. Let a, b, c in A such that $a + c \leq b + c$ and $c \preceq b$. By definition, there is m in $\mathbb{N}$ such that $c \leq mb$, thus, since A is preminimal, $a + mb \leq (m+1)b$, whence $a \leq b$ by claim 2. Moreover, suppose that $a + c = b + c$ and $c \preceq a, b$; by the previous result, $b \leq a$, and again by preminimality of A, $a + mb = (m+1)b$. By claim 2, we obtain $a = b$. Thus we have proved that (a) and (b) are equivalent.

Finally, it is trivial that (c) implies (a). Conversely, assume (a). For all a in A, put $A_a = \dfrac{A|a}{a}$. Since A is separative, it is easy to verify that A_a is cancellative, thus it embeds into its group of differences G_a equipped with the canonical preordering defined by "($x - y \leq x' - y'$ if and only if $x + y' \leq x' + y$)" (x, x', y, y' in A_a). Let $C_a = (G_a)_+ \cup \{\infty\}$. Then C_a is a cone with infinity, and A_a is a sub-P.O.M. of C_a. Define a map e_a from A to C_a by

$$e_a(x) = \begin{cases} [x]_a & (x \in A|a) \\ \infty & (\text{otherwise}). \end{cases}$$

It is straightforward to verify that e_a is a P.O.M.-homomorphism from A to C_a. Now, let e be the map from A to $\prod_{a \in A} C_a$ defined by $e(x) = (e_a(x))_{a \in A}$. Then e is a P.O.M.-homomorphism, thus, to conclude, it suffices to prove that e is an embedding. So let a, b in A, such that $e(a) \leq e(b)$. Then $e_b(a) \leq e_b(b)$, thus $a \in A|b$ and $a + b \leq 2b$, thus $a \leq b$ since A is separative. Moreover, if $e(a) = e(b)$, then $e_a(a) = e_a(b)$, whence $b \in A|a$ and $2a = a + b$; similarly, $2b = a + b$, whence $a = b$ since A is separative. Therefore, e is an embedding, which concludes the proof. ∎

1.5. Remark. Note that this theorem proves that separativeness is finitely axiomatizable, which was not appearent in definition 1.2.

1.6. Remark. Note that the same proof shows that *any antisymmetric separative P.O.M. embeds into a product of antisymmetric cones with infinity.*

The notion we shall present now, the notion of D.P.O.M. (the 'D' stands here for "difference") can be viewed as a 'functional' aspect of separative P.O.M.'s, or the analogue for separative P.O.M.'s of the group of differences for cancellative semigroups (there is more than one analogy, see next chapter). Due to the much better computational convenience offered by D.P.O.M.'s, several propositions about separative P.O.M.'s appearing in this work will be proved using D.P.O.M.'s.

1.7. Definition. *A* D.P.O.M. *is a structure* $(A,+,0,\leq,-)$ *such that* $(A,+,0,\leq)$ *is a P.O.M. and* $-$ *is a partial binary operation on* A *(the 'difference' of the D.P.O.M.), defined on* $\mathrm{H}(A)=\{(a,b)\in A\times A : a\geq b\}$*, satisfying both following conditions:*

(D1) $\big(\forall(a,b)\in \mathrm{H}(A)\big)\big(b+(a-b)=a\big)$;

(D2) $(\forall a\in A)\big(\forall(b,c)\in \mathrm{H}(A)\big)\big(a+(b-c)=(a+b)-c\big)$.

If a is an element of a D.P.O.M. A, we will write $\frac{a}{\infty}$ instead of $a-a$. From (D1), it results immediately that A is minimal. From (D2), it results immediately that for all a, b, c in A, $a+c=b+c$ implies $a+\frac{c}{\infty}=b+\frac{c}{\infty}$ and that $a+c\leq b+c$ implies $a+\frac{c}{\infty}\leq b+\frac{c}{\infty}$.

Let us first state some arithmetical properties of D.P.O.M.'s. From lemmata 1.8 to 1.10, let A be a D.P.O.M..

1.8. Lemma. *Let* a, b, c *in* A*. Then* $b+c\leq a$ *if and only if* $b\leq a$ *and* $c\leq a-b$*.*

Proof. If $b+c\leq a$, then $b\leq a$; put $d=a-(b+c)$. We have $c\leq c+d+(b-b)$ $=(b+c+d)-b=a-b$ (we use (D2)). Conversely, if $b\leq a$ and $c\leq a-b$, then $b+c\leq b+(a-b)=a$. ∎

The following lemma justifies the notation $\frac{1}{\infty}A$ for the set of idem-multiple elements of A.

1.9. Lemma. *Let* a *in* A*. Then* $\frac{a}{\infty}$ *is the unique largest element* x *of* A *such that* $x\ll a$*, and it is idem-multiple. Furthermore,* a *is idem-multiple if and only if* $a=\frac{a}{\infty}$*.*

Proof. We have $\frac{a}{\infty}+a=a$ by (D1), thus, using (D2), we get $\frac{a}{\infty}+\frac{a}{\infty}=(\frac{a}{\infty}+a)-a=a-a=\frac{a}{\infty}$, whence $\frac{a}{\infty}$ is idem-multiple $\ll a$. If $x\in A$ and $x\ll a$, then $x+\frac{a}{\infty}=\frac{a}{\infty}$ by (D2), thus $\frac{a}{\infty}$ is maximum in the set of x in A such that $x\ll a$; if x is another such element of A, then $x\equiv\frac{a}{\infty}$ by definition, thus $x=\frac{a}{\infty}+(x-\frac{a}{\infty})$ by (D1), whence $\frac{a}{\infty}+x=x$; but we have seen that $\frac{a}{\infty}+x=\frac{a}{\infty}$, whence $x=\frac{a}{\infty}$. So the first assertion is proved. The second one follows immediately. ∎

1.10. Lemma. *Let* (a,b) *in* $\mathrm{H}(A)$*. Then* $a-b$ *is the unique largest element* x *of* A *such that* $b+x=a$*.*

Proof. Let $c=a-b$. Then $a=b+c$ by (D1), and for all x in A such that $b+x=a$, we have $x+\frac{b}{\infty}=c+\frac{b}{\infty}$ by (D2). But $a=b+c$, thus $a+\frac{b}{\infty}=a$, thus, by (D2), $\frac{b}{\infty}+c=c$, whence $x\leq c$. Furthermore, if x is also a maximum element of A such that $b+x=a$, then $x\equiv c$, thus $\frac{b}{\infty}\leq x$, whence $\frac{b}{\infty}+x=x$. But $x+\frac{b}{\infty}=c+\frac{b}{\infty}$, thus $x=c$, which concludes the proof. ∎

From lemma 1.10, we get immediately the following corollary:

1.11. Corollary. *Let* A *be a P.O.M.. Then there is at most one difference on* A *making* A *a D.P.O.M..* ∎

The reader can easily verify as an exercise that a P.O.M. A can be structured as a D.P.O.M. if and only if the following holds:

(i) *A is minimal;*

(ii) *For all a in A, there is a unique largest element x of A such that $x \ll a$, denoted by $\frac{a}{\infty}$ (Necessarily, $\frac{a}{\infty}$ is idem-multiple);*

(iii) *For all $a,\ b,\ c$ in A, $a+c=b+c$ implies $a+\frac{c}{\infty}=b+\frac{c}{\infty}$.*

(In this situation, $b-a$ is defined as $c+\frac{a}{\infty}$ for all c in A such that $a+c=b$). We will not use this characterization in this paper.

The following lemma summarizes some more arithmetical properties of D.P.O.M.'s:

1.12. Lemma. *Let A be a D.P.O.M.. Then the following holds:*

(i) *For all $a \leq b$ in A, $\frac{b-a}{\infty}=\frac{b}{\infty}=\frac{b}{\infty}-\frac{a}{\infty}$;*

(ii) *For all $a,\ b,\ c$ in A such that $b+c \leq a$, we have $a-(b+c)=(a-b)-c$;*

(iii) *For all (b,a) and (b',a') in $\mathrm{H}(A)$, we have $(b+b')-(a+a')=(b-a)+(b'-a')$;*

(iv) *For all $a,\ b$ in A, $\frac{a+b}{\infty}=\frac{a}{\infty}+\frac{b}{\infty}$;*

(v) *For all $a,\ b,\ c$ in A such that $b \geq c$ and $a \geq b-c$, we have $a-(b-c)=(a+c)-b$.*

Proof. (i) $\frac{b}{\infty} \ll b-a$ by (D2) and $b-a \leq b$, thus the conclusion follows from 1.9 and the easily checked fact that since A is minimal, the restriction of $\leq$ to $\frac{1}{\infty}A$ is antisymmetric.

(ii) Put $x=a-(b+c)$ and $y=(a-b)-c$. Then $x+(b+c)=y+(b+c)=a$, thus $x+\frac{b+c}{\infty}=y+\frac{b+c}{\infty}$. But $\frac{b+c}{\infty}$ is $\ll a$, thus $\ll x$ and $\ll y$ by (D2); it follows that $x=y$.

(iii) A simple computation, using previous results:

$$\begin{aligned}(b+b')-(a+a') &= ((b+b')-a)-a' \text{ by (ii)}\\ &= (b'+(b-a))-a' \text{ by (D2)}\\ &= (b-a)+(b'-a') \text{ by (D2).}\end{aligned}$$

(iv) follows immediately from (iii).

(v) Put $x=a-(b-c)$ and $y=(a+c)-b$. Then $x+b=y+b=a+c$, thus $x+\frac{b}{\infty}=y+\frac{b}{\infty}$; but $\frac{b}{\infty}=\frac{b-c}{\infty}$ is $\ll x$ (use (i)) and $\frac{b}{\infty} \ll y$, whence $x=y$. ■

1.13. Example. Let A be an idem-multiple, minimal P.O.M.. Then A is a D.P.O.M., the difference being defined by $b-a=b$ for all $a,\ b$ in A such that $a \leq b$.

1.14. Example. In [16], chapter 2, we introduced *complete P.O.M.'s*, and we proved that *every complete P.O.M. is a D.P.O.M..*

1.15. Example. *Every cone or every cone with infinity is a D.P.O.M.. In particular, every abelian group is a D.P.O.M..*

The case of abelian groups is trivial, but it shows that the preordering of a D.P.O.M. is not necessarily antisymmetric.

The reader can verify that *every D.P.O.M.'s is a sub-D.P.O.M. of a product of cones with infinity* (the proof of this fact proceeds as the proof of theorem 1.4). We will not need this result here, but one cannot not notice the similarity with separative P.O.M.'s (*via* theorem 1.4). The following lemma shows the 'trivial direction' of this connection.

1.16. Lemma. *Every D.P.O.M. is a separative P.O.M..*

Proof. Let A be a D.P.O.M.. First, A is minimal by (D1), thus preminimal. Let a, b be in A. If $a+b \leq 2b$, then, using (D2) and minimality of A, we obtain $a + \frac{b}{\infty} \leq b + \frac{b}{\infty}$, whence $a \leq b$. Now, if $a + b = 2a = 2b$, then $a + \frac{b}{\infty} = b$ by (D2); but $a \equiv b$ by previous result, thus $\frac{a}{\infty} = \frac{b}{\infty}$ by lemma 1.9 and minimality of A, whence $a + \frac{b}{\infty} = a$; it follows that $a = b$. ∎

Now, we shall prove a sort of converse of lemma 1.16. Let A be a sub-P.O.M. of a P.O.M. B. Put $H = \{(b, a) \in B \times A : \ a \leq b\}$. We introduce two binary relations $\equiv_\bullet$ and $\leq_\bullet$ on H by the following definitions:

$$(b, a) \equiv_\bullet (b', a') \Leftrightarrow \big(b \asymp b' \textbf{ and } (\exists c \in A)(c \preceq b' \textbf{ and } c + a' + b = c + a + b')\big),$$

and

$$(b, a) \leq_\bullet (b', a') \Leftrightarrow \big(b \preceq b' \textbf{ and } (\exists c \in A)(c \preceq b' \textbf{ and } c + a' + b \leq c + a + b')\big).$$

1.17. Lemma. *$\equiv_\bullet$ is an equivalence on H, $\leq_\bullet$ is a preordering of H containing $\equiv_\bullet$, and both are compatible with the addition.*

Proof. A straightforward verification. ∎

This lemma entitles us to define the quotient P.O.M. of $(H, +, (0,0), \leq_\bullet)$ by $\equiv_\bullet$; we will denote it by $B \dot{-} A$. For all (b, a) in H, we will denote the equivalence class of (b, a) modulo $\equiv_\bullet$ by $b \dot{-} a$. The *natural map* from B to $B \dot{-} A$ is by definition the map $(x \mapsto x \dot{-} 0)$. Note finally that the canonical map from $A \dot{-} A$ to $B \dot{-} A$ (that is, $(b \dot{-} a)_A \mapsto (b \dot{-} a)_B$) is obviously a P.O.M.-embedding; thus, we will identify $A \dot{-} A$ to a sub-P.O.M. of $B \dot{-} A$.

Now, we shall equip $(A \dot{-} A, B \dot{-} A)$ with a structure similar to the structure of D.P.O.M..

1.18. Lemma. *Let x, y in $B \dot{-} A$ such that $x \leq y$. Then for all (b, a) in x, there is (b', a') in y such that $a' + b \leq a + b'$; furthermore, if $b \in A$, then the element $(a + b') \dot{-} (a' + b)$ of $B \dot{-} A$ depends only on x and y. Denote this element by $y - x$. Then $A \dot{-} A$ is closed under this operation, which structures it as a D.P.O.M.. Furthermore, for all x, x' in $A \dot{-} A$ and all y in $B \dot{-} A$, $x + y \leq x'$ implies $y \leq x' - x$.*

Proof. An easy (but tedious) verification. ∎

We will need later one more proposition about D.P.O.M.'s:

1.19. Lemma. *Let f be a P.O.M.-homomorphism from a P.O.M. A to a D.P.O.M. E. Then there is a unique D.P.O.M.-homomorphism $\overline{f}$ from $A \dot{-} A$ to E such that $(\forall x \in A)(\overline{f}(x \dot{-} 0) = f(x))$.*

Proof. $\overline{f}$ is defined by $\overline{f}(b \dot{-} a) = f(b) - f(a)$; all the verifications are straightforward. ■

Let us finally complete the connection between separative P.O.M.'s and D.P.O.M.'s:

1.20. Lemma. *Let A be a P.O.M.. Then A is separative if and only if the natural map from A to $A \dot{-} A$ is a P.O.M.-embedding.*

Proof. By lemma 1.16, the necessary condition is immediate. Conversely, assume that A is separative. Let a, b be in A. If $a \dot{-} 0 \leq b \dot{-} 0$, then, by definition, $a \preceq b$ and there is c in A such that $c \preceq b$ and $a + c \leq b + c$. Since A is separative, we obtain $a \leq b$. Similarly, $a \dot{-} 0 = b \dot{-} 0$ implies $a = b$. ■

1.21. Lemma. *Let A be a sub-P.O.M. of a preminimal P.O.M. B. Then the natural map from $A \dot{-} A$ to $B \dot{-} B$ is a D.P.O.M.-embedding.*

Proof. Denote by e the natural map from $A \dot{-} A$ to $B \dot{-} B$. It is straightforward to check that e is a D.P.O.M.-homomorphism. Let $x = b \dot{-} a$ and $y = b' \dot{-} a'$ in $A \dot{-} A$ such that $e(x) \leq e(y)$. By definition, $b \preceq b'$ and there is c in B such that $c \preceq b'$ and $c + a' + b \leq c + a + b'$. Let m in $\mathbb{N}$ such that $c \leq mb$. Since B is preminimal, we have $mb' + a' + b \leq mb' + a + b'$, whence $x \leq y$ since $mb' \in A$. Similarly, $e(x) = e(y)$ implies $x = y$. ■

§2. AMALGAMATION PROPERTIES OF P.O.M.'s.

