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Restricted isometry properties and nonconvex compressive sensing

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Abstract. In previous work, numerical experiments showed that ℓ^p minimization with $0 < p < 1$ recovers sparse signals from fewer linear measurements than does ℓ^1 minimization. It was also shown that a weaker restricted isometry property is sufficient to guarantee perfect recovery in the ℓ^p case. In this work, we generalize this result to an ℓ^p variant of the restricted isometry property, and then determine how many random, Gaussian measurements are sufficient for the condition to hold with high probability.

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1. Introduction

Recent papers [1, 2] have introduced the concept known as *compressive sensing* (among other related terms). The basic principle is that sparse or compressible signals can be reconstructed from a surprisingly small number of linear measurements, provided that the measurements satisfy an *incoherence* property (see, e.g., [3] for an explanation of incoherence). Such measurements can then be regarded as a compression of the original signal, which can be recovered if it is sufficiently compressible. A few of the many potential applications are medical image reconstruction [4], image acquisition [5], and sensor networks [6].

If the goal is to reconstruct sparse signals from the measurements, a natural approach is to find the sparsest signal consistent with the measurements. Let Φ be an $M \times N$ measurement matrix, and $\Phi x = b$ the vector of M measurements of an N-dimensional signal x. The approach would be to solve the following optimization problem:

$$\min_u \|u\|_0, \quad \text{subject to } \Phi u = b. \tag{1}$$

Here, the ℓ^0 norm $\|\cdot\|_0$ simply counts the number of nonzero components. (This is a standard abuse of terminology: $\|\cdot\|_0$ is not a norm, not being positive homogeneous.) In principle, this strategy is effective. For example, in the particular case of random measurements, where the entries of Φ are drawn from a Gaussian distribution, and a signal x with $\|x\|_0 = K$, then with probability 1 the problem (1) will have a unique solution that is exactly $u^* = x$, as long as $M > K$. If we have $M \geq 2K$, we can strengthen this statement to say that with probability 1, our choice of Φ will allow (1) to perfectly recover all signals x satisfying $\|x\|_0 \leq K$. Proofs of these statements can be found in [6].

Unfortunately, solving (1) would appear to require combinatorial optimization, and be utterly intractable to solve. In fact, it is provably NP-hard [7]. However, a remarkable result of Candès and Tao [8] for random, Gaussian measurements is that we can recover x with $\|x\|_0 = K$ with high probability as the unique solution solution of the convex, *basis pursuit* problem [9]:

$$\min_u \|u\|_1, \quad \text{subject to } \Phi u = b, \tag{2}$$

provided $M \geq CK \log(N/K)$ for some constant C. The required C depends on the desired probability of success, which in any case tends to one as $N \to \infty$. Because (2) is convex, it can be solved efficiently, a much better situation than that of (1). The cost is that more measurements are required, depending logarithmically on N. (Note that the above is only a sufficient condition, but Donoho and Tanner [10] have computed sharp reconstruction thresholds, so that for any choice of sparsity K and signal size N, the required number of measurements M for (2) to recover x with high probability can be determined precisely. Their results replace $\log(N/K)$ with $\log(N/M)$, showing that logarithmic growth in N is necessary.)

Variants of these results have included Φ being a random Fourier submatrix, or having values $\pm 1/\sqrt{N}$ with equal probability. More general matrices are considered in [3, 11]. Also, x can be sparse with respect to any basis, with u replaced with Ψu for suitable unitary Ψ.

A family of iterative greedy algorithms [12, 13, 14] have been shown to enjoy a similar exact reconstruction property, generally with less computational complexity. However, these algorithms require more measurements for exact reconstruction than the basis pursuit method.

In the other direction, it was shown in [15] that a nonconvex variant of basis pursuit will produce exact reconstruction with fewer measurements. Specifically, the ℓ^1 norm is replaced with the ℓ^p norm, where $0 < p < 1$ (in which case $\|\cdot\|_p$ isn't actually a norm, though $d(x,y) = \|x-y\|_p^p$ is a metric):

$$\min_u \|u\|_p^p, \quad \text{subject to } \Phi u = b. \tag{3}$$

That fewer measurements are required for exact reconstruction than when $p = 1$ was demonstrated by numerical experiments in [15], with random and nonrandom Fourier measurements. A theorem was also proven in terms of the restricted isometry constants of Φ (see section 2), generalizing a result of [16] to show that a condition sufficient for (3) to recover x exactly is weaker for smaller p. In this paper, we will show for the case of random, Gaussian measurements that the above condition of Candès and Tao generalizes to

$$M \geq C_1(p)K + pC_2(p)K\log(N/K), \tag{4}$$

where C_1, C_2 are determined explicitly, and are bounded in p. Thus, the dependence of the sufficient number of measurements M on the signal size N vanishes as $p \to 0$.