We first present a possible (standard) construction of the amalgamation of two P.O.M.'s above a third one. So let $f : A \to B$ and $g : A \to C$ be P.O.M.-homomorphisms. Let $\to$ be the binary relation defined on $B \times C$ by

$$(b, c) \to (b', c') \text{ if and only if } (\exists a \in A)\big(b = b' + f(a) \textbf{ and } c' = c + g(a)\big).$$

Let $\leftarrow$ be the inverse of $\to$, let $\equiv_*$ be the transitive closure of $(\to) \cup (\leftarrow)$, and let $\leq_*$ be the transitive closure of $(\leq) \cup (\to) \cup (\leftarrow)$ (where $\leq$ is of course $\leq_{B \times C}$). Then $\equiv_*$ is an equivalence on $B \times C$, $\leq_*$ is a preordering on $B \times C$, both are compatible with the addition. This entitles us to define the quotient P.O.M. D of $(B \times C, +, (0,0), \leq_*)$ by $\equiv_*$. For all (b, c) in $B \times C$, denote by $[b, c]$ its equivalence class modulo $\equiv_*$, and let $\overline{f}$ (resp. $\overline{g}$) be the map from C (resp. B) to D defined by $\overline{f}(c) = [0, c]$ and $\overline{g}(b) = [b, 0]$.

2.1. Lemma. *$(D, \overline{f}, \overline{g})$ is the amalgamation of (A, f, B, g, C), that is, it is an initial object in the category of all (X, u, v) such that X is a P.O.M., u is a P.O.M.-homomorphism from C to X, v is a P.O.M.-homomorphism from B to X and $u \circ g = v \circ f$.* ■

Therefore, from now on, we will denote the P.O.M. D which we just constructed by $B \amalg_{f,g} C$. It is often difficult to deduce properties of $B \amalg_{f,g} C$ from properties of B and C, but let us note the following elementary fact, which will be of importance in the sequel:

2.2. Lemma. *If B and C are minimal, then $B \amalg_{f,g} C$ is minimal.*

Proof. We have to prove that for all (b,c) and (b',c') in $B \times C$ such that $(b,c) \leq_* (b',c')$, there is (x,y) in $B \times C$ such that $(b,c)+(x,y) \equiv_* (b',c')$. Actually, an easy induction argument shows that it suffices to prove the conclusion when $(b,c) \to (b',c')$ or $(b,c) \leftarrow (b',c')$ or $(b,c) \leq (b',c')$. But in the first two cases, $(b,c) \equiv_* (b',c')$ and in the third case, the conclusion follows immediately from minimality of B and C. ∎

We turn now to a definition which will play an essential role throughout this work. Our terminology is borrowed from [7], where it covers similar notions.

2.3. Definition.
(i) *Let $e : A \to B$ be a P.O.M.-homomorphism. Then e is* transferable *when for every P.O.M.-homomorphism from A to a P.O.M. C, the natural map from C to $B \amalg_{e,f} C$ is a P.O.M.-embedding;*
(ii) *Let $\mathcal{E}$ be a class of embeddings. Then a P.O.M. E has the* transfer property *relatively to $\mathcal{E}$ when for every P.O.M.-homomorphism f from a P.O.M. A to E and every P.O.M.-embedding e from A to a P.O.M. B such that e is in $\mathcal{E}$, the natural map from E to $B \amalg_{e,f} E$ is a P.O.M.-embedding; we will drop the mention of $\mathcal{E}$ when $\mathcal{E}$ is the class of all P.O.M.-embeddings.*

It is obvious that every transferable P.O.M.-homomorphism is itself a P.O.M.-embedding (take $C = A$, $f = id_A$ in (i)). But unlike many other situations (abelian groups, Boolean algebras...), P.O.M.-homomorphisms are *not* always transferable — or, which is equivalent, all P.O.M.'s do not have the transfer property. Our goal in this chapter will be to characterize those P.O.M.'s which have the transfer property (theorem 2.9), and to show that in that case, the transfer property is true in a sort of 'hereditary' way (lemma 2.15).

2.4. Lemma. *Let A be a sub-P.O.M. of a P.O.M. B such that the inclusion map from A into B satisfies the following condition:*

$$\text{(UT)} \qquad (\forall a_0, a_1 \in A)(\forall b \in B)\big(a_0 + b \leq a_1 \Rightarrow (\exists x \in A)(b \leq x \textbf{ and } a_0 + x \leq a_1)\big).$$

Let f be a P.O.M.-homomorphism from A to a P.O.M. C. Then the natural map $\overline{e}$ from C to $B \amalg_{e,f} C$ satisfies the following statement:

$$(\forall c_0, c_1 \in C)(\overline{e}(c_0) \leq \overline{e}(c_1) \Rightarrow c_0 \leq c_1).$$

Here, (UT) stands for 'Upper Transferability'; there is a corresponding notion of 'lower transferability', but we will not use it in this work.

Proof. Consider the following binary relation $\prec$ on $B \times C$ defined by

$$(b,c) \prec (b',c') \Leftrightarrow (\forall a' \in A)\big(b' \leq a' \Rightarrow (\exists a \in A)(b \leq a \textbf{ and } f(a) + c \leq f(a') + c')\big).$$

Obviously, $\prec$ is a preordering of $B \times C$. Consider the binary relations $\rightarrow$, $\leftarrow$, $\leq_*$ defined at the beginning of this chapter.

Claim. $\prec$ *contains* $\leq_*$.

Proof of claim. Since $\prec$ is transitive, it suffices to prove that for all (b, c) and (b', c') in $B \times C$, $(b, c) \leq (b', c')$ or $(b, c) \rightarrow (b', c')$ or $(b, c) \leftarrow (b', c')$ implies $(b, c) \prec (b', c')$.

Case 1. $(b, c) \leq (b', c')$. Let a' in A such that $b' \leq a'$. Put $a = a'$. Then $b \leq a$ and $f(a) + c \leq f(a') + c'$.

Case 2. $(b, c) \rightarrow (b', c')$. Let $\overline{a}$ in A such that $b = b' + \overline{a}$ and $c' = c + f(\overline{a})$. Let a' in A such that $b' \leq a'$. Put $a = a' + \overline{a}$. Then $b \leq a$ and $f(a) + c \leq f(a') + c'$.

Case 3. $(b, c) \leftarrow (b', c')$. Let $\overline{a}$ in A such that $b' = b + \overline{a}$ and $c = c' + f(\overline{a})$. Let a' in A such that $b' \leq a'$. This means that $\overline{a} + b \leq a'$, thus, by (UT), there is a in A such that $b \leq a$ and $\overline{a} + a \leq a'$; thus $f(a) + c \leq f(a') + c'$.

In all three cases, $(b, c) \prec (b', c')$. ■ Claim .

Now we can finish the proof of lemma 2.4. Let c_0, c_1 in C such that $\overline{e}(c_0) \leq \overline{e}(c_1)$, *i.e.* $(0, c_0) \leq_* (0, c_1)$. By the claim, it follows that $(0, c_0) \prec (0, c_1)$. Taking $a' = 0$, we see that there is a in A such that $f(a) + c_0 \leq f(0) + c_1 = c_1$, whence $c_0 \leq c_1$. ■

Now, we will state and prove a sufficient condition for (full) transferability of a P.O.M.-homomorphism:

2.5. Proposition. *Let A be a sub-P.O.M. of a P.O.M. B such that A is a D.P.O.M. and the inclusion map from A into B satisfies (UT). Then the inclusion map from A into B is transferable.*

Proof. Denote by e be the inclusion map from A into B. Let f be a P.O.M.-homomorphism from A to a P.O.M. C, and let $\overline{e}$ be the natural map from C to $B \amalg_{e,f} C$. Since e satisfies (UT), by the result of lemma 2.4, it suffices to prove that $\overline{e}$ is one-to-one.

For all x in B, put $\delta(x) = \{(a, a') \in A \times A : a + x = a'\}$. Define a binary relation $\sqsubseteq$ on $B \times C$ by

$$(b, c) \sqsubseteq (b', c') \Leftrightarrow \big(\forall (a_0, a_1) \in \delta(b)\big)\big(\exists (a'_0, a'_1) \in \delta(b')\big)\big(f(a_1 - a_0) + c = f(a'_1 - a'_0) + c'\big).$$

Claim. $\sqsubseteq$ *contains* $\equiv_*$.

Proof of claim. Since $\sqsubseteq$ is obviously transitive, it suffices to prove that for all (b, c) and (b', c') in $B \times C$, $(b, c) \rightarrow (b', c')$ implies $(b, c) \sqsubseteq (b', c')$ and $(b', c') \sqsubseteq (b, c)$. So let a in A such that $b = b' + a$ and $c' = c + f(a)$. Let (a_0, a_1) in $\delta(b)$. Thus $(a'_0, a'_1) \in \delta(b')$ where $a'_0 = a_0 + a$ and $a'_1 = a_1$. But we have

$$\begin{aligned} a + (a'_1 - a'_0) &= (a_1 + a) - (a_0 + a) \text{ (by (D2))} \\ &= (a_1 - a_0) + \frac{a}{\infty} \text{ (by lemma 1.12, (iii))} \\ &= a_1 - a_0 \text{ (by (D2) and } a \leq a_1). \end{aligned}$$

It follows that $f(a_1 - a_0) + c = f(a) + f(a'_1 - a'_0) + c = f(a'_1 - a'_0) + c'$, whence $(b, c) \sqsubseteq (b', c')$.

Conversely, let (a'_0, a'_1) in $\delta(b')$. Thus $(a_0, a_1) \in \delta(b)$ where $a_0 = a'_0$ and $a_1 = a'_1 + a$. But $a_1 - a_0 = a + (a'_1 - a'_0)$ by (D2). It follows that $f(a_1 - a_0) + c = f(a) + f(a'_1 - a'_0) + c = f(a'_1 - a'_0) + c'$, whence $(b', c') \sqsubseteq (b, c)$. ■ Claim .

We can now finish the proof of proposition 2.5. Let c_0, c_1 in C such that $(0, c_0) \equiv_* (0, c_1)$. By the claim, $(0, c_0) \sqsubseteq (0, c_1)$, thus, since $(0, 0) \in \delta(0)$, there is (a, a') in $\delta(0)$ such that $f(0) + c_0 = f(a' - a) + c_1$; but necessarily $a = a'$, thus $c_0 = \alpha + c_1$ where $\alpha = f(\frac{a}{\infty})$, so that α is idem-multiple. Similarly, there is β in A, idem-multiple, such that $c_1 = \beta + c_0$. It follows that

$$c_1 = c_1 + \alpha + \beta = c_1 + 2\alpha + \beta = c_1 + \alpha = c_0.$$

Thus we have proved that $\overline{e}$ is one-to-one, which concludes the proof. ■

2.6. Corollary. *Let A be a sub-P.O.M. of a P.O.M. B. Then the natural embedding from $A \dot{-} A$ into $B \dot{-} A$ is transferable.*

Proof. Immediate from proposition 2.5 and lemma 1.18. ■

As another application of proposition 2.5, let us mention a generalization to P.O.M.'s of a result of H. Dobbertin (*see* [5]):

2.7. Corollary. *For every P.O.M. A, there is a P.O.M. B containing A (and constructed from A in a canonical way) satisfying the following* finite refinement property*:*

$$(\forall_{i<2} a_i, b_i)\big(a_0 + a_1 = b_0 + b_1 \Rightarrow (\exists_{i,j<2} c_{ij})(\bigwedge_{i<2}(a_i = c_{i0} + c_{i1} \text{ and } b_i = c_{0i} + c_{1i}))\big).$$

Furthermore, if A is minimal, then B is minimal.

Proof. Let R be the sub-P.O.M. of $\mathbb{N}^4$ generated by $\alpha = (1, 1, 0, 0)$, $\alpha' = (0, 0, 1, 1)$, $\beta = (1, 0, 1, 0)$ and $\beta' = (0, 1, 0, 1)$.

Claim 1. $R = \{(x, y, x', y') \in \mathbb{N}^4 : x + y' = x' + y\}$. Thus, R is a sub-D.P.O.M. of $\mathbb{N}^4$.

Proof of claim 1. The first fact can be easily proved by a straightforward induction on $x + y + x' + y'$. The second fact follows immediately. ■ Claim 1.

From claim 1, it is easy to deduce the

Claim 2. Let a, a', b, b' in A such that $a + a' = b + b'$. Then there is a [unique] P.O.M.-homomorphism from R to A sending α on a, α' on a', β on b, β' on b'. ■ Claim 2.

Now, let $I = \{(a, a', b, b') \in A^4 : a + a' = b + b'\}$; if $i = (a, a', b, b')$ in I, put $a_i = a$, $a'_i = a'$, $b_i = b$, $b'_i = b'$. Let X (resp. Y) be the direct sum of I copies of R (resp. $\mathbb{N}^4$). For all i in I, let α_i, α'_i, β_i, β'_i be the canonical generators of the i^{th} component of X (isomorphic to R), and let f be the P.O.M.-homomorphism from X to A sending α_i on a_i,

α_i' on a_i', β_i on b_i, β_i' on b_i', this for all i in I. Since X is a sub-D.P.O.M. of Y (by claim 1), it results from proposition 2.5 that the natural embedding e from X into Y is transferable. Thus the natural map from A into $\mathrm{R}(A) = Y \amalg_{e,f} A$ is a P.O.M.-embedding. Then, let B be the increasing union (more precisely, direct limit) of all the $\mathrm{R}^n(A)$ for $n \in \omega$. By lemma 2.2, if A is minimal, then $\mathrm{R}(A)$ is minimal; thus, B is minimal. By construction, B satisfies the finite refinement property. ■

2.8. Remark. The analogue of proposition 2.5 for separative P.O.M.'s is *false*: there are very easy examples where the inclusion map from a separative P.O.M. into a separative P.O.M. is *not* transferable. The proof of the following theorem should make clear which examples to consider.

2.9. Theorem. [definitions 1.2, 2.3] *Let E be a P.O.M.. Then the following are equivalent:*

(i) *E is separative;*

(ii) *E has the transfer property with respect to the class of all inclusion maps from a sub-P.O.M. of $\mathbb{N}^4$ with at most 4 generators into $\mathbb{N}^4$;*

(iii) *E has the transfer property.*

Proof. Let us first prove that (ii)⇒(i). So let us assume that E satisfies (ii). Using theorem 1.4, we prove that E is separative. First, note that (ii) remains true after having replaced $\mathbb{N}^4$ by $\mathbb{N}^2$ or $\mathbb{N}^3$. Then, we prove three claims.

Claim 1. E is preminimal.

Proof of claim 1. Let a, b, c, d in E such that $c \leq d$. Let M be the sub-P.O.M. of $\mathbb{N}^2$ generated by $\gamma = (1, 0)$ and $\delta = (1, 1)$. It is easy to verify that there is a [unique] P.O.M.-homomorphism f from M to E sending γ on c and δ on d. By assumption, f extends to a P.O.M.-homomorphism g from $\mathbb{N}^2$ to some P.O.M. F containing E. Let $e = g((0, 1))$. Then by construction, $c + e = d$. It follows immediately that $a + c = b + c$ implies $a + d = b + d$ and $a + c \leq b + c$ implies $a + d \leq b + d$. ■ Claim 1.

Claim 2. Let a, b in E such that $a + b \leq 2b$. Then $a \leq b$.

Proof of claim 2. Let P be the sub-P.O.M. of $\mathbb{N}^4$ generated by $\alpha = (1, 1, 0, 0)$, $\beta_0 = (1, 0, 1, 0)$, $\beta_1 = (0, 1, 0, 1)$ and $\beta = (0, 0, 0, 1)$. It is not difficult to prove that there is a [unique] P.O.M.-homomorphism f from P to E sending α to a and β_0, β_1 and β on b. Thus, by assumption, f extends to a P.O.M.-homomorphism g from $\mathbb{N}^4$ to some P.O.M. F containing E. Let $\gamma = (0, 1, 0, 0)$. Put $c = g(\gamma)$. Since $\alpha \leq \beta_0 + \gamma$ and $\beta_1 = \beta + \gamma$, we have $a \leq b + c$ and $b = b + c$, whence $a \leq b$. ■ Claim 2.

Claim 3. Let a, b in E such that $a + b = 2a = 2b$. Then $a = b$.

Proof of claim 3. Let Q be the sub-P.O.M. of $\mathbb{N}^3$ generated by $\alpha = (1, 1, 0)$, $\beta_0 = (1, 0, 0)$, $\beta_1 = (0, 1, 1)$ and $\beta = (0, 0, 1)$. It is not difficult to prove that there is a [unique] P.O.M.-homomorphism f from Q to E sending α on a and β_0, β_1 and β to b. By assumption, f extends to a P.O.M.-homomorphism g from $\mathbb{N}^3$ to some P.O.M. F containing E. Let $\gamma = (0, 1, 0)$, let $c = g(\gamma)$. Since $\alpha = \beta_0 + \gamma$ and $\beta_1 = \beta + \gamma$, we have $a = b + c$ and $b = b + c$, whence $a = b$. ■ Claim 3.

Now, separativeness of E results immediately from claims 1, 2 and 3. Thus we have proved that (ii) implies (i). Since (iii) trivially implies (ii), it remains to prove that (i) implies (iii).

So assume that E is separative. Let A be a sub-P.O.M. of a P.O.M. B, let e be the inclusion map from A into B, and let f be a P.O.M.-homomorphism from A to E. Our goal is to extend f to a P.O.M.-homomorphism from B to some P.O.M. containing E. First, since E is separative, E embeds into $E \dot{-} E$ by lemma 1.20. Now, since $E \dot{-} E$ is a D.P.O.M. (lemma 1.18), there is, by lemma 1.19, a D.P.O.M.-homomorphism f' from $A \dot{-} A$ to $E \dot{-} E$ such that $f'(y \dot{-} x) = f(y) \dot{-} f(x)$ for all $x \leq y$ in A. But by corollary 2.6, the natural embedding from $A \dot{-} A$ into $B \dot{-} A$ is transferable, thus f' extends to a P.O.M.-homomorphism g' from $B \dot{-} A$ to some P.O.M. F containing $E \dot{-} E$. Now define a map g from B to F by $g(x) = g'(x \dot{-} 0)$. It is immediate that g is a P.O.M.-homomorphism from B to F extending f. Thus we have proved that E has the transfer property, which concludes the proof. ∎

2.10. Remark. A similar but substantially simpler proof (using a construction similar to the construction of the D.P.O.M.'s $B \dot{-} A$ seen in chapter 1, but where the conditions "$b \preceq b'$" or "$b \asymp b'$" and "$c \preceq b'$" have been dropped in the definitions of $\leq_\bullet$ and $\equiv_\bullet$) would have given the corresponding result in commutative monoids:

2.11. Proposition. *A commutative monoid has the transfer property (in the class of commutative monoids) if and only if it is cancellative.* ∎

So we see that in fact, the transfer property is 'much more common' in the class of P.O.M.'s than in the class of commutative monoids. This slightly paradoxical situation (most of the P.O.M.'s we study are minimal, there is not much difference between these and commutative monoids except for the mention of the preordering, the transfer property for the preordering looks like one more condition to satisfy so the transferability should be more difficult to realize...) is essentially due to the fact that if A is a submonoid of a commutative monoid B, then the minimal preordering of A is *strictly contained* in the restriction to A of the minimal preordering of B.

2.12. Example. For every n in $\mathbb{N} \setminus \{0\}$, let $\mathcal{P}_n$ be the P.O.M. of equidecomposability classes of polyhedra of $\mathbb{R}^n$ modulo the [affine] isometries: here, only polyhedral pieces are allowed in the decompositions and sets of non zero codimension are identified to zero (*see* [2] for more about this). By Zylev's theorem, $\mathcal{P}_n$ is cancellative (so it is a *cone*); thus it is separative, thus it has the transfer property by theorem 2.9.

2.13. Example. Let G be a group operating on a set X, let $\mathcal{B}$ be a Boolean subalgebra of $\mathcal{P}(X)$ which is invariant by G. Consider the P.O.M. of equidecomposability types of elements of $\mathcal{B}$ modulo G, let us denote it by $\mathrm{S}(\mathcal{B})/G$ (*see* [14], [15]). Tarski proved in [13] the following theorem:

If G is exponentially bounded, then $\mathrm{S}(\mathcal{B})/G$ satisfies the following statement:

$$(\forall a, b, c)(a + c = b + 2c \Rightarrow a = b + c).$$

In fact, Tarski proves his theorem for commutative groups, but his proof applies as well for exponentially bounded groups (*see* [14], chapter 12). But this implies immediately that if G is exponentially bounded, then $\mathrm{S}(\mathcal{B})/G$ satisfies the statement $(\forall a, b)(a + b = 2b \Rightarrow a = b)$. Since $\mathrm{S}(\mathcal{B})/G$ is minimal, it follows immediately that it is separative. Therefore,

If G is exponentially bounded, then $\mathrm{S}(\mathcal{B})/G$ *has the transfer property.*

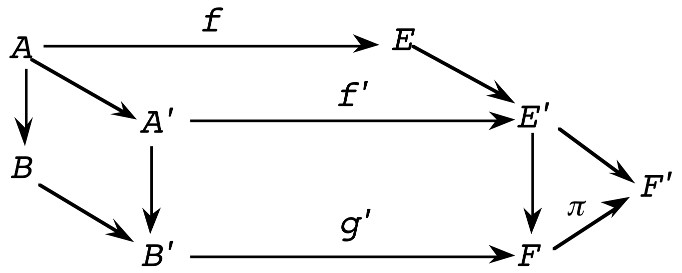
For $\mathcal{B} = \mathcal{P}(X)$, it results from the cancellation law (*see* [14], theorem 8.7) that the underlying semigroup of $\mathrm{S}(\mathcal{B})/G$ is always separative, even if G is not exponentially bounded. However, we do not know whether *the P.O.M.* $\mathrm{S}(\mathcal{B})/G$ is separative in general. A positive answer would be interesting, since it would allow us to remove the assumption that G is exponentially bounded in the statement of corollary 4.16.

2.14. Example. At the beginning of chapter 1, we gave the definition of the *pseudo-cancellation property.* Then it is immediate that every antisymmetric P.O.M. with the pseudo-cancellation property is separative; this is the case for the class of *strong refinement P.O.M.'s*, which are the antisymmetric, minimal P.O.M.'s satisfying the pseudo-cancellation property and the finite refinement property. Thus, *every strong refinement P.O.M. has the transfer property.* In particular, *every weak cardinal algebra (*i.e. *its associated P.O.M.) has the transfer property.*

Note that most of the P.O.M.'s of these examples, although they have the transfer property, do not embed into any injective P.O.M.. We will find in the next chapter weaker definitions of injectivity for which this irregular behaviour cannot happen. For this, an important lemma is the following.

2.15. Lemma. *Let A be a sub-P.O.M. of a* preminimal *P.O.M. B, let f be a P.O.M.-homomorphism from A to a separative P.O.M. E. Then f extends to a P.O.M.-homomorphism from B to some* separative *and* minimal *P.O.M. containing E.*

Proof. Put $A' = A \dot{-} A$, $B' = B \dot{-} B$ and $E' = E \dot{-} E$. By lemma 1.18, all three of them are D.P.O.M.'s. Denote by e' the natural embedding from A' into B' (this uses lemma 1.21). By lemma 1.19, there is a D.P.O.M.-homomorphism f' from A' to E' such that $f'(y \dot{-} x) = f(y) \dot{-} f(x)$ for all $x \leq y$ in A. Since E' and B' are minimal, it results from lemma 2.2 that $F = B' \amalg_{e',f'} E'$ is minimal; since E' is separative (lemmata 1.16 and 1.18), it results from theorem 2.9 that the natural P.O.M.-homomorphism from E' to F is a P.O.M.-embedding, so that we may identify E' with its image in F. Let g' be the natural P.O.M.-homomorphism from B' to F. Define a map g from B to F by putting $g(x) = g'(x \dot{-} 0)$. By construction, g is a P.O.M.-homomorphism from B to F extending f. Since F is preminimal, the natural map from E' to $F' = F \dot{-} F$ is a P.O.M.-embedding (lemma 1.21), so that we may identify E with its natural image in F'. Let π be the natural map from F to F', let $h = \pi \circ g$. The picture is as follows:



Then h is a P.O.M.-homomorphism from B to F' extending f. Furthermore, F' is separative and minimal (lemmata 1.16 and 1.18). ∎

2.16. Remark. Among the hypotheses of lemma 2.15, the assumption that B is preminimal cannot be dropped; we will have confirmation of this in chapter 4, as a consequence of theorem 4.4.

§3. RESTRICTED PREMINIMAL INJECTIVITY.

Let us first introduce some notations. Let κ be a cardinal, let $\mathcal{C}$ be a class of P.O.M.'s. Then we will denote by $\mathcal{C}_\kappa$ the class of all structures in $\mathcal{C}$ whose underlying set has size at most κ; furthermore, we will denote by $\mathcal{C}^*$ the class of *antisymmetric* elements of $\mathcal{C}$. Similarly, if $\mathcal{E}$ is a class of P.O.M.-homomorphisms, then we will denote by $\mathcal{E}_\kappa$ the class of all P.O.M.-homomorphisms $f : A \to B$ in $\mathcal{E}$ such that $|A| \leq \kappa$ and $|B| \leq \kappa$; furthermore, we will denote by $\mathcal{E}^*$ the class of homomorphisms in $\mathcal{E}$ between *antisymmetric* P.O.M.'s. If A is a sub-P.O.M. of a P.O.M. B, then we will sometimes denote by $(A \hookrightarrow B)$ the inclusion map from A into B.

3.1. Definition. *Let E be a P.O.M., let $\mathcal{E}$ be a class of P.O.M.-embeddings. Then E is* $\mathcal{E}$-injective *when for every $e : A \to B$ in $\mathcal{E}$ and every P.O.M.-homomorphism f from A to E, there is a P.O.M.-homomorphism g from B to E such that $g \circ e = f$. If $\mathcal{C}$ is a class of P.O.M.'s, then we will say that E is* $\mathcal{C}$-injective *when it is injective relatively to the class of all inclusion maps $A \hookrightarrow B$ where B is in $\mathcal{C}$.*

It is obvious on this definition that if E is $\mathcal{E}$-injective, then it has the transfer property relatively to $\mathcal{E}$. The converse is false, as injectivity appears as a notion of completeness.

Injective P.O.M.'s, *i.e.* P.O.M.'s which are injective relatively to the class of all P.O.M.'s, have been completely characterized in several ways in [16], [17]. One of these characterizations is that injective P.O.M.'s are exactly the retracts of the powers of $\overline{\mathbb{P}} = ([0, \infty], +, 0, \leq)$.

From now on, we will make frequent use of the following classes of P.O.M.'s:

POM = the class of all P.O.M.'s,

$PREM$ = the class of all preminimal P.O.M.'s,

SEP = the class of all separative P.O.M.'s,

and the following classes of P.O.M.-embeddings:

$\mathcal{IM}$ = class of all $(A \hookrightarrow B)$ such that B is preminimal and *idem-multiple generated over* A, *i.e.* $B = A + \frac{1}{\infty}B$,

$\mathcal{CO}$ = class of all $(A \hookrightarrow B)$ such that B is a cone.

The first theorem of this chapter proves that there are 'enough injectives' relatively to small classes in $PREM$:

3.2. Theorem. [definitions 1.2, 2.3, 3.1] *Let κ be an infinite cardinal, let A be a P.O.M.. Then the following are equivalent:*
(i) *A is separative;*
(ii) *A has the transfer property;*
(iii) *A embeds into a $PREM_\kappa$-injective P.O.M..*

Proof. We have already seen in theorem 2.9 that (i) and (ii) are equivalent; also, (iii) implies (i) by characterization (ii) of separativeness in theorem 2.9. So it remains to prove that (i) implies (iii).

If A and B are P.O.M.'s, then write $A <_\kappa B$ the following statement:

"A is a sub-P.O.M. of B, and for all $(X \hookrightarrow Y)$ such that Y is in $PREM_\kappa$, every P.O.M.-homomorphism from X to A extends to a P.O.M.-homomorphism from Y to B."

Claim. For every separative P.O.M. A, there is a separative P.O.M. B such that $A <_\kappa B$.

Proof of claim. Let Δ be a *set* of representatives of all triples (X, Y, f) such that X is a sub-P.O.M. of Y, Y is in $PREM_\kappa$ and f is a P.O.M.-homomorphism from X to A, modulo the relation of isomorphy; write $\Delta = \{(X_i, Y_i, f_i) : i \in I\}$. Let $X = \coprod_{i\in I} X_i$, $Y = \coprod_{i\in I} Y_i$, and let $f : X \to A$, $(x_i)_{i\in I} \mapsto \sum_{i\in I} f_i(x_i)$. Then X is a sub-P.O.M. of Y and Y is preminimal. By lemma 2.15, there is a separative P.O.M. B containing A such that f extends to a P.O.M.-homomorphism from Y to B. By construction, we have $A <_\kappa B$. ■ Claim .

Now, let A be a separative P.O.M.. We construct inductively a κ^+-sequence $(A_\xi)_{\xi\leq\kappa^+}$ as follows:

$$\begin{cases} A_0 = A; \\ A_{\xi+1} = \text{some separative P.O.M. } B \text{ such that } A <_\kappa B \ (\xi < \kappa^+); \\ A_\lambda = \bigcup_{\xi<\lambda} A_\xi \ (\lambda \text{ limit } \leq \kappa^+). \end{cases}$$

Finally, put $B = A_{\kappa^+}$.

Now, let X be a sub-P.O.M. of a P.O.M. Y in $PREM_\kappa$, let f be a P.O.M.-homomorphism from X to B. So there is $\xi < \kappa^+$ such that the range of f is contained in A_ξ. Since $A_\xi <_\kappa A_{\xi+1}$, f extends to a P.O.M.-homomorphism g from Y to $A_{\xi+1}$. Thus g is an extension of f from Y to B. Therefore, B is $PREM_\kappa$-injective. ■

3.3. Remark. By following the proof of theorem 3.11 of [16], one can prove that sub-P.O.M.'s of $PREM$-injective P.O.M.'s satisfy *e.g.* the statement

$$(\forall a, b)\left(\bigwedge_{n\in\mathbb{N}}(na \leq (n+1)b) \Rightarrow a \leq b\right),$$

which is not the case for all separative P.O.M.'s, as *e.g.* $\mathcal{P}_3$ seen in example 2.12 (take a=equidecomposability class of the cube of volume 1, b=equidecomposability class of the regular tetrahedron of volume 1).

In this chapter, we shall present an 'arithmetical' characterization of $PREM_\kappa$-injective P.O.M.'s, as close as possible to the one presented in [16] (see theorems 3.6, 3.10, 3.13).

Let us introduce the following definition.