2. Restricted isometry properties

In [16], Candès and Tao introduce the notion of *restricted isometry constants* of a matrix. Let Φ be an $M \times N$ matrix, where $M < N$, and L a positive number. Then δ_L is the smallest number such that

$$(1-\delta_L)\|x\|_2^2 \leq \|\Phi x\|_2^2 \leq (1+\delta_L)\|x\|_2^2 \tag{5}$$

for all x such that $\|x\|_0 \leq L$. Thus, δ_L quantifies how close to isometrically Φ acts on L-sparse vectors, or how close to isometric must $M \times L$ submatrices of Φ be. The following theorem illustrates the relevance of these constants.

Theorem 2.1 (Candès-Tao). *Let $x \in \mathbb{R}^N$ have sparsity $\|x\|_0 = K$, and suppose Φ is a matrix satisfying the following:*

$$\delta_{3K} + 3\delta_{4K} < 2. \tag{6}$$

Then x is the unique minimizer of (2).

It is clear from their proof that the constants 3, 4, and 2 can be replaced with b, $b+1$, and $b-1$ for any $b>1$. There is a tradeoff between the increase of δ_{bK} and the weakening of the condition on the constants, with the resulting optimal value of b not known.

The above sufficient condition is then shown [16] to be met with high probability for random, Gaussian Φ provided $M \geq CK\log(N/K)$ for some constant C.

In [17], theorem 2.1 was generalized to the case of ℓ^p minimization:

Theorem 2.2. *[17] Let $x \in \mathbb{R}^N$ have sparsity $\|x\|_0 = K$, $0 < p \leq 1$, $b > 1$, and $a = b^{p/(2-p)}$. Suppose that Φ satisfies*

$$\delta_{aK} + b\delta_{(a+1)K} < b - 1. \tag{7}$$

Then the unique minimizer of (3) is exactly x.

Since $b^{p/(2-p)} < b$ for $p < 1$, the sufficient condition (7) is weaker than (6) when $p < 1$. The following corollary appears in [15]:

Corollary 2.3. *Given x with $\|x\|_0 = K$, suppose Φ is such that $\delta_{2K+1} < 1$. Then there is $p > 0$ such that the unique minimizer of (3) is exactly x.*

The corollary says that the limiting case of theorem 2.2 as $p \to 0$ is essentially the ℓ^0 case. It follows from the theorem by simply choosing b sufficiently large, then p sufficiently small.

In this paper, we consider a different notion of restricted isometry constant, based on the fact that we are working with ℓ^p norms. For an $M \times N$ matrix Φ, $L > 0$, and $0 < p \leq 1$, we define the *restricted p-isometry constant* δ_L to be the smallest number such that

$$(1-\delta_L)\|x\|_2^p \leq \|\Phi x\|_p^p \leq (1+\delta_L)\|x\|_2^p \tag{8}$$

for all x such that $\|x\|_0 \leq L$. We do not explicitly indicate the dependence of δ_L on p (or Φ, as before), which should not cause confusion. Also, for the rest of the paper, the definition of δ_L will be given by (8), and not (5). This newer notion quantifies how close Φ is to an isometric embedding of L-dimensional subspaces of $\ell^2(\mathbb{R}^N)$ into $\ell^p(\mathbb{R}^M)$. A similar definition in the case of $p = 1$ appears in [18], and is related to the Banach-Mazur distance of Banach space theory.

We now generalize theorem 2.2 to the new setting.

Theorem 2.4. *Let Φ be an $M \times N$ matrix with $M < N$, $x \in \mathbb{R}^N$, and let $K = \|x\|_0$ be the size of the support of x. Let $0 < p \leq 1$, $b > 1$, $a = b^{2/(2-p)}$. Suppose that Φ satisfies*

$$\delta_{aK} + b\delta_{(a+1)K} < b - 1. \tag{9}$$

Then the unique minimizer of (3) is exactly x.

Proof. We will prove something slightly stronger, that $\|\Phi x\|_p^p$ can be replaced with $(1/c)\|\Phi x\|_p^p$ in (8) for any $c > 0$. Although the isometry constants are not scale invariant, the sufficient condition is. The proof generally modifies that of [19], but with a simplification. (Specifically, equation (2.2) therein is not required.) We consider a solution u of (3) (that one exists is geometrically obvious). Let $h = u - x$; we wish to show that $h = 0$. For $T \subset \{1, \ldots, N\}$, Φ_T will denote the matrix equalling Φ in those columns whose indices belong to T, and otherwise zero, and similarly for the vector h_T. Let T_0 be the support of x. By the triangle inequality for $\|\cdot\|_p^p$, we have

$$|\|x\|_p^p - \|-h_{T_0}\|_p^p| \leq \|x + h_{T_0}\|_p^p. \tag{10}$$

Since $T_0 \cap T_0^c = \emptyset$, we have

$$\begin{aligned} \|x\|_p^p - \|h_{T_0}\|_p^p + \|h_{T_0^c}\|_p^p \leq \|x + h_{T_0} + h_{T_0^c}\|_p^p = \|x+h\|_p^p = \|u\|_p^p \\ \leq \|x\|_p^p, \end{aligned} \tag{11}$$

the last inequality holding because u solves (3). The result is that

$$\|h_{T_0^c}\|_p^p \leq \|h_{T_0}\|_p^p. \tag{12}$$

In other words, although u need not be sparse, a bound exists on the portion of u outside the support of x.

Let $L = aK$. Arrange the elements of T_0^c in order of decreasing magnitude of $|h|$ and partition into $T_0^c = T_1 \cup T_2 \cup \cdots \cup T_J$, where each T_j has L elements (except possibly T_J). We do this because the restricted isometry condition gives us control over the action of Φ on small sets. Denote $T_{01} = T_0 \cup T_1$. We decompose Φh:

$$\begin{aligned} 0 = \|\Phi u - \Phi x\|_p^p = \|\Phi h\|_p^p &= \left\| \Phi_{T_{01}} h_{T_{01}} + \sum_{j=2}^{J} \Phi_{T_j} h_{T_j} \right\|_p^p \\ &\geq \|\Phi_{T_{01}} h_{T_{01}}\|_p^p - \left\| \sum_{j=2}^{J} \Phi_{T_j} h_{T_j} \right\|_p^p \geq \|\Phi_{T_{01}} h_{T_{01}}\|_p^p - \sum_{j=2}^{J} \|\Phi_{T_j} h_{T_j}\|_p^p \\ &\geq c(1 - \delta_{L+K}) \|h_{T_{01}}\|_2^p - c(1 + \delta_L) \sum_{j=2}^{J} \|h_{T_j}\|_2^p. \end{aligned} \tag{13}$$

Now we need to control the size of the $\|h_{T_j}\|_2$. We aim to use (12), for which we must estimate the ℓ^2 norm in terms of the ℓ^p norm. For each $t \in T_j$ and $s \in T_{j-1}$, $|h(t)| \leq |h(s)|$, so that

$$|h(t)|^p \leq \|h_{T_{j-1}}\|_p^p / L. \tag{14}$$

Then

$$|h(t)|^2 \leq \|h_{T_{j-1}}\|_p^p / L^{2/p}, \tag{15}$$

$$\|h_{T_j}\|_2^2 \leq L \|h_{T_{j-1}}\|_p^2 / L^{2/p}, \tag{16}$$

$$\|h_{T_j}\|_2^p \leq \|h_{T_{j-1}}\|_p^p / L^{1-p/2}, \tag{17}$$

so that

$$\sum_{j=2}^{J} \|h_{T_j}\|_2^p \le \Big(\sum_{j=1}^{J} \|h_{T_j}\|_p^p\Big)/L^{1-p/2} = \Big(\sum_{j=1}^{J}\sum_{t\in T_j} |h(t)|^p\Big)/L^{1-p/2}$$
$$= \|h_{T_0^c}\|_p^p/L^{1-p/2}. \tag{18}$$

Now we may use (12), and then convert back from ℓ^p to ℓ^2 by means of Hölder's inequality:

$$\|h_{T_0}\|_p^p = \sum_{t\in T_0} |h(t)|^p \cdot 1 \le \left(\sum_{T_0} |h(t)|^2\right)^{\frac{p}{2}} \left(\sum_{T_0} 1\right)^{1-\frac{p}{2}}$$
$$= \|h_{T_0}\|_2^p K^{1-p/2}. \tag{19}$$