3.4. DEFINITION. *Let κ be an infinite cardinal. A preminimal P.O.M. E is said to be κ-*smooth *when it satisfies the following conditions:*

(SM 1) *Let a, b in E and let X be a subset of E of size at most κ such that $(\forall x \in X)(a + x \leq b + x)$; then there exists $c \leq X$ in $\frac{1}{\infty}E$ such that $a + c \leq b + c$;*

(SM 2) *Let a, b in E and let X be a subset of E of size at most κ such that $(\forall x \in X)(a + x = b + x)$; then there exists $c \leq X$ in $\frac{1}{\infty}E$ such that $a + c = b + c$;*

(SM 3) *For all subsets X and Y of size at most κ of E such that $X \ll Y$, there is c in $\frac{1}{\infty}E$ such that $X \ll c$ and $c \leq Y$.*

Note that in the context of (SM 3), $c \leq Y$ is equivalent to $c \ll Y$.

Smoothness of P.O.M.'s will be one of the notions leading to simple formulations of existence of solutions of *linear systems*, which we shall show now. Say that a *linear system* with parameters from a P.O.M. E is a set of atomic formulas (in the sense of model theory, *see e.g.* [3]) with parameters from E, and unrestricted number of variables (called here *unknowns*). For example, linear systems with only one unknown x may be written in the following general form:

$$\begin{cases} a_i + m_i x \leq b_i + n_i x & \text{(all } i \text{ in } I) \\ a_j + m_j x = b_j + n_j x & \text{(all } j \text{ in } J) \end{cases}$$

In connection with smoothness of embeddings, we will consider linear systems whose unique unknown has to be thought as idem-multiple. Such systems can always be written in the following form:

$$\begin{cases} a_i \leq b_i + x & \text{(all } i \text{ in } I_\leq) \\ a_i = b_i + x & \text{(all } i \text{ in } I_=) \\ a_i \geq b_i + x & \text{(all } i \text{ in } I_\geq) \\ a_i + x \leq b_i + x & \text{(all } i \text{ in } J) \\ a_i + x = b_i + x & \text{(all } i \text{ in } K) \\ x = 2x \end{cases} \tag{3.1}$$

where the a_i, b_i are all in E. But now, observe that when E is preminimal, for all a, b in E and all idem-multiple x in E, we have

$$a \leq b + x \Leftrightarrow a + x \leq b + x,$$
$$a = b + x \Leftrightarrow b \leq a \textbf{ and } x \leq a \textbf{ and } a + x = b + x,$$

and

$$a \geq b + x \Leftrightarrow b \leq a \textbf{ and } x \leq a \textbf{ and } a + x \geq b + x,$$

so that in E, the linear system (3.1) is equivalent (in an effective way) to a linear system of the following form:

$$(3.2) \qquad \begin{cases} \text{(S)} & \text{(some system without } x\text{)} \\ a_i + x \leq b_i + x & \text{(all } i \text{ in } I'\text{)} \\ a_j + x = b_j + x & \text{(all } j \text{ in } J'\text{)} \\ x \leq c_k & \text{(all } k \text{ in } K'\text{)} \\ x = 2x & \end{cases}$$

Now, define the *resolvent* of the system (3.1) to be the following system:

$$(3.3) \qquad \begin{cases} \text{(S)} & \\ a_i + c_k \leq b_i + c_k & \text{(all } (i,k) \text{ in } I' \times K'\text{)} \\ a_j + c_k = b_j + c_k & \text{(all } (j,k) \text{ in } J' \times K'\text{)} \end{cases}$$

3.5. Lemma. *Let κ be an infinite cardinal, let E be a P.O.M., let (S) be a linear system of the form (3.1), of size at most κ with parameters from E. If (S) admits a solution in some preminimal extension of E, then its resolvent is satisfied in E. Conversely, if E is κ-smooth and satisfies the resolvent of (S), then (S) admits a solution in E.*

Proof. By what we have just seen, it suffices to consider the case of a linear system of the form

$$(3.4) \qquad \begin{cases} a_i + x \leq b_i + x & \text{(all } i \text{ in } I\text{)} \\ a_j + x = b_j + x & \text{(all } j \text{ in } J\text{)} \\ x \leq c_k & \text{(all } k \text{ in } K\text{)} \\ x = 2x & \end{cases}$$

where $I,\ J,\ K$ are of size at most κ. Then the resolvent of (3.4) is the following linear system:

$$(3.5) \qquad \begin{cases} a_i + c_k \leq b_i + c_k & \text{(all } (i,k) \text{ in } I \times K\text{)} \\ a_j + c_k = b_j + c_k & \text{(all } (j,k) \text{ in } J \times K\text{)} \end{cases}$$

If (3.4) holds in some preminimal extension of E, it is immediate that (3.5) holds. Conversely, assume that E is κ-smooth and that it satisfies (3.5). Let i in I; by (3.5) and (SM 1), there is e_i in $\frac{1}{\infty}E$ such that $(\forall k \in K)(e_i \leq c_k)$ and $a_i + e_i \leq b_i + e_i$; similarly, by (3.5) and (SM 2), for all j in J, there is e_j in $\frac{1}{\infty}E$ such that $(\forall k \in K)(e_j \leq c_k)$ and $a_j + e_j = b_j + e_j$. By preminimality of E, we have $(\forall (i,k) \in (I \cup J) \times K)(e_j \ll c_k)$, thus, by (SM 3), there is x in $\frac{1}{\infty}E$ such that $(\forall i \in I \cup J)(e_i \ll x)$ and $(\forall k \in K)(x \ll c_k)$. By preminimality of E, it is immediate that x satisfies (3.4). ■

Say that a linear system of the form (3.1) (with parameters from some P.O.M. E) is *compatible* when its resolvent is satisfied by E.

Lemma 3.5 is very useful in the proof of the following theorem:

3.6. Theorem. [definitions 3.1, 3.4] *Every ω-smooth P.O.M. is separative. Furthermore, let κ be an infinite cardinal and let E be a preminimal P.O.M.. Then the following are equivalent:*

(i) *E is κ-smooth;*

(ii) *Every compatible linear system of the form (3.1) of size at most κ with parameters from E admits a solution in E;*

(iii) *E is $\mathcal{IM}_\kappa$-injective;*

(iv) *E is $\mathcal{IM}_\kappa^*$-injective.*

Proof. (i)⇒(ii) is immediate from lemma 3.5.

(ii)⇒(iii) Let E satisfy (ii). We first prove that E is separative (the first result listed in the theorem follows). So let a, b in E. Assume first that $a + b \leq 2b$. Consider the following linear system:

$$\begin{cases} a + x \leq b + x \\ x \leq b \\ 2x = x \end{cases}$$

Then the resolvent of this linear system is just $a+b \leq 2b$, which holds by assumption; thus, by hypothesis, it admits a solution in E, say x. But $x = 2x \leq b$ and E is preminimal, thus $x + b = b$; hence $a \leq b$. Suppose now that $a + b = 2a = 2b$. Consider the following linear system:

$$\begin{cases} a + x = b + x \\ x \leq a \\ x \leq b \\ x = 2x \end{cases}$$

Then the resolvent of this linear system is just $a + b = 2a = 2b$, which holds by assumption, thus it admits a solution, say x. As in the previous case, $x \ll a$ and $x \ll b$, whence $a = a + x = b + x = b$. So we have proved that E is separative. Now, let us prove that E is $\mathcal{IM}_\kappa$-injective. It suffices to prove that for every sub-P.O.M. A of E of size at most κ and every P.O.M. B containing A such that $B = A + \mathbb{N}b$ for some b in $\frac{1}{\infty}B$, $(A \hookrightarrow B)$ extends to a P.O.M.-homomorphism r from B to E. But an element x of E is the value $r(b)$ for such an r if and only if x satisfies all the equations and inequations with parameters from A satisfied by b; let (S) be the corresponding linear system. Then b is idem-multiple, $b \in B$, B is preminimal and b is a solution of (S), thus the resolvent of (S) holds by lemma 3.5. Thus, by hypothesis, (S) admits a solution in E, which solves the extension problem. Thus (iii) holds.

The fact that (iii)⇒(iv) is trivial. Finally, assume (iv). For all $\alpha \leq \kappa$, denote by $(\delta_i)_{i<\alpha}$ the canonical basis of $\mathbb{N}^{(\alpha)}$. Let C_α be the P.O.M. $\mathbb{N} \cup \{s\} \cup (\mathbb{N}^{(\alpha)} \setminus \{0\})$ equipped with the addition defined by $1 + s = s = 2s$ and $s + \delta_i = \delta_i$ for all $i < \alpha$, and its natural (minimal) ordering.

Let first $\alpha \leq \kappa$, a, b, c_i $(i < \alpha)$ in E such that $a + c_i \leq b + c_i$ (all $i < \kappa$). Consider the sub-P.O.M. A of C_α^4 generated by $u = (1, 1, 0, 0)$, $v = (1, 0, 1, 1)$, $w_i = (0, \delta_i, \delta_i, 0)$ (all $i < \alpha$). It is not difficult to verify that there is a [unique] P.O.M.-homomorphism f from A to E sending u on a, v on b and w_i on c_i (all $i < \alpha$). Let B be the sub-P.O.M. of C_α^4 generated by A and $w = (0, s, s, 0)$; thus B is minimal. Then $(A \hookrightarrow B)$ is in $\mathcal{IM}_\kappa^*$, thus, by hypothesis, f extends to a P.O.M.-homomorphism g from B to E. Let $c = g(w)$. Then $a + c \leq b + c$ and $c = 2c$ and $c \leq c_i$ for all i in α. Thus (SM 1) is satisfied. One checks similarly (SM 2): if a, b, c_i $(i < \kappa)$ are elements of E such that $a + c_i = b + c_i$ (all $i < \alpha$), let A be the sub-P.O.M. of C_α^3 generated by $u = (1, 1, 0)$, $v = (1, 0, 1)$, $w_i = (0, \delta_i, \delta_i)$ (all $i < \alpha$); it is not difficult to verify that there is a [unique] P.O.M.-homomorphism f from A to E sending u on a, v on b and w_i on c_i (all $i < \alpha$). Let B be the sub-P.O.M. of C_α^3 generated by A and $w = (0, s, s)$; thus B is minimal. Then $(A \hookrightarrow B)$ is in $\mathcal{IM}_\kappa^*$, thus, by hypothesis, f extends to a P.O.M.-homomorphism g from B to E. Let $c = g(w)$. Then $a + c = b + c$ and $c = 2c$ and $c \leq c_i$ for all i in α, which verifies (SM 2). It remains to check (SM 3). So let X, Y be subsets of E of size at most κ such that $X \ll Y$. Let B be the P.O.M. $(\mathbb{N}^{(X)} \times \{0\}) \cup \{s\} \cup ((\mathbb{N}^{(Y)} \setminus \{0\}) \times \{1\})$, equipped with the addition defined componentwise on $\mathbb{N}^{(X)}$ and on $\mathbb{N}^{(Y)} \setminus \{0\}$, and by $x + s = s$ if $x \in \mathbb{N}^{(X)} \times \{0\}$, $s + y = y$ if $y \in (\mathbb{N}^{(Y)} \setminus \{0\}) \times \{1\}$ and $2s = s$, and with its natural (minimal) ordering, and let A be the sub-P.O.M. of B defined by $A = (\mathbb{N}^{(X)} \times \{0\}) \cup ((\mathbb{N}^{(Y)} \setminus \{0\}) \times \{1\})$. It is easy to verify that there is a [unique] P.O.M.-homomorphism f from A to E sending $(\delta_x, 0)$ on x (all $x \in X$) and $(\delta_y, 1)$ on y (all $y \in Y$). By hypothesis, f extends to a P.O.M.-homomorphism g from B to E. Put $c = g(s)$. Then $c = 2c$ and $X \ll c$ and $c \leq Y$. So we have verified (SM 3). Thus (i) holds, which concludes the proof. ■

From theorem 3.6, we can deduce an immediate corollary:

3.7. Corollary. *Let κ be an infinite cardinal. Then every κ-smooth P.O.M. is injective relatively to the class of all natural embeddings from some separative P.O.M. A of size at most κ into $A \dot{-} A$.*

Proof. Let E be a κ-smooth P.O.M., let f be a P.O.M.-homomorphism from some P.O.M. A of size at most κ to E. Let B be a minimal sub-P.O.M. of E of size at most κ containing fA. By 1.19, f extends to a P.O.M.-homomorphism f' from $A' = A \dot{-} A$ to $B' = B \dot{-} B$. Since B is separative, we shall as usual (using lemma 1.20) identify B with its natural image in B'.

Claim. $(B \hookrightarrow B')$ *is in* $\mathcal{IM}$.

Proof of claim. Let x in $B \dot{-} B$; write $x = b \dot{-} a$ where a, b are in B and $a \leq b$. Since B is minimal, there is c in B such that $a + c = b$, whence $x = (a + c) \dot{-} a = c + (a \dot{-} a)$. Thus $B \dot{-} B$ is generated by B and the set of all $a \dot{-} a$, $a \in B$. The conclusion follows.■ Claim .

By theorem 3.6, it follows that $(B \hookrightarrow B')$ extends to a P.O.M.-homomorphism r from B' to E. Then $r \circ f'$ is a P.O.M.-homomorphism from A' to E extending f. ■

We shall now study another kind of linear system, *via* another form of injectivity. We will first need some preliminary constructions. Let I, J, K be arbitrary sets such that

$I \cap J = K$, and let $\vec{n}$ be in $\mathbb{N}^{I \cup J}$ (not necessarily with finite support). Put $\overline{n} = (I, J, K, \vec{n})$. We will associate with $\overline{n}$ a certain P.O.M.-inclusion $(A_{\overline{n}} \hookrightarrow B_{\overline{n}})$, whose construction we now show.

First, let $\mathcal{B}_{\overline{n}}$ be the P.O.M. $\mathbb{N}^{(I \cup J)} \times \mathbb{N}^{(I \cup J)} \times \mathbb{N}$. We equip $\mathcal{B}_{\overline{n}}$ with binary relations $\equiv_*$ and $\leq_*$ respectively defined by

$$(\vec{p}, \vec{q}, m) \equiv_* (\vec{p'}, \vec{q'}, m') \Leftrightarrow (\exists \vec{r}, \vec{s} \in \mathbb{N}^{(K)}) \begin{cases} \vec{p} + \vec{s} = \vec{p'} + \vec{r} \\ \vec{q} + \vec{r} = \vec{q'} + \vec{s} \\ m + \vec{s} \cdot \vec{n} = m' + \vec{r} \cdot \vec{n} \end{cases} \tag{3.6}$$

($\cdot$ denotes the canonical 'scalar product'), and

$$(\vec{p}, \vec{q}, m) \leq_* (\vec{p'}, \vec{q'}, m') \Leftrightarrow (\exists \vec{r} \in \mathbb{N}^{(I)})(\exists \vec{s} \in \mathbb{N}^{(J)}) \begin{cases} \vec{p} + \vec{s} \leq \vec{p'} + \vec{r} \\ \vec{q} + \vec{r} \leq \vec{q'} + \vec{s} \\ m + \vec{s} \cdot \vec{n} \leq m' + \vec{r} \cdot \vec{n} \end{cases} \tag{3.7}$$

3.8. Lemma. *$\equiv_*$ is an equivalence on $\mathcal{B}_{\overline{n}}$, $\leq_*$ is a preordering on $\mathcal{B}_{\overline{n}}$ containing $\equiv_*$, both are compatible with the addition. Furthermore, $\mathcal{B}_{\overline{n}}$ satisfies the three following statements:*

$$(\forall x, y)(x \equiv_* y \Leftrightarrow x \leq_* y \textbf{ and } y \leq_* x),$$
$$(\forall x, y, z)(x + z \leq_* y + z \Rightarrow x \leq_* y),$$

and

$$(\forall x, y, z)(x + z \equiv_* y + z \Rightarrow x \equiv_* y).$$

Proof. A straightforward verification. ∎

Consequently, one can define the quotient P.O.M. $B_{\overline{n}}$ of $(\mathcal{B}_{\overline{n}}, +, 0, \leq_*)$ by $\equiv_*$, and it is a cancellative, antisymmetric P.O.M.. Therefore, it embeds canonically into some antisymmetric cone (the positive cone of the ordered group of differences of $B_{\overline{n}}$), say $C_{\overline{n}}$. For each $(\vec{p}, \vec{q}, m)$ in $\mathcal{B}_{\overline{n}}$, denote by $[\vec{p}, \vec{q}, m]$ its equivalence class modulo $\equiv_*$. Let $A_{\overline{n}} = \{[\vec{p}, \vec{q}, 0] : \vec{p}, \vec{q} \in \mathbb{N}^{(I \cup J)}\}$. Define monoids $\mathcal{O}_{\overline{n}}$ and $\mathcal{E}_{\overline{n}}$ by

$$\mathcal{O}_{\overline{n}} = \left\{ (\vec{p}, \vec{q}, \vec{p'}, \vec{q'}) \in (\mathbb{N}^{(I \cup J)})^4 : [\vec{p}, \vec{q}, 0] \leq [\vec{p'}, \vec{q'}, 0] \right\}, \tag{3.8}$$

and

$$\mathcal{E}_{\overline{n}} = \left\{ (\vec{p}, \vec{q}, \vec{p'}, \vec{q'}) \in (\mathbb{N}^{(I \cup J)})^4 : [\vec{p}, \vec{q}, 0] = [\vec{p'}, \vec{q'}, 0] \right\}. \tag{3.9}$$

Choose sets of generators $\mathcal{O}'_{\overline{n}}$ of $\mathcal{O}_{\overline{n}}$, $\mathcal{E}'_{\overline{n}}$ of $\mathcal{E}_{\overline{n}}$. Using the fact that for every k in ω, the monoid of solutions in $\mathbb{N}^k$ of every linear equation system (with k unknowns and coefficients in $\mathbb{N}$) is finitely generated (*see e.g.* [4], Vol. 2, page 130, corollary 9.19), one can easily suppose that if I and J are finite, then $\mathcal{O}'_{\overline{n}}$ and $\mathcal{E}'_{\overline{n}}$ are finite — one has to find finite sets of generators for the $(\vec{p}, \vec{q}, \vec{p'}, \vec{q'}, \vec{r}, \vec{s})$ satisfying the right hand side (without existential quantifiers) of (3.6) and (3.7), and then take the projection on the first four coordinates.

Now, for all i in $I \cup J$, let $u_i = [\delta_i, \vec{0}, 0]$ and $v_i = [\vec{0}, \delta_i, 0]$; put $\xi = [\vec{0}, \vec{0}, 1]$. Then u_i, v_i are all in $A_{\overline{n}}$ and it is immediate, using the definition of $\equiv_*$ and $\leq_*$, that the following linear system is satisfied by $B_{\overline{n}}$:

$$(3.10) \qquad \begin{cases} u_i + n_i\xi \leq v_i & \text{(all } i \text{ in } I) \\ v_j \leq u_j + n_j\xi & \text{(all } j \text{ in } J) \\ u_k + n_k\xi = v_k & \text{(all } k \text{ in } K) \end{cases}$$

We shall now prove that in a certain sense, this linear system is fundamental among all the linear systems of this form. So let E be a P.O.M., let I, J, K be arbitrary sets; consider the following linear system:

$$(3.11) \qquad \begin{cases} a_i + n_i x \leq b_i & \text{(all } i \text{ in } I) \\ b_j \leq a_j + n_j x & \text{(all } j \text{ in } J) \\ a_k + n_k x = b_k & \text{(all } k \text{ in } K) \end{cases}$$

where the a_i, b_i (i in $I \cup J$) are elements of E. It is easy to see that (3.11) is, in a canonical way, equivalent to a similar linear system where this time, $K = I \cap J$; such a system will be said to be in *normal form*. So we may assume without loss of generality that (3.11) is put in normal form, *i.e.* $K = I \cap J$. Now, say that a *resolvent* of (3.11) is any linear system of the following form:

$$(3.12) \qquad \begin{cases} \vec{p} \cdot \vec{a} + \vec{q} \cdot \vec{b} \leq \vec{p'} \cdot \vec{a} + \vec{q'} \cdot \vec{b} & \text{(all } (\vec{p}, \vec{q}, \vec{p'}, \vec{q'}) \text{ in } \mathcal{O}'_{\overline{n}}) \\ \vec{p} \cdot \vec{a} + \vec{q} \cdot \vec{b} = \vec{p'} \cdot \vec{a} + \vec{q'} \cdot \vec{b} & \text{(all } (\vec{p}, \vec{q}, \vec{p'}, \vec{q'}) \text{ in } \mathcal{E}'_{\overline{n}}) \end{cases}$$

Obviously, all the resolvents of a given linear system of the form (3.11) are equivalent, so that we will sometimes speak about *the* resolvent of the linear system. The interest of not necessarily taking $\mathcal{O}'_{\overline{n}} = \mathcal{O}_{\overline{n}}$ or $\mathcal{E}'_{\overline{n}} = \mathcal{E}_{\overline{n}}$ is essentially in the case where I and J are *finite*, so that what we describe here can be in fact used as an *algorithm.* Note that (3.12) is satisfied if and only if there exists a [necessarily unique] P.O.M.-homomorphism from $A_{\overline{n}}$ to E sending u_i on a_i, v_i on b_i (all i in $I \cup J$). This remark will be used in the following lemma:

3.9. Lemma. *Let E be a P.O.M., let κ be an infinite cardinal, let (S) be a linear system of the form (3.11) and of size at most κ. If (S) admits a solution in some extension of E, then E satisfies the resolvent of (S). Conversely, if E is $\mathcal{CO}^*_\kappa$-injective and satisfies the resolvent of (S), then (S) admits a solution in E.*

Proof. Without loss of generality, we may assume that (S) is just (3.11), written under normal form. Assume first that (S) admits a solution, say x, in some extension, say F, of E. We prove that any resolvent of (S), say (3.12), is satisfied in E. So let first $(\vec{p}, \vec{q}, \vec{p}', \vec{q}')$ in $\mathcal{E}'_{\vec{n}}$. By definition of $\equiv_*$, there are $\vec{r}$ and $\vec{s}$ in $\mathbb{N}^{(K)}$ such that the following holds:

$$
\begin{cases} \vec{p} + \vec{s} = \vec{p}' + \vec{r} \\ \vec{q} + \vec{r} = \vec{q}' + \vec{s} \\ \vec{s} \cdot \vec{n} = \vec{r} \cdot \vec{n} \end{cases} \tag{3.13}
$$

By possibly substracting $\vec{r} \wedge \vec{s}$ from $\vec{r}$ and $\vec{s}$, we may assume without loss of generality that $\vec{r}$ and $\vec{s}$ are incompatible. Therefore, (3.13) implies that $\vec{s} \leq \vec{p}'$ and $\vec{r} \leq \vec{p}$, thus, using the first equation of (3.13), there exists $\vec{h}$ in $\mathbb{N}^{(I \cup J)}$ such that $\vec{p} = \vec{r} + \vec{h}$ and $\vec{p}' = \vec{s} + \vec{h}$. Similarly, there is $\vec{k}$ in $\mathbb{N}^{(I \cup J)}$ such that $\vec{q} = \vec{s} + \vec{k}$ and $\vec{q}' = \vec{r} + \vec{k}$. It follows that

$$
\begin{aligned}
\vec{r} \cdot \vec{a} + \vec{s} \cdot \vec{b} &= \vec{r} \cdot \vec{a} + \vec{s} \cdot (\vec{a} + \vec{n}x) \\
&= \vec{r} \cdot \vec{a} + (\vec{s} \cdot \vec{n})x + \vec{s} \cdot \vec{a} \\
&= \vec{r} \cdot (\vec{a} + \vec{n}x) + \vec{s} \cdot \vec{a} \\
&= \vec{r} \cdot \vec{b} + \vec{s} \cdot \vec{a},
\end{aligned}
$$

the first and the last step being justified by the fact that $\vec{r}$ and $\vec{s}$ are in $\mathbb{N}^{(K)}$,

from which it follows easily that $\vec{p} \cdot \vec{a} + \vec{q} \cdot \vec{b} = \vec{p}' \cdot \vec{a} + \vec{q}' \cdot \vec{b}$. Thus the first part of (3.12) is satisfied. Now, let $(\vec{p}, \vec{q}, \vec{p}', \vec{q}')$ in $\mathcal{O}'_{\vec{n}}$. By definition of $\leq_*$, there are $\vec{r}$ in $\mathbb{N}^{(I)}$ and $\vec{s}$ in $\mathbb{N}^{(J)}$ such that the following holds:

$$
\begin{cases} \vec{p} + \vec{s} \leq \vec{p}' + \vec{r} \\ \vec{q} + \vec{r} \leq \vec{q}' + \vec{s} \\ \vec{s} \cdot \vec{n} \leq \vec{r} \cdot \vec{n} \end{cases} \tag{3.14}
$$

By possibly substracting $\vec{r} \wedge \vec{s}$ from $\vec{r}$ and $\vec{s}$, we may assume without loss of generality that $\vec{r}$ and $\vec{s}$ are incompatible. Therefore, (3.14) implies that $\vec{s} \leq \vec{p}'$, thus, using the first inequation of (3.14), there exists $\vec{h}$ in $\mathbb{N}^{(I \cup J)}$ such that $\vec{p} \leq \vec{r} + \vec{h}$ and $\vec{p}' = \vec{s} + \vec{h}$. Similarly, there is $\vec{k}$ in $\mathbb{N}^{(I \cup J)}$ such that $\vec{q} \leq \vec{s} + \vec{k}$ and $\vec{q}' = \vec{r} + \vec{k}$. It follows that

$$
\begin{aligned}
\vec{r} \cdot \vec{a} + \vec{s} \cdot \vec{b} &\leq \vec{r} \cdot \vec{a} + \vec{s} \cdot (\vec{a} + \vec{n}x) \\
&\leq \vec{r} \cdot \vec{a} + (\vec{s} \cdot \vec{n})x + \vec{s} \cdot \vec{a} \\
&\leq \vec{r} \cdot (\vec{a} + \vec{n}x) + \vec{s} \cdot \vec{a} \\
&\leq \vec{r} \cdot \vec{b} + \vec{s} \cdot \vec{a},
\end{aligned}
$$

the first and the last step being justified by the fact that $\vec{r} \in \mathbb{N}^{(I)}$ and $\vec{s} \in \mathbb{N}^{(J)}$,

from which it follows easily that $\vec{p} \cdot \vec{a} + \vec{q} \cdot \vec{b} \leq \vec{p'} \cdot \vec{a} + \vec{q'} \cdot \vec{b}$. Hence, (3.12) is satisfied by E.

Conversely, suppose that (3.12) is satisfied by E and that E is $\mathcal{CO}_\kappa^*$-injective. Since E satisfies (3.12), there exists, as remarked before, a [unique] P.O.M.-homomorphism f from $A_{\overline{n}}$ to E such that for all i in $I \cup J$, $f(u_i) = a_i$ and $f(v_i) = b_i$. Since $\mathcal{C}_{\overline{n}}$ is an antisymmetric cone containing $\mathcal{A}_{\overline{n}}$, f extends by hypothesis to a P.O.M.-homomorphism g from $\mathcal{C}_{\overline{n}}$ to E. Put $x = g(\xi)$. Since ξ satisfies (3.10), x satisfies (3.11), which concludes the proof. ■

Now, say that a linear system of the form (3.11) with parameters from some P.O.M. E is *compatible* when E satisfies its resolvent. From lemma 3.9, we can deduce the following consequence, which is the analogue of theorem 3.6 for linear systems of the form (3.11):

3.10. Theorem. [definition 3.1] *Let κ be an infinite cardinal, let E be a P.O.M.. Then the following are equivalent:*

(i) *Every compatible linear system of the form (3.11) of size at most κ with parameters from E admits a solution in E;*

(ii) *E is $\mathcal{CO}_\kappa$-injective;*

(iii) *E is $\mathcal{CO}_\kappa^*$-injective.*

Proof. Assume that E satisfies (i). Let A be a sub-P.O.M. of a cone B and let f be a P.O.M.-homomorphism from A to E; we try to extend f to a P.O.M.-homomorphism from B to E; by an easy application of Zorn's lemma, it suffices to prove that for all ξ in B, f extends to a P.O.M.-homomorphism from $C = A + \mathbb{N}\xi$ to E. Consider the linear system (S) of all equations (resp. inequations) of the form $a + mx = b + nx$ (resp. $a + mx \leq b + nx$) where a, b are in A, m, n are in $\mathbb{N}$, $m = 0$ or $n = 0$ and which are satisfied in B by ξ. Let (R) be some resolvent of (S). By lemma 3.9, A satisfies (R), thus E satisfies f(R); since f(R) is a resolvent of f(S) and E satisfies (i), f(S) admits a solution in E, say x. But then, since B is cancellative, it is easy to verify that for *all* a, b in A and m, n in $\mathbb{N}$ such that $a + m\xi \leq b + n\xi$ (resp. $a + m\xi = b + n\xi$), we have $f(a) + mx \leq f(b) + nx$ (resp. $f(a) + mx = f(b) + nx$). Thus x is the value at ξ of some P.O.M.-homomorphism from C to E extending f; hence E is $\mathcal{CO}_\kappa$-injective, so that (i)⇒(ii). Now, (ii)⇒(iii) is trivial, and (iii)⇒(i) is an immediate consequence of lemma 3.9. ■

From what precedes, one can immediately get the following corollary:

3.11. Corollary. *Let E be a separative P.O.M., let (S) be a linear system either of the form (3.1) or of the form (3.11) with parameters from E. Then (S) admits a solution in some separative extension of E if and only if E satisfies the resolvent of (S).*

Proof. Suppose first that (S) admits a solution in some separative extension F of E. By lemma 3.5 or lemma 3.9, according to the case, E satisfies the resolvent of (S). Conversely, assume that E satisfies the resolvent of (S). Let κ be an infinite cardinal majorating the size of (S). By theorem 3.2, there is a $PREM_\kappa$-injective P.O.M. F containing E. Thus, again by lemma 3.5 or lemma 3.9, (S) admits a solution in F. ■

Our next step is now to define resolvents of any arbitrary linear system with one unknown. Since the natural context for solving such linear systems is the context of *separative* P.O.M.'s, such systems can always be written in the following *canonical form*:

$$(3.15)\qquad \begin{cases} a_i + n_i x \leq b_i & \text{(all } i \text{ in } I) \\ b_j \leq a_j + n_j x & \text{(all } j \text{ in } J) \\ a_k + n_k x = b_k & \text{(all } k \text{ in } K) \\ a_s + (n_s+1)x \leq b_s + x & \text{(all } s \text{ in } S) \\ a_t + (n_t+1)x = b_t + x & \text{(all } t \text{ in } T) \end{cases}$$

where the a_i, b_i are elements of a given separative P.O.M.. Now, to (3.15), we can associate the following linear system with two unknowns (obtained by 'thinking that $y = \frac{x}{\infty}$'):

$$(3.16)\qquad \begin{cases} a_i + n_i x \leq b_i & \text{(all } i \text{ in } I) \\ b_j \leq a_j + n_j x & \text{(all } j \text{ in } J) \\ a_k + n_k x = b_k & \text{(all } k \text{ in } K) \\ a_s + n_s x \leq b_s + y & \text{(all } s \text{ in } S) \\ a_t + n_t x = b_t + y & \text{(all } t \text{ in } T) \end{cases}$$

Denote by $(\mathrm{R}_{x,y})$ this system. Then any of its resolvents in x (as defined in (3.12)) is a linear system 'with idem-multiple unknown', thus of the form (3.1) (with y instead of x); denote it by (S_y). Finally, let (T) be the resolvent of (S_y), as defined in (3.3). *We will say by definition that* (T) *is a resolvent of* (3.15).