Combining, we obtain

$$\sum_{j=2}^{J} \|h_{T_j}\|_2^p \le \|h_{T_0}\|_p^p/L^{1-p/2} \le \|h_{T_0}\|_2^p \Big(\frac{K}{L}\Big)^{1-\frac{p}{2}} = \|h_{T_0}\|_2^p/a^{1-p/2}$$
$$= \|h_{T_0}\|_2^p/b. \tag{20}$$

Putting together with (13), we have

$$0 \ge c(1-\delta_{L+K})\|h_{T_{01}}\|_2^p - c(1+\delta_L)\|h_{T_0}\|_2/b$$
$$\ge c\left(1-\delta_{L+K} - (1+\delta_L)/b\right)\|h_{T_{01}}\|_2^p. \tag{21}$$

The condition (9) of the theorem ensures that the scalar factor is positive, so $h_{T_{01}} = 0$. In particular, $h_{T_0} = 0$; then $h = 0$ follows from (12). □

Since $2/(2-p) = 1 + p/(2-p)$, the dependence of (9) on p is the same as that of (7). In the next section, we will determine how many random, Gaussian measurements are needed for (9) to be satisfied with high probability.

3. Restricted p-isometry property of random, Gaussian matrices

Henceforth, Φ will denote an $M \times N$ matrix whose entries are i.i.d. Gaussian random variables, specifically $\varphi_{ij} \sim N(0, \sigma^2)$. Our results will not depend on the choice of σ.

Note that for $x \in \mathbb{R}^N$, we can write

$$\|\Phi x\|_p^p = \sum_{i=1}^{N} |(\Phi x)_i|^p = \sum_{i=1}^{N} |W_i|^p, \tag{22}$$

where each $W_i = \sum_{j=1}^{M} x_j \varphi_{ij}$ is a Gaussian random variable of mean zero and variance $\|x\|_2^2\sigma^2$. We have that $\|\Phi x\|_p^p$ is a sum of independent random variables $X_i = |W_i|^p$, having an identical distribution whose properties are straightforward to calculate. For example, its mean is $\mu = \mathbb{E}(X) = \|x\|_2^p \sigma^p 2^{p/2}\Gamma(\frac{p+1}{2})/\sqrt{\pi}$, and its density function is

$$f_X(t) = \frac{\sqrt{2}}{p\|x\|_2\sigma\sqrt{\pi}} t^{\frac{1}{p}-1} \mathrm{e}^{-\frac{t^{2/p}}{2\|x\|_2^2\sigma^2}} \tag{23}$$

for $t > 0$ and otherwise zero. These can be obtained via changes of variable in the integrals defining them in terms of the Gaussian distribution. In the sequel we shall find it simpler to work with the Gaussian density than with f_X.

We thus adopt the perspective of $\|\Phi x\|_p^p$ as a random variable, and will use probabilistic methods to prove the main theorem of the paper:

Theorem 3.1. *Let Φ be an $M \times N$ matrix whose elements are i.i.d. random variables distributed normally with mean zero and variance σ^2, where $M < N$. Then there are constants $C_1(p)$ and $C_2(p)$ such that whenever $0 < p \leq 1$ and*

$$M \geq C_1(p)K + pC_2(p)K \log(N/K), \tag{24}$$

the following is true with probability exceeding $1/\binom{N}{K}$: For any $x \in \mathbb{R}^N$ with sparsity $\|x\|_0 = K$, x is the unique solution of (3) (where $b = \Phi x$).

The main approach of the proof will be as follows:

(a) for a *fixed* sparse x, bound the probability that (8) fails;

(b) deduce bounds on the probability that (8) fails (as a condition uniform in x); then

(c) determine M sufficiently large for theorem 2.4 to hold with high probability.

Similar approaches can be found in [16, 18], but with substantially different methods used to fulfill them. Since (6) can be regarded as a statement about the singular values of Φ, Candès and Tao [16] invoke powerful concentration of measure results [20] concerning singular values in order to achieve (a). Since we are working in the context of (9), such an approach is not available to us. Similarly, Donoho [18] uses concentration of measure applied to a weighted ℓ^1 norm, possible since this defines a Lipschitz function. The analog for our case would involve a pth power of a weighted ℓ^p norm, which is not Lipschitz. (It is Lipschitz with respect to the metric induced by $\|\cdot\|_p^p$; however, applying concentration of measure would require knowing the concentration function of Gaussian measure with respect to this metric; see [20]. Determining this concentration function would require solving a difficult isoperimetric problem, which may be of independent interest.) Like Donoho's, our approach to (b) will generalize the proof of Dvoretzky's theorem found in Pisier's book [21], but following Donoho's argument would not yield a sufficiently sharp result when p is small.