3.12. Lemma. *Let κ be an infinite cardinal, let E be a separative P.O.M., let (S) be a linear system of the form (3.15), of size at most κ and with parameters from E. If (S) admits a solution in some preminimal extension of E, then E satisfies any resolvent of (S). Conversely, if E is both $\mathcal{IM}_\kappa$-injective and $\mathcal{CO}_\kappa$-injective and E satisfies some resolvent of (S), then (S) admits a solution in E.*

Proof. So suppose that (S) is just (3.15). Assume first that (S) admits a solution, say x, in some preminimal extension F of E; using lemmas 1.20 and 1.21, it is easy to see that one may replace F by $F \dot{-} F$, so that without loss of generality, F is a D.P.O.M.. Let $y = \frac{x}{\infty}$. Then, using the definition of a D.P.O.M., it is immediate that (x, y) is a solution of (3.16) in F. By lemma 3.9, y is a solution in F of the resolvent (S_y) of (3.16) (taking x as unknown). But now, (S_y) is a linear system of the form (3.1), thus, by lemma 3.5, F satisfies its resolvent, which is (T); since all the parameters from (T) are in E, E satisfies (T). Conversely, suppose that E satisfies (T). Since E is $\mathcal{IM}_\kappa$-injective, it results from lemma 3.5 that E satisfies (S_y) for some y in $\frac{1}{\infty}F$. Since E is $\mathcal{CO}_\kappa$-injective, it results from lemma 3.9 that E satisfies $(\mathrm{R}_{x,y})$ for some x in E; it is then immediate that E satisfies (3.15). ■

It follows that all resolvents of (3.15) are (in separative P.O.M.'s) equivalent, so that we will sometimes just speak about *the* resolvent of (S).

A first striking consequence of lemma 3.12 is the following characterization of $PREM_\kappa$-injectivity:

3.13. Theorem. [definitions 3.1, 3.4] *Let κ be an infinite cardinal, let E be a P.O.M.. Then the following are equivalent:*

(i) *E is $PREM_\kappa$-injective;*

(ii) *E is $PREM^*_\kappa$-injective;*

(iii) *E is κ-smooth and $\mathcal{CO}_\kappa$-injective;*

(iv) *E is $\mathcal{IM}_\kappa$-injective and $\mathcal{CO}_\kappa$-injective.*

(v) *E is $\mathcal{IM}^*_\kappa$-injective and $\mathcal{CO}^*_\kappa$-injective.*

Proof. It follows immediately from theorem 2.9 that every $\mathcal{CO}_\kappa$-injective P.O.M. is separative; therefore, the equivalence between (iii), (iv) and (v) results immediately from theorems 3.6 and 3.10. Furthermore, it is trivial that (i) implies (ii) implies (v). Finally, assume (iv); let us prove (i). It is sufficient to show that if A is a sub-P.O.M. of E of size at most κ and B is a preminimal *monogenic* extension of A, then $(A \hookrightarrow E)$ extends to a P.O.M.-homomorphism r from B to E. But A is separative, thus (lemmata 1.20 and 1.21) the natural map from A to $B \dot{-} B$ is a P.O.M.-embedding: thus we may assume without loss of generality that B is separative. Write $B = A + \mathbb{N}b$ for some b in B. Consider the linear system (S) of all equations and inequations with parameters from A of which b is a solution (put under the canonical form (3.15)). By lemma 3.12, since B is separative, A satisfies the resolvent of (S); thus E satisfies the resolvent of (S). By lemma 3.12, (S) admits a solution, say x, in E. By definition of (S), x is the value at b of some extension of $(A \hookrightarrow E)$ to B. Thus E satisfies (i). ■

We can also define resolvents of arbitrary linear systems (with any number — possibly infinite — of unknowns) the following way: let (S) be a linear system with parameters in some P.O.M. E. Suppose first that (S) has finitely many unknowns x_i ($i < n$) for some n in ω. One defines inductively linear systems (S_k) ($k \leq n$) by (S_0)=(S), and (S_{k+1})=some resolvent of (S_k) with respect to the unknown x_k; then, we say that (S_n) is a resolvent of (S). In the general case, where (S) has an arbitrary, not necessarily finite, set of unknowns, say $\{x_i : i \in I\}$, for every finite subset p of I, let (S^p) be a resolvent of the set of equations or inequations in (S) whose unknowns belong to p; then say that $\bigcup_p(\mathrm{S}^p)$ is a resolvent of (S). Then we have the following

3.14. Theorem. [definition 1.2] *Let E be a separative P.O.M., let (S) be an arbitrary linear system with parameters from E and of size at most κ. Then (S) admits a solution in some separative extension of E if and only if E satisfies some [any] resolvent of (S).*

Proof. When (S) admits finitely many unknowns, this is an immediate consequence of lemma 3.12. Now assume that (S) has an arbitrary set of unknowns, say $(x_i)_{i\in I}$. By the previous case, for every finite subset p of I, the linear system (S^p) of all equations and inequations in (S) with unknowns among $\{x_i : i \in p\}$ admits a solution, say $(a_i^p)_{i\in p}$, in some separative extension F_p of E. Let P be the set of all finite subsets of I, and for all p in P, let $P_p = \{q \in P : p \subseteq q\}$. Now, let $\mathcal{F}$ be the filter on P with basis $\{P_p : p \in P\}$, and let F be the reduced product of $(F_p)_{p\in P}$ relatively to $\mathcal{F}$. For all i in I, let a_i be the class modulo $\mathcal{F}$ of $(a_i^p)_{p\in P}$ (it is well-defined since a_i^p is defined $\mathcal{F}$-everywhere on P). Then it is easy to verify that $(a_i)_{i\in I}$ is a solution of (S) in F; furthermore, F is a separative extension of E. ■

It follows again that in any *separative* P.O.M.'s, all resolvents of a given linear system (S) are equivalent; thus we will sometimes just speak about *the* resolvent of (S).

3.15. REMARK. Theorem 3.14 actually *characterizes* separative P.O.M.'s. For example, in general, if a and b are two elements of some P.O.M. E, then the existence of an x in some extension of E such that $a + x = b$ is not expressed only by the resolvent (the quantifier-free formula $a \leq b$), but also by both following universal formulas

$$(\forall x, y)(a + x \leq a + y \Rightarrow b + x \leq b + y),$$

and

$$(\forall x, y)(a + x = a + y \Rightarrow b + x = b + y).$$

Let $\theta(a, b)$ be the conjunction of $a \leq b$ and both formulas above. As mentioned without proof after the definition of preminimality (definition 1.2), $\theta(a, b)$ is equivalent to "the equation $a + x = b$ admits a solution in some extension of E". However, when *e.g.* $E = \mathbb{N} \times_{lex} \overline{\mathbb{N}}$, there is no parameter-free, quantifier-free formula equivalent to $\theta(a, b)$. In general, existence of a solution of a given linear system in some extension of the base P.O.M. is expressed by a universal formula, not necessarily quantifier-free.

The following theorem follows immediately from theorem 3.14 and the effectiveness of the construction of the resolvent:

3.16. Theorem. [definition 1.2] *Let E be a separative P.O.M. equipped with a notion of recursivity for which the equality, preordering and addition of E are recursive. Let (S) be a given finite linear system with parameters from E. Then the existence of a solution of (S) in some extension of E is decidable.* ∎

§4. RESTRICTED INJECTIVITY WITH MULTIPLICATIVE CANCELLATION; CASE OF CONES WITH INFINITY.

It is time now to harvest some results yielded by the previous three chapters. Our first result will be a characterization of $PREM_\kappa$-injective P.O.M.'s (κ=some infinite cardinal) with the restriction that they are antisymmetric and satisfy the multiplicative $\leq$-cancellation property. Hence, this will allow us to give an exact characterization of all POM_κ-injective P.O.M.'s. This will yield for example that [the P.O.M. associated with] any divisible weak cardinal algebra is POM_ω-injective. We will first need a definition, which generalizes the definition of strong refinement P.O.M. seen in [16], chapter 1.

4.1. DEFINITION. *Let κ be an infinite cardinal, let E be a P.O.M.. Then E is a κ-strong refinement P.O.M. when the following holds:*

(SR 1) *E is minimal, antisymmetric and satisfies the pseudo-cancellation property;*

(SR 2) *E satisfies the κ-absorption property: for every subset X of E of size at most κ and every a in E such that $X \ll a$, there exists b in E such that $X \leq b$ and $b \ll a$;*

(SR 3) *For all a, b in E and all $X \subseteq E$ of size at most κ such that $b \leq a + X$, there is $c \leq X$ in E such that $b \leq a + c$;*

(SR 4) *E satisfies the (κ, κ)-interpolation property: for all subsets X and Y of E of size at most κ such that $X \leq Y$, there is c in E such that $X \leq c$ and $c \leq Y$.*

Although we will not use this fact here, one can show that κ-strong refinement P.O.M.'s satisfy the finite refinement property. This can also easily be derived from the following

4.2. Lemma. *Let E be a κ-strong refinement P.O.M., let I and J be sets of size at most κ, let a_i, b_i ($i \in I \cup J$) be elements of E. For the inequation system*

$$\begin{cases} a_i + x \leq b_i & \text{(all } i \text{ in } I) \\ b_j \leq a_j + x & \text{(all } j \text{ in } J) \end{cases} \tag{4.1}$$

to admit a solution in E, it is necessary and sufficient that the following holds:

$$\begin{cases} a_i \leq b_i & \text{(all } i \text{ in } I) \\ a_i + b_j \leq a_j + b_i & \text{(all } (i,j) \text{ in } I \times J) \end{cases} \tag{4.2}$$

Proof. The fact that (4.1) implies (4.2) is trivial. Now, assume (4.2). If $J = \emptyset$, then $x = 0$ is a solution. If $I = \emptyset$, then (4.1) follows by interpolation between $\{b_j : j \in J\}$ and $\emptyset$. Now, suppose that $I \neq \emptyset$ and $J \neq \emptyset$. For all i in I, let c_i in E such that $a_i + c_i = b_i$. Thus for all (i,j) in $I \times J$, we have $a_i + b_j \leq a_i + a_j + c_i$, thus, by pseudo-cancellation, $b_j \leq a_i + c_i + d_{ij}$ for some $d_{ij} \ll a_i$; by (SR 2) (κ-absorption property), there is $d_i \ll a_i$ such that $d_{ij} \leq d_i$ for all j in J. But then, replacing c_i by $c_i + d_i$ does not affect the definition of c_i, thus we may as well assume that $b_j \leq a_j + c_i$. Now, fixing j and using (SR 3), we find e_j such that $e_j \leq c_i$ for all i in I and $b_j \leq a_j + e_j$. By (SR 4) ((κ, κ)-interpolation), we find x such that $(\forall (i,j) \in I \times J)(e_j \leq x \leq c_i)$. It follows easily that x is a solution of (4.1). ∎

From this lemma, we can now deduce the following

4.3. Theorem. [definitions 3.1, 4.1] *Let κ be an infinite cardinal. For an antisymmetric P.O.M. E satisfying the multiplicative $\leq$-cancellation property to be $PREM_\kappa$-injective, it is necessary and sufficient that the following holds:*

(i) *E is a κ-strong refinement P.O.M.;*

(ii) *For all a in E and every $X \subseteq E$ of size at most κ such that $a \ll X$, there is b in $\frac{1}{\infty}E$ such that $a \leq b$ and $b \leq X$.*

(iii) *E is divisible,* i.e. *for all m in $\mathbb{N} \setminus \{0\}$, it satisfies $(\forall x)(\exists y)(my = x)$.*

Proof. Assume first that E satisfies (i), (ii) and (iii). We first prove that E is κ-smooth (see definition 3.4). Since it is antisymmetric and satisfies the pseudo-cancellation property, it is separative. Now let $a, b \in E$, let $X \subseteq E$ of size at most κ such that $(\forall x \in X)(a + x \leq b + x)$. By the pseudo-cancellation property (in (SR 1)), there is a map $(x \mapsto \overline{x})$ from X to E such that $(\forall x \in X)(a \leq b + \overline{x}$ **and** $\overline{x} \ll x)$. By (SR 3), there is c in E such that $a \leq b + c$ and $(\forall x \in X)(c \leq \overline{x})$. Thus $c \ll x$ for all x in X (we use antisymmetry), hence, by (ii), there is d in $\frac{1}{\infty}E$ such that $c \leq d$ and $d \leq X$; thus $d \in \frac{1}{\infty}E$, $d \leq X$ and $a+d \leq b+d$, so that (SM 1) holds. Let now $a, b \in E$, $X \subseteq E$ of size at most κ such that $(\forall x \in X)(a+x=b+x)$. By (SM 1), there are $c, d \leq X$ in $\frac{1}{\infty}E$ such that $a+c \leq b+c$ and $b+d \leq a+d$; since c and d are idem-multiple $\leq X$, $c+d$ is idem-multiple $\leq X$, and, by antisymmetry, $a+(c+d)=b+(c+d)$. Thus (SM 2) holds. Finally, let X and Y be two subsets of E of size at most κ such that $X \ll Y$. By (SR 2), for every y in Y, there is $\overline{y} \ll y$ in E such that $X \leq \overline{y}$. By (SR 4), there is c in E such that $X \leq c$ and $(\forall y \in Y)(c \leq \overline{y})$. Thus, by (ii), there is d in $\frac{1}{\infty}E$ such that $c \leq d$ and $(\forall y \in Y)(d \leq y)$; thus $X \ll d$ and $d \leq Y$. Thus E satisfies (SM 3). So we have proved that E is κ-smooth. Now, let (S) be an arbitrary compatible linear system of type (3.11) (thus with one unknown), with parameters from E and of size at most κ; thus (S) admits a solution in some separative extension F of E (corollary 3.11), which we may assume antisymmetric by quotienting F by $\equiv_F$ (by antisymmetry of E). Thus in F, (S) is equivalent to a system of the following form:

$$\begin{cases} a_i + n_i x \leq b_i & \text{(all } i \text{ in } I) \\ b_j \leq a_j + n_j x & \text{(all } j \text{ in } J) \end{cases} \tag{4.3}$$

where $|I|, |J| \leq \kappa$, $a_i, b_i \in E$ and m_i, n_i are in $\mathbb{N} \setminus \{0\}$ for all i. Put $a'_i = (1/n_i)a_i$ and $b'_i = (1/n_i)b_i$ (we use the hypotheses on E). Since (4.3) admits a solution in F, it is immediate that the following holds:

$$\begin{cases} a'_i \leq b'_i & \text{(all } i \text{ in } I) \\ a'_i + b'_j \leq a'_j + b'_i & \text{(all } (i,j) \text{ in } I \times J) \end{cases} \tag{4.4}$$

Thus, by lemma 4.2, there is x in E satisfying the following linear system:

$$\begin{cases} a'_i + x \leq b'_i & \text{(all } i \text{ in } I) \\ b'_j \leq a'_j + x & \text{(all } j \text{ in } J) \end{cases}$$

But this implies immediately that x satisfies (4.3). By theorem 3.10, E is $\mathcal{CO}_\kappa$-injective. Finally, by theorem 3.13, E is $PREM_\kappa$-injective.