We begin with (a), which we regard as a large deviation inequality for the random variable $\|\Phi x\|_p^p$. Since this is the sum of M independent random variables, what is needed can be thought of as a quantitative, nonasymptotic form of the central limit theorem. Known bounds on the tails of $\sum_i X_i$ in terms of the tails of the X_i, such as those arising from Hermitian or Edgeworth expansions [22], were not quite sharp enough. Instead we will make use of the theory of subgaussian variables (see [23]), those having tails dominated by a Gaussian density function. The density f_X above shows that the tails of the random variable $X = |W|^p = |\sigma\|x\|_2 Z|^p$, where Z is standard-normally distributed, are much thinner than Gaussian. The consequence is that X is φ-subgaussian with $\varphi(t) = t^{2/(2-p)}$ for large t, but in the end this turns out to yield an inferior bound [23, Corollary 2.4.2].

Lemma 3.2. *Let $0 < p \le 1$, $\eta > 0$, $1 \le L \le N$, $x \in \mathbb{R}^L$, Ψ an $M \times L$ submatrix of Φ as in theorem 3.1. Define $\mu_p = \mu/\|x\|_2^p$, which is independent of x. Then*

$$(1-\eta)M\mu_p\|x\|_2^p \le \|\Psi x\|_p^p \le (1+\eta)M\mu_p\|x\|_2^p, \tag{25}$$

with probability exceeding $1 - P_{M,p}(\eta) = 1 - 2\mathrm{e}^{-\frac{\eta^2 M}{2pc_p^2}}$, where

$$c_p \le (31/40)^{1/4}\Big[1.15 + \sqrt{p}\Big(\frac{\Gamma(\frac{p+1}{2})}{\sqrt{\pi}}\Big)^{-1/p}\Big]. \tag{26}$$

Note that $\alpha_p := (\Gamma(\frac{p+1}{2})/\sqrt{\pi})^{1/p}$ is an increasing function of p, bounded below by $\mathrm{e}^{-\gamma}/2 \approx 0.377$.

Proof. We approach (25) as a large-deviation inequality for the random variable $\|\Psi x\|_p^p$, a sum of M independent copies of the random variable $X = |\sigma\|x\|_2 Z|^p$, $Z \sim N(0,1)$ as described above in the case of the full matrix Φ. Such inequalities are simple to establish for random variables satisfying $\mathbb{E}\mathrm{e}^{\lambda(X-\mu)} \le \mathrm{e}^{\tau^2\lambda^2/2}$ for all λ. The left side of this inequality is the moment-generating function of $X - \mu$, and the inequality is the definition of $X - \mu$ being subgaussian. We now seek to determine an upper bound for such a τ. We will employ theorem 1.3 of [23], which gives the bound

$$\tau \le \sup_{k\ge 1}(3.1)^{1/4}\Big[\frac{2^k k!}{(2k)!}\mathbb{E}(X-\mu)^{2k}\Big]^{\frac{1}{2k}}. \tag{27}$$

Examination of the proof of this theorem shows that the constant $(3.1)^{1/4}$ can be lowered for $k = 1$ to $\sqrt{7/6}$.

We estimate $\mathbb{E}(X-\mu)^{2k}$:

$$\begin{aligned}\mathbb{E}(X-\mu)^{2k} &= \int_{-\infty}^{\infty}(|x|^p-\mu)^{2k}\frac{\mathrm{e}^{-\frac{x^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}\,\mathrm{d}x \\ &= \frac{1}{\sigma\sqrt{\pi/2}}\int_0^{\infty}(x^p-\mu)^{2k}\mathrm{e}^{-\frac{x^2}{2\sigma^2}}\,\mathrm{d}x.\end{aligned} \tag{28}$$

We break this integral into two parts: I_1 from 0 to $\mu^{1/p}$, I_2 the rest.

$$\begin{aligned}I_1 &\le \frac{1}{\sigma\sqrt{\pi/2}}\int_0^{\mu^{1/p}}(x^p-\mu)^{2k}\,\mathrm{d}x = \frac{1}{\sigma\sqrt{\pi/2}}\int_0^1(\mu u-\mu)^{2k}\frac{\mu^{1/p}}{p}u^{1/p-1}\,\mathrm{d}u \\ &= \frac{\mu^{2k+1/p}}{p\sigma\sqrt{\pi/2}}\int_0^1(1-u)^{2k}u^{1/p-1}\,\mathrm{d}u = \frac{\mu^{2k+1/p}}{p\sigma\sqrt{\pi/2}}B(2k+1,1/p).\end{aligned} \tag{29}$$

In bounding the Beta function, we will need to strike a balance between desirable dependence on p, and controlling the growth in k:

$$\begin{aligned}B(2k+1,1/p) &= \frac{\Gamma(2k+1)\Gamma(1/p}{\Gamma(2k+1+1/p)} = p\prod_{j=1}^{2k}\frac{j}{j+1/p} \\ &= p^{k+1}\prod_{j=1}^{k}\frac{j}{pj+1}\prod_{j=k+1}^{2k}\frac{j}{j+1/p} \le p^{k+1}\prod_{j=1}^{k}j\prod_{j=k+1}^{2k}1 \\ &= p^{k+1}k!.\end{aligned} \tag{30}$$

Thus

$$I_1 \leq \frac{\mu^{2k+1/p} p^k k!}{\sigma\sqrt{\pi/2}}. \tag{31}$$

For I_2, we will apply the mean value theorem to $g(t) = t^p$ on the interval $[x, \mu^{1/p}]$, obtaining

$$x^p - \mu = (x - \mu^{1/p}) g'(\xi_x) \leq (x - \mu^{1/p}) p\mu^{1-1/p}. \tag{32}$$

Then

$$\begin{aligned} I_2 &\leq \frac{p^{2k}\mu^{2k-2k/p}}{\sigma\sqrt{\pi/2}} \int_{\mu^{1/p}}^{\infty} (x - \mu^{1/p})^{2k} \mathrm{e}^{-\frac{x^2}{2\sigma^2}} \,\mathrm{d}x \\ &= \frac{p^{2k}\mu^{2k-2k/p}}{\sigma\sqrt{\pi/2}} \int_0^{\infty} x^{2k} \mathrm{e}^{-\frac{(x+\mu)^2}{2\sigma^2}} \,\mathrm{d}x \leq \frac{p^{2k}\mu^{2k-2k/p}}{\sigma\sqrt{\pi/2}} \int_0^{\infty} x^{2k} \mathrm{e}^{-\frac{x^2}{2\sigma^2}} \,\mathrm{d}x \\ &= \frac{p^{2k}\mu^{2k-2k/p}}{\sigma\sqrt{\pi/2}} \int_0^{\infty} (\sigma\sqrt{2u})^{2k-1} \mathrm{e}^{-u} 2\sigma^2 \,\mathrm{d}u \\ &= p^{2k} 2^k \mu^{2k-2k/p} \sigma^{2k} \Gamma(k+1/2)/\sqrt{\pi} = p^{2k}\mu^{2k}\alpha_p^{-2k} \frac{(2k)!}{2^{2k}k!}, \end{aligned} \tag{33}$$

noting $\mu^{1/p}/\sigma = \sqrt{2}\alpha_p$. Thus

$$\mathbb{E}(X-\mu)^{2k} \leq 2p^k\mu^{2k}\alpha_p k!/\sqrt{\pi} + p^{2k}\mu^{2k}\alpha_p^{-2k}\frac{(2k)!}{2^{2k}k!}. \tag{34}$$

Now we multiply through by $2^k k!/(2k)!$. For the first term, we use Stirling's approximation [24]:

$$\frac{2^k (k!)^2}{(2k)!} \leq \frac{2^k 2\pi k^{2k+1} \mathrm{e}^{-2k} \mathrm{e}^{\frac{1}{6k}}}{\sqrt{2\pi}(2k)^{2k+1/2}\mathrm{e}^{-2k}} = \frac{\sqrt{\pi k}\mathrm{e}^{\frac{1}{6k}}}{2^k}. \tag{35}$$