Conversely, suppose that E is $PREM_\kappa$-injective. Thus it is separative (by theorem 2.9), and κ-smooth (theorem 3.6), thus it satisfies (SR 1) (use (SM 1)). Let R be the sub-P.O.M. of $\mathbb{N}^4$ used in the proof of corollary 2.7. Then every P.O.M.-homomorphism from R to E extends to a P.O.M.-homomorphism from $\mathbb{N}^4$ to E, thus, by the result of claim 2 of the proof of corollary 2.7, E satisfies the finite refinement property; thus it is a strong refinement P.O.M. (as defined in [16], chapter 1), and thus, it satisfies the finite interpolation property, *i.e.* for all finite subsets X and Y of E such that $X \leq Y$, there is c in E such that $X \leq c$ and $c \leq Y$ (the proof of [13], theorem 2.28, applies). Now, let X and Y be two subsets of E of size at most κ. By theorem 3.14, for every linear system (S) of size at most κ with parameters from E, (S) admits a solution if and only if every finite subsystem of (S) admits a solution (this can also be proved directly using reduced powers, *see* [16], proof of theorem 3.11). Applying this to the following system (with unknown c)

$$\begin{cases} x \leq c & \text{(all } x \text{ in } X) \\ c \leq y & \text{(all } y \text{ in } Y) \end{cases}$$

we obtain that E satisfies (SR 4). A similar proof, using the fact that any strong refinement P.O.M. satisfies the statement

$$(\forall a, b, c, d)\big(b \leq a + c, a + d \Rightarrow (\exists x \leq c, d)(b \leq a + x)\big)$$

shows that E satisfies (SR 3). ■

Theorem 4.3 will now allow us to characterize all POM_κ-injective P.O.M.'s. We will need to prove that POM_ω-injective P.O.M.'s actually embed into injective P.O.M.'s.

4.4. Theorem. [definition 3.1] *Let κ be an infinite cardinal. Then a P.O.M. E is POM_κ-injective if and only if it is $PREM_\kappa$-injective and* Archimedean, *the latter condition meaning that*

$$(\forall a, b)\left(\bigwedge_{n \in \mathbb{N}} (na \leq b) \Rightarrow a \ll b\right).$$

In particular, every POM_ω-injective P.O.M. embeds into a power of $\overline{\mathbb{P}}$.

Proof. Assume first that E is $PREM_\kappa$-injective and Archimedean. Thus it is antisymmetric (using the Archimedean property and minimality), and it satisfies the finite refinement property. Furthemore, it satisfies (SR 3) (for countable X). Thus it is what we called in [16], chapter 2, a relatively σ-complete P.O.M.; thus ([17], corollary 2.17) it embeds into some injective P.O.M., say F. Let A be a sub-P.O.M. of a P.O.M. B of size at most κ, let f be a P.O.M.-homomorphism from A to E. Since F is injective, f extends to a P.O.M.-homomorphism g from B to F. But gB is a preminimal extension of fA of size at most κ, thus, by assumption, $(fA \hookrightarrow E)$ extends to a P.O.M.-homomorphism r from gB to E. Then $r \circ g$ is a P.O.M.-homomorphism from B to E extending f. Thus E is POM_κ-injective.

Conversely, assume that E is POM_κ-injective. So E is $PREM_\kappa$-injective, so that all we have to prove is that E is Archimedean. First, the proof presented in [16], theorem 3.11, applies here to show that E is antisymmetric. Now let e, b in E such that for all n in $\mathbb{N}$, $ne \leq b$. Put $a = b + e$, let A be the sub-P.O.M. of E generated by $\{a, b\}$.

Claim 1. Suppose that there is an extension B of A where lies an element x such that $a + x \leq 2x$ and $x + b \leq 2b$. Then $e \ll b$.

Proof of claim . Without loss of generality, B is countable. Since E is POM_κ-injective, $(A \hookrightarrow E)$ extends to a P.O.M.-homomorphism r from B to E. Let $c = r(x)$. Then $a + c \leq 2c$ and $c + b \leq 2b$, thus, since E is separative, $a \leq c$ and $c \leq b$, whence $a \leq b$; since $b \leq a$ and E is antisymmetric, $a = b$, thus the conclusion follows. ■ Claim 1.

Now, we shall construct an extension as in claim 1. Equip $A \times \mathbb{N}$ with the binary relation $\rightarrow$ containing only the following ordered pairs:

$$(x+a,m) \to (x,m+1)\ (x \in A,\ m \in \mathbb{N} \setminus \{0\}),$$
$$(x,m+1) \to (x+b,m)\ (x \in A,\ x \geq b,\ m \in \mathbb{N}),$$
$$(x,m) \to (x',m')\ (x,x' \in A,\ m,m' \in \mathbb{N},\ x \leq x',\ m \leq m').$$

In the first case, we will write $(x+a,m) \xrightarrow{\mu} (x,m+1)$; in the second case , we will write $(x,m+1) \xrightarrow{\nu} (x+b,m)$; in the third case, we will write $(x,m) \xrightarrow{\tau} (x',m')$. For each n in ω, let $\to_n$ be the binary relation on $A \times \mathbb{N}$ defined by

$$s \to_n t \Leftrightarrow (\exists_{i \leq n} s_i)\big(s_0 = s \textbf{ and } s_n = t \textbf{ and } (\bigwedge_{i<n} s_i \to s_{i+1})\big).$$

Finally, let $\leq_*$ be the union of all $\to_n$ $(n \in \omega)$. Since $\leq_*$ contains $\xrightarrow{\tau}$, it is easy to see that $\leq_*$ is a P.O.M.-preordering of $A \times \mathbb{N}$, containing the componentwise preordering $\leq$. Let $\equiv_*$ be the equivalence associated with $\leq_*$; let B be the quotient P.O.M. of $(A \times \mathbb{N}, +, 0, \leq_*)$ by $\equiv_*$. So by definition, B is an antisymmetric P.O.M.. For all (x,m) in $A \times \mathbb{N}$, denote by $[x,m]$ its equivalence class modulo $\equiv_*$. Let j be the natural P.O.M.-homomorphism from A to B, defined by $(x \mapsto [x,0])$. We shall prove that j is an embedding.

Claim 2. Let (x,p), (y,q) in $A \times \mathbb{N}$ such that $(x,p) \leq_* (y,q)$. Then there are h, k in $\mathbb{N}$ such that $p+h \leq q+k$ and $x+kb \leq y+ha$.

Proof of claim . We prove by induction on n that if $(x,p) \to_n (y,q)$, then there are h, k as above. If $n=0$ take $h=k=0$. Suppose the result is proved for n, and assume $(x,p) \to_{n+1} (y,q)$; thus there is (z,r) such that $(x,p) \to_n (z,r)$ and $(z,r) \to (y,q)$. By induction hypothesis, there are h, k in $\mathbb{N}$ such that $p+h \leq r+k$ and $x+kb \leq z+ha$. Now, we have three cases to consider:

Case 1. $(z,r) \xrightarrow{\mu} (y,q)$. Then it is easy to verify that $(h+1,k)$ solves the problem for (x,p) and (y,q).

Case 2. $(z,r) \xrightarrow{\nu} (y,q)$. Then it is easy to verify that $(h,k+1)$ solves the problem for (x,p) and (y,q).

Case 3. $(z,r) \xrightarrow{\tau} (y,q)$. Then it is easy to verify that (h,k) solves the problem for (x,p) and (y,q).

This concludes the proof of the claim. ■ Claim 2.

We can now prove the

Claim 3. j is a P.O.M.-embedding.

Proof of claim. Since B is antisymmetric, it suffices to prove that for all x, y in A, $(x,0) \leq_* (y,0)$ implies $x \leq y$. First, we prove by induction on n that if $(x,0) \to_n (y,0)$ and $b \not\leq y$, then $x \leq y$. For $n=0$ it is trivial. Assume that it is proved for n, and suppose $(x,0) \to_{n+1} (y,0)$; thus there is (z,r) such that $(x,0) \to_n (z,r)$ and $(z,r) \to (y,0)$. But by definition of $\to$ and since $b \not\leq y$, we have $(z,r) \xrightarrow{\tau} (y,0)$, thus $z \leq y$ and $r=0$; thus

$b \not\leq z$, hence, by induction hypothesis, $x \leq z$; since $z \leq y$, we get $x \leq y$. So the problem is solved for $b \not\leq y$.

Suppose now that $b \leq y$ and $(x, 0) \leq_* (y, 0)$. By definition, there are n in ω and s_i $(i \leq n)$ in $A \times \mathbb{N}$ such that $s_0 = (x, 0)$, $s_n = (y, 0)$ and for all $i < n$, $s_i \to s_{i+1}$. Let m be the largest $i \leq n$ such that $(x, 0) \xrightarrow{\tau} s_i$. If $m = n$, then $x \leq y$ and we are done, so suppose $m < n$. Then we can only have $s_m \xrightarrow{\mu} s_{m+1}$ or $s_m \xrightarrow{\nu} s_{m+1}$, thus $s_m = (z, r)$ where $r \neq 0$ and $x \leq z$. Since $(z, r) \leq_* (y, 0)$, there are (by claim 2) h, k in $\mathbb{N}$ such that $r + h \leq k$ and $z + kb \leq y + ha$. Therefore,

$$\begin{aligned} x + (h+1)b &\leq x + (h+r)b \text{ (because } r \neq 0) \\ &\leq x + kb \\ &\leq z + kb \\ &\leq y + ha \\ &\leq y + (h+1)b. \end{aligned}$$

Now, since E satisfies the pseudo-cancellation property and since $b \leq y$, we get $x \leq y$. ■ Claim 3.

Now, let $x = [0, 1]$. Then $(a, 1) \xrightarrow{\mu} (0, 2)$ implies $j(a) + x \leq 2x$, and $(b, 1) \xrightarrow{\nu} (2b, 0)$ implies $j(b) + x \leq 2j(b)$. Thus j satisfies the conditions of claim 1: the conclusion follows. ■

The 'if' part of this theorem has an immediate corollary:

4.5. Corollary. *The P.O.M. associated with any divisible weak cardinal algebra is POM_ω-injective.* ■

The 'only if' part of this theorem justifies remark 2.16: namely, if the assumption that B is preminimal could be dropped from the hypotheses in lemma 2.15, then one could prove as in the proof of theorem 3.2 that every separative P.O.M. can be embedded into a POM_ω-injective P.O.M., thus into an injective P.O.M.. However, $\mathcal{P}_3$ (see example 2.12) cannot be embedded into any injective P.O.M..

Now, we shall study briefly the $PREM_\kappa$-injectivity of P.O.M.'s which do not necessarily enjoy neither the antisymmetry of $\leq$, nor the multiplicative $\leq$-cancellation property, but are fundamental enough in view of theorem 1.4; these are the cones with infinity.

4.6. Definition. *A cone E is said to be* positively existentially closed *(write $\exists^+$-closed) when every positive existential formula with parameters from E and one free variable which admits a solution in some cone containing E admits a solution in E.*

4.7. Lemma. *Let E be a cone. Then E is $\exists^+$-closed if and only if every compatible finite linear system of the form (3.11) with parameters from E admits a solution in E.*

Proof. In a given cone, every positive existential formula with one free variable is equivalent to a finite disjunction of systems of the form (3.11); thus, the condition of the lemma implies $\exists^+$-closure of E. Conversely, assume that E is $\exists^+$-closed. Consider a finite linear system (S) of the form (3.11), with parameters from E; assume that it admits a solution, say x, in some preminimal extension F of E. If $I \cup K \neq \emptyset$, then x is bounded by some element of E, thus we may assume without loss of generality that $(\forall v \in F)(\exists u \in E)(v \leq u)$. Define on F the preordering $\leq_*$ and the equivalence $\equiv_*$ by

$$u \leq_* v \Leftrightarrow (\exists w \in E)(u + w \leq v + w),$$
$$u \equiv_* v \Leftrightarrow (\exists w \in E)(u + w = v + w).$$

Since F is preminimal and E is cofinal in F, the quotient P.O.M. F' of $(F, +, 0, \leq_*)$ by $\equiv_*$ is cancellative; since E is cancellative, the natural map from E to F is a P.O.M.-embedding. Thus, we may assume without loss of generality that F is cancellative. Thus F embeds into a cone, so that (S) admits a solution in a cone containing E, thus in E by assumption. Suppose now that $I = J = \emptyset$ (so that (3.11) is always compatible). Then any large enough element of E satisfies (S), so we are done. ■

From this lemma, we could prove easily that in definition 4.6 of $\exists^+$-closure, one may as well have considered formulas with an arbitrary finite number of free variables. We will not use this result here.

Note that any finite system of the form (3.11) admits a finite resolvent. From this we deduce immediately the following fact:

4.8. Corollary. *Positive existential closure is a first-order property (in the language of P.O.M.'s).* ■

4.9. Theorem. [definitions 3.1, 4.6] *Let κ be an infinite cardinal, let E be an $\exists^+$-closed cone. Then there is an elementary extension of E which is a $\mathcal{CO}_\kappa$-injective cone.*

Proof. For all cones A and B, write $A <_\kappa B$ the following statement:

"A is a sub-cone of B and every consistent linear system of the form (3.11) of size at most κ with parameters from A admits a solution in B."

Claim. Every $\exists^+$-closed cone A admits an ultrapower B such that $A <_\kappa B$.

Proof of claim. Let P be the set of all finite subsets of κ; for all p in P, put $P_p = \{q \in P : p \subseteq q\}$; let $\mathcal{U}$ be an ultrafilter on P containing $\{P_p : p \in P\}$. Let B be the ultrapower of A by $\mathcal{U}$. We prove that $A <_\kappa B$. So let (S) be a linear system of the form (3.11) of size at most κ with parameters from A; write (S)$= \{\phi_i(x) : i < \kappa\}$. For every p in P, there is x_p in A such that A satisfies $\phi_i(x_p)$ for all i in p. Let $x = [x_p : p \in P]_{\mathcal{U}}$. Then x is a solution of (S) in B; the conclusion follows. ■ Claim .

By corollary 4.8, the B above is still $\exists^+$-closed. We conclude by a κ^+-elementary chain argument, similar to the one used in the proof of theorem 3.2. ■

In fact, the fundamental class of objects we have to study is the class of cones with infinity (definition 1.3):

4.10. Proposition. *Let κ be an infinite cardinal, let E be a $\mathcal{CO}_\kappa$-injective cone. Then $E \cup \{\infty\}$ is $PREM_\kappa$-injective.*

Proof. Let A be a sub-P.O.M. of a cone B of sise at most κ, let f be a P.O.M.-homomorphism from A to $E\cup\{\infty\}$. Let $A' = f^{-1}E$ and $B' = \{x \in B :\ (\exists y \in A')(x \leq y)\}$. Then, by hypothesis, $f\restriction_{A'}$ extends to a P.O.M.-homomorphism g' from B' to E. Extend g' by the value ∞ on $B \setminus B'$: the map we obtain is a P.O.M.-homomorphism from B to E extending f. Thus $E \cup \{\infty\}$ is $\mathcal{CO}_\kappa$-injective. But it is trivially κ-smooth (see definition 3.4). The conclusion follows by theorem 3.13. ■

The picture about cones is completed by the following

4.11. Proposition. *Every cone embeds into a $\exists^+$-closed cone.*

Proof. An easy increasing chain argument (see the proof of theorem 3.2). ■

But it is trivial that every cone with infinity is κ-smooth for all κ (see definition 3.4). Using theorem 1.4, and propositions 4.10 and 4.11, we immediately get the following

4.12. Corollary. *Let κ be an infinite cardinal. Then every separative P.O.M. embeds into a product of $PREM_\kappa$-injective cones with infinity.* ■

We shall now discuss a case where the structure of the cones of corollary 4.12 is particularly simple. We start with the

4.13. Lemma. *The cone $\mathbb{R}_+$ of positive reals is $\exists^+$-closed.*

Proof. For every cone E containing $\mathbb{R}_+$, define binary relations $\leq_*$ and $\equiv_*$ on E by

$$x \leq_* y \Leftrightarrow \bigvee_{m\in\mathbb{N}\setminus\{0\}} mx \leq my,$$

and

$$x \equiv_* y \Leftrightarrow\ x \leq_* y \textbf{ and } y \leq_* x.$$

Then the natural embedding from $\mathbb{R}_+$ to the quotient-P.O.M. F of $(E, +, 0, \leq_*)$ by $\equiv_*$ is a P.O.M.-embedding, and F is antisymmetric and satisfies the multiplicative $\leq$-cancellation property; and finally, if (S) is a linear system with parameters from $\mathbb{R}_+$ admitting a solution in E, then it admits a solution in F. So the only linear systems to consider can be put under the form

$$\begin{cases} a_i \leq x & \text{(all } i \text{ in } I) \\ x \leq b_j & \text{(all } j \text{ in } J) \end{cases}$$

where the a_i and the b_j are in $\mathbb{R}_+$. However, such a system is compatible if and only if $a_i \leq b_j$ for all (i, j) in $I \times J$, and then, it admits a solution in $\mathbb{R}_+$. ■

A similar proof shows in fact that an antisymmetric cone satisfying the multiplicative $\leq$-cancellation property is $\exists^+$-closed if and only if it satisfies the finite interpolation property (example: $\mathbb{Q}_+$).