Then we have

$$\begin{aligned} \left[\frac{2^k k!}{(2k)!}\mathbb{E}(X-\mu)^{2k}\right]^{\frac{1}{2k}} &\leq \mu\sqrt{p/2}[2\alpha_p\sqrt{k}\mathrm{e}^{\frac{1}{6k}} + p^k\alpha_p^{-2k}]^{\frac{1}{2k}} \\ &\leq \mu\sqrt{p/2}[(2\alpha_p)^{\frac{1}{2k}} k^{\frac{1}{4k}} \mathrm{e}^{\frac{1}{12k^2}} + \sqrt{p}/\alpha_p]. \end{aligned} \tag{36}$$

The first term does not depend strongly on p, so we bound α_p by its largest value $\alpha_1 = 1\sqrt{\pi} > 1/2$. Hence $(2\alpha_p)^{\frac{1}{2k}}$ is decreasing in k, as is $\mathrm{e}^{\frac{1}{12k^2}}$. Since $\log k/k$ is decreasing for $k > e$, in maximizing (36) we need only consider $k = 1$, 2, and 3. Taking into account the remark following (27), the largest value is calculated to be for $k = 3$. We thus have

$$\begin{aligned} \tau &\leq \mu\sqrt{p/2}(3.1)^{1/4}[(2\alpha_1)^{\frac{1}{12}} 3^{\frac{1}{12}} \mathrm{e}^{\frac{1}{108}} + \sqrt{p}/\alpha_p] \\ &\leq \mu\sqrt{p}(31/40)^{1/4}[1.15 + \sqrt{p}/\alpha_p]. \end{aligned} \tag{37}$$

Finally, we apply theorem 1.5 of [23], which applied to our setting gives that

$$P\{|\sum_{i=1}^{M}(X_i - \mu)| \geq t\} \leq 2\mathrm{e}^{-\frac{t^2}{2M\tau^2}}. \tag{38}$$

This theorem comes from estimating the moment-generating function of the sum, making use of the special form for the estimate of the MGF of $X-\mu$, and then applying a standard argument using Markov's inequality. An entirely equivalent approach is to apply Cramér's theorem, using the simple form of the Legendre-Fenchel transform of the logarithm of the estimate of the MGF; see [25]. In any case, applying (38) with $t=\eta M\mu$ completes the proof of the lemma. $\square$

Now we turn to (b) in the proof strategy for theorem 3.1. Our approach is a generalization of the proof of Dvoretzky's theorem in [21]. Dvoretzky's theorem was generalized by Dilworth to the case of the ℓ^p quasi-norm [26]; however, better bounds result by considering the metric induced by $\|\cdot\|_p^p$ instead.

Lemma 3.3. *Let $0<p\leq 1$, Ψ an $M\times L$ submatrix of Φ as in theorem 3.1. Let $\delta>0$. Choose $\eta,\epsilon>0$ such that $\frac{\eta+\epsilon^p}{1-\epsilon^p}\leq\delta$. Then*

$$M\mu_p(1-\delta)\|x\|_2^p\leq\|\Psi x\|_p^p\leq M\mu_p(1+\delta)\|x\|_2^p \tag{39}$$

holds uniformly for $x\in\mathbb{R}^L$ with probability exceeding $1-(1+2/\epsilon)^L P_{M,p}(\eta)$.

Proof. Let S be the unit sphere of the ℓ^2 norm in $\mathbb{R}^L$. Let A be an ϵ-net of S (with respect to the ℓ^2 metric) having at most $(1+2/\epsilon)^L$ points. Then the probability that (25) fails for any $x\in A$ is at most $(1+2/\epsilon)^L P_{M,p}(\eta)$. Assume now that Ψ is such that the tail bound (25) holds uniformly on A.

First, let $x\in S$. Then we can find $x_0\in A$ such that $\|x-x_0\|_2\leq\epsilon$. Letting $\epsilon_1=\|x-x_0\|_2$, we have that $(x-x_0)/\epsilon_1\in S$. Then we can find $x_1\in A$ within ϵ of this quantity; continuing in this fashion, we obtain sequences (ϵ_n) and $(x_n)\subset A$ such that $|\epsilon_n|\leq\epsilon^n$, and $\|x-\sum_{n=0}^N\epsilon_n x_n\|_2\leq\epsilon^{N+1}$, where $\epsilon_0=1$ for notational convenience. Therefore $x=\sum_{n=0}^\infty\epsilon_n x_n$. (Note that if any ϵ_n is zero, we can terminate the series at the preceding term and obtain an element of A, which only strengthens what follows.)

Now we calculate, denoting $c=M\mu_p$, using that we know the tail bounds hold for each x_n, and that $x_n\in S$:

$$\begin{aligned}\|\Psi x\|_p^p&=\left\|\sum_{n=0}^\infty\epsilon_n\Psi x_n\right\|_p^p\leq\sum_{n=0}^\infty\|\epsilon_n\Psi x_n\|_p^p=\sum_{n=0}^\infty|\epsilon_n|^p\|\Psi x_n\|_p^p\\&\leq\sum_{n=0}^\infty\epsilon^{np}(1+\eta)c=\frac{(1+\eta)c}{1-\epsilon^p}=\left(1+\frac{\eta+\epsilon^p}{1-\epsilon^p}\right)c.\end{aligned}\tag{40}$$

Also, since by the triangle inequality

$$\big|\|\Psi x\|_p^p-\|\Psi x_0\|_p^p\big|\leq\|\Psi x-\Psi x_0\|_p^p, \tag{41}$$

we obtain similarly

$$\begin{aligned}\|\Psi x\|_p^p&\geq\|\Psi x_0\|_p^p-\left\|\sum_{n=1}^\infty\epsilon_n\Psi x_n\right\|_p^p\geq(1-\eta)c-\frac{(1+\eta)c\epsilon^p}{1-\epsilon^p}\\&=\left(1-\frac{\eta+\epsilon^p}{1-\epsilon^p}\right)c.\end{aligned}\tag{42}$$

Now let let $x \neq 0$ be arbitrary. Then $x/\|x\|_2 \in S$, so

$$\left(1 - \frac{\eta + \epsilon^p}{1 - \epsilon^p}\right) c \leq \|\Psi x/\|x\|_2\|_p^p \leq \left(1 + \frac{\eta + \epsilon^p}{1 - \epsilon^p}\right) c, \tag{43}$$

so

$$(1 - \delta)c\|x\|_2^p \leq \|\Psi x\|_p^p \leq (1 + \delta)c\|x\|_2^p. \tag{44}$$

We therefore have that (44) holds uniformly on $\mathbb{R}^K$ with probability exceeding $1 - (1 + 2/\epsilon)^L P_{M,p}(\eta)$. □

We can now bring the pieces together and complete the proof of our main theorem.

Proof of theorem 3.1. We need to determine how large M must be for (9) to hold with high probability, which we have chosen to mean failure probability at most $1/\binom{N}{K}$. In applying our lemmas, we will make use of the remark beginning the proof of theorem 2.4, with $c = M\mu_p$. It will be simpler to show the stronger condition that $\delta_{(a+1)K} < (b-1)/(b+1)$. Leaving b undetermined for the moment, we let $L = (a+1)K = (b^{\frac{2}{2-p}} + 1)K$, let $\eta = r(b-1)/(b+1)$ for $r \in (0,1)$ to be chosen shortly, and let $\epsilon^p = (1-r)(b-1)/2b$. We have $(\eta + \epsilon^p)/(1 - \epsilon^p) \leq (b-1)/(b+1)$. (We should really require strict inequality, perhaps by letting $\eta = r_0(b-1)/(b+1)$ for $r_0 < r$, but the lack of sharpness in our estimates will render this unnecessary.) Then by lemma 3.3, an upper bound for the probability that any $M \times L$ submatrix of Φ fails to satisfy (39) is

$$\binom{N}{L}(1 + 2/\epsilon)^L 2\mathrm{e}^{-\frac{\eta^2 M}{2pc_p^2}}. \tag{45}$$

We want this quantity to be bounded above by $1/\binom{N}{K} \geq \frac{K^K}{N^K \mathrm{e}^K}$. For this it suffices that

$$\begin{aligned}
M &\geq \frac{2pc_p^2}{\eta^2}\Big[L\Big(\log\frac{N}{L} + 1 + \log\frac{3}{\epsilon}\Big) + \log 2 + K\Big(\log\frac{N}{K} + 1\Big)\Big] \\
&= \frac{2pc_p^2}{r^2\frac{(b-1)^2}{(b+1)^2}}\Big[K(b^{\frac{2}{2-p}} + 1)\Big(\log\frac{N}{K} + \log\frac{1}{b^{\frac{2}{2-p}} + 1} + 1 + \log 3 + \frac{1}{p}\log\frac{2b}{(1-r)(b-1)}\Big) \\
&\qquad + \log 2 + K\Big(\log\frac{N}{K} + 1\Big)\Big] \\
&= \