4.14. Theorem. [definition 1.2] *Let A be a separative, antisymmetric P.O.M. satisfying the multiplicative $\leq$-cancellation property. Then there is an elementary extension E of $\mathbb{R}_+$ such that A embeds into a power of $E \cup \{\infty\}$.*

Proof. First of all, let ${}^*\mathbb{R}_+$ be an ultrapower of $\mathbb{R}_+$ with respect to some non trivial ultrafilter over ω. Then ${}^*\mathbb{R}_+$ is $\exists^+$-closed, and there is $\varepsilon > 0$ in ${}^*\mathbb{R}_+$ such that $(\forall n \in \mathbb{N} \setminus \{0\})(\varepsilon < 1/n)$. Now let κ be an infinite cardinal such that $|A| \leq \kappa$. By theorem 4.9, there is a $\mathcal{CO}_\kappa$-injective cone E which is an elementary extension of ${}^*\mathbb{R}_+$. Let $F = E \cup \{\infty\}$. By proposition 4.10, F is $PREM_\kappa$-injective. Put $A' = \mathrm{Hom}(A, F)$ (=P.O.M. of P.O.M.-homomorphisms from A to F), $A'' = \mathrm{Hom}(A', F)$, and let T be the canonical evaluation map from A to A''. Since A is antisymmetric, it suffices to show that for all a, b in A, $T(a) \leq T(b)$ implies $a \leq b$. So suppose $T(a) \leq T(b)$. One can define u in A' by $u(x) = 0$ if $x \in A|b$, $u(x) = \infty$ if $x \notin A|b$. Since $u(a) \leq u(b) = 0$, we have $a \in A|b$. Thus if $2b = b$, then $a \leq b$ and we are done. Now suppose that $2b \neq b$. Since A is separative and antisymmetric, one can define a P.O.M.-homomorphism u from $\mathbb{N}b$ to $\mathbb{R}_+$ by $u(nb) = n$. Let $S = \{(p, q, n) \in \mathbb{N} \times \mathbb{N} \times (\mathbb{N} \setminus \{0\}) : na + pb \leq qb\}$. Since $a \in A|b$, $S \neq \emptyset$, thus $\alpha = \bigwedge\{\frac{q-p}{n} :\ (p, q, n) \in S\}$ is an element of $\mathbb{R}_+$. It is known (*see* [16], lemma 3.7) that α is the value at a of some P.O.M.-homomorphism v from $\mathbb{N}a + \mathbb{N}b$ to $\mathbb{R}_+$ extending u. Since F is $PREM_\kappa$-injective, v extends to a P.O.M.-homomorphism, still denoted by v, from A to F. By assumption, $v(a) \leq v(b)$, *i.e.* $\alpha \leq 1$. Now two cases can occur:

Case 1. There exists (p, q, n) in S such that $\alpha = \dfrac{q-p}{n}$.

Then $na + pb \leq qb \leq (p + n)b$, thus, by pseudo-cancellation property and since $n \neq 0$, $na \leq nb$, thus, by the multiplicative $\leq$-cancellation property, $a \leq b$ so we are done.

Case 2. For all (p, q, n) in S, $\alpha < \dfrac{q-p}{n}$.

Let $\beta = \alpha + \varepsilon$. We prove that β is the value at a of some P.O.M.-homomorphism from A to F extending u. Since F is antisymmetric and $PREM_\kappa$-injective, it suffices to prove that for all m, n, m', n' in $\mathbb{N}$ such that $ma + nb \leq m'a + n'b$, we have $m\beta + n \leq m'\beta + n'$. Since $m\alpha + n \leq m'\alpha + n$, the conclusion is immediate if $m \leq m'$. Now suppose that $m > m'$. We have

$$m'a + [(m - m')a + nb] \leq m'a + n'b$$

and there is k in $\mathbb{N}$ such that $m'a \leq kb$, thus, since A is preminimal, we have

$$kb + [(m - m')a + nb] \leq kb + n'b,$$

i.e. $(k + n, k + n', m - m') \in S$. Thus, by hypothesis, $\alpha < \dfrac{n' - n}{m - m'}$, thus also $\beta < \dfrac{n' - n}{m - m'}$ since α is real and ε is infinitely small. It follows easily that $m\beta + n < m'\beta + n'$.

So, let w be a P.O.M.-homomorphism from A to F extending u such that $w(a) = \beta$. We have $w(a) \leq w(b)$, *i.e.* $\beta \leq 1$; thus $\alpha < 1$, thus there is (p, q, n) in S such that $\dfrac{q-p}{n} < 1$. Then we conclude as in case 1 that $a \leq b$.

Hence T is a P.O.M.-embedding, which concludes the proof. ∎

4.15. Example. It is known (*see e.g.* [8]) that the equidecomposability types P.O.M.'s seen in example 2.13 do not necessarily embed into a power of $\overline{\mathbb{P}}$ ("finitely additive positive invariant measures do not separate equidecomposability types"). However, when $\mathrm{S}(\mathcal{B})/G$ is separative (this is the case when G is exponentially bounded, see example 2.13) and satisfies the multiplicative $\leq$-cancellation property (this is the case for $\mathcal{B} = \mathcal{P}(X)$ — the proof presented in [14], theorem 8.7, needs only a minor modification to apply to $\leq$ instead of just $=$), then theorem 4.14 shows that finitely additive invariant measures with values in a certain 'non-standard version' of $\mathbb{R}_+ \cup \{\infty\}$ separate equidecomposability types. To summarize, we have the

4.16. Corollary. *Let G be an exponentially bounded group acting on a set X. Then the equidecomposability type P.O.M. $\mathrm{S}(\mathcal{P}(X))/G$ embeds into a power of $E \cup \{\infty\}$, where E is some elementary extension of $\mathbb{R}_+$.* ∎

Laconically, this could be expressed by *positive invariant non-standard measures separate equidecomposability types.*

One may object that in the conclusion of theorem 4.14, the elementary extension E of $\overline{\mathbb{P}}$ in the powers of which we embed A grows at the same rate as A; could it be possible to embed A into a power of a certain separative P.O.M. which does *not* depend on A? We shall now give a strong negative answer to this question. For this purpose, we shall construct a family $(A_\xi)_{\xi \in ON}$ of simply defined antisymmetric cones satisfying the multiplicative $\leq$-cancellation property such that there is no P.O.M. S such that every A_ξ embeds into a power of S. The definition of A_ξ is the following: consider the abelian group $G_\xi = \mathbb{Z}^{(\aleph_\xi + 1)}$, equipped with the positive cone A_ξ defined by

$$A_\xi = \{x \in G_\xi :\ x = 0 \textbf{ or } (\exists \alpha \leq \aleph_\xi)\big(x(\alpha) > 0 \textbf{ and } (\forall \beta \geq \alpha)(x(\beta) = 0)\big)\}.$$

Finally, if A and B are two P.O.M.'s, write $A \lhd B$ the statement "A embeds into a power of B".

4.17. Theorem. *There is no P.O.M. S such that $(\forall \xi \in ON)(A_\xi \lhd S)$.*

Proof. For every P.O.M. A and all a, b in A, let $[b : a]$ be the set of all ordinals θ such that there is an increasing $\theta + 1$-sequence $(x_\xi)_{\xi \leq \theta}$ such that $x_\theta \leq b$ and $(\forall \xi < \theta)(x_{\xi+1} \geq x_\xi + a)$; put $b/a = \bigvee [b : a]$ if it exists, ∞ otherwise. Thus b/a depends on A, but in what follows, the A under consideration will always be clear from the context. Finally, we put $\|A\| = \bigvee \{b/a :\ (a, b) \in A,\ b/a < \infty\}$.

Claim 1. Let A be a preminimal P.O.M.. Then for all a, b in A such that $a + b \not\leq b$, we have $b/a \leq |A|^+$. Thus, $\|A\| \leq |A|^+$.

Proof of claim. Let θ in $[b : a]$, let $(x_\xi)_{\xi\leq\theta}$ be an increasing $\theta+1$-sequence in A such that $x_\theta \leq b$ and $(\forall\xi<\theta)(x_{\xi+1} \geq x_\xi + a)$. If $\theta \geq |A|^+$, then there are $\xi < \eta$ in $\theta+1$ such that $x_\xi = x_\eta$. Thus $x_\xi + a \leq x_\xi$, thus $a + b \leq b$ since A is preminimal, a contradiction. Thus $\theta < |A|^+$; the conclusion follows. ■ Claim 1.

Claim 2. Let f be a P.O.M.-homomorphism from a P.O.M. A to a P.O.M. B, let a, b in A. Then $b/a \leq f(b)/f(a)$.

Proof of claim. Straightforward. ■ Claim 2.

Claim 3. Let A and B be two P.O.M.'s with B preminimal such that $A \triangleleft B$. Then $\|A\| \leq \|B\|$.

Proof of claim. Let a, b in A such that $b/a < \infty$; thus $a+b \not\leq b$ (because of the constant sequences with value b and of arbitrary length), thus there is a P.O.M.-homomorphism f from A to B such that $f(a)+f(b) \not\leq f(b)$. Since B is preminimal, $f(b)/f(a) < \infty$ by claim 1, thus $f(b)/f(a) \leq \|B\|$. The conclusion follows by claim 2. ■ Claim 3.

Claim 4. For all ξ in ON, $\|A_\xi\| \geq \aleph_\xi$.

Proof of claim. For all $\alpha \leq \aleph_\xi$, let x_α be the characteristic function of $\{\alpha\}$. Then $(x_\alpha)_{\alpha\leq\aleph_\xi}$ is increasing, and for all $\alpha < \aleph_\xi$, $x_{\alpha+1} \geq x_\alpha + x_0$. Thus $x_{\aleph_\xi}/x_0 \geq \aleph_\xi$.■ Claim 4.

Now we can conclude: let S be a P.O.M. such that for every ordinal ξ, $A_\xi \triangleleft S$. Since all the A_ξ's are minimal, the P.O.M. S_m obtained by replacing the preordering of S by the *minimal* preordering of S satisfies the same property; thus we may assume without loss of generality that S is minimal. By claims 3, 4 and 1, we obtain that for every ξ, $\aleph_\xi \leq \|S\| \leq |S|^+$, a contradiction. ■

4.18. Remark. This last result is to be put in complete opposition with several connected categories: for example, in the category of Boolean algebras, we have $(\forall A)(A\triangleleft\mathbf{2})$; in the category of abelian groups, we have $(\forall A)(A \triangleleft \mathbb{Q}/\mathbb{Z})$; in the category of normed linear spaces, we have $(\forall A)(A\triangleleft\mathbb{R})$; in the category of all P.O.M.'s which can be embedded into injective P.O.M.'s, we have $(\forall A)(A \triangleleft \overline{\mathbb{P}})$; in the category of separative commutative semigroups, we have $(\forall A)(A\triangleleft (\mathbb{Q}/\mathbb{Z}) \cup \{\infty\})$ — this last fact being a counterpart for the generally better 'categorical simplicity' of P.O.M.'s versus commutative semigroups.

4.19. Remark. There are other ways, not investigated here, to decompose P.O.M.'s into smaller pieces. One of the most important seems to be the decomposition into a product of *subdirectly irreducible* components. One of the most remarkable (and almost trivial) results using these techniques is that any subdirectly irreducible P.O.M. is either coarse (*i.e.* $\leq_A = A \times A$) or antisymmetric; consequently, *every P.O.M. embeds into the product of an antisymmetric P.O.M. and a coarse P.O.M.*. However, there are only two antisymmetric, separative subdirectly irreducible P.O.M.'s up to isomorphism, these are $\mathbf{1} = \{0\}$ and $\mathbf{2} = \{0, \infty\}$, so there is little connection here with corollary 4.12 or theorem 4.14. But still, the elementary theory of decomposition into subdirectly irreducible components can be easily carried out in the class of models of a given theory consisting only on universal Horn axioms; this is the case *e.g.* for separativeness. However, we have not at present investigated this aspect of things for P.O.M.'s.

BIBLIOGRAPHY

[1] G. BIRKHOFF, "Lattice Theory", *American Mathematical Society, Providence, Rhode Island*, 1961.

[2] W.G. BOLTIANSKII, "Hilbert's third problem", *translated by* R. Silverman, *Washington, D.C.: Winston,* 1978.

[3] C. C. CHANG and H. J. KEISLER, "Model Theory", *North Holland*, 1973.

[4] A.H. CLIFFORD and G.B. PRESTON, "The algebraic theory of semigroups", *Mathematical Surveys* 7, *American Mathematical Society, Providence, R.I.*, Vol. 1 (1961) and Vol. 2 (1967).

[5] H. DOBBERTIN, "Refinement monoids, Vaught Monoids, and Boolean Algebras", *Mathematische Annalen* 265, 475-487, 1983.

[6] L. FUCHS, "Infinite abelian groups", Vol. 1 (1970) and Vol. 2 (1973), *Academic Press.*

[7] E. W. KISS, L. MARKI, P. PROHLE, and W. THOLEN, "Categorical algebraic properties. A compendium on amalgamation, congruence extension, epimorphisms, residual smallness, and injectivity", *Studia Scientiarum Mathematicarum Hungarica* 18, pp. 79-141, 1983.

[8] M. LACZKOVICH, "Invariant signed measures and the cancellation law", *Proceedings of the American Mathematical Society*, Vol. 111, Number 2, February 1991.

[9] K.P.S. BHASKARA RAO *and* R.M. SHORTT, "The dual of a refinement algebra", *Lecture Notes in pure and applied mathematics, General Topology and Applications, 5th North East Conference* (Marcel Dekker), vol. 134, pp. 335-367, 1991.

[10] K.P.S. BHASKARA RAO *and* R.M. SHORTT, "Weak cardinal algebras", *Annals of the New York Academy of Sciences*, vol. 659, pp. 156-162, 1992.

[11] K.P.S. BHASKARA RAO *and* R.M. SHORTT, "Extensions of semigroup valued, finitely additive measures", *to appear in Semigroup Forum.*

[12] R.M. SHORTT, "Duality for cardinal algebras", *Forum Mathematicum* 2, pp. 433-450, 1990.

[13] A. TARSKI, "Cardinal algebras", *New York: Oxford*, 1949.

[14] S. WAGON, "The Banach-Tarski paradox", *Cambridge University Press*, 1985.

[15] F. WEHRUNG, "Théorème de Hahn-Banach et paradoxes continus et discrets", *Comptes Rendus de l'Académie des Sciences*, t. 310, série I, pp. 303-306, 1990.

[16] F. WEHRUNG, "Injective positively ordered monoids I", *Journal of Pure and Applied Algebra* 83, pp. 43-82 (1992).

[17] F. WEHRUNG, "Injective positively ordered monoids II", *Journal of Pure and Applied Algebra* 83, pp. 83-100 (1992).

[18] F. WEHRUNG, "Metric properties of positively ordered monoids", *Forum Mathematicum* 5 (1993), pp. 183-201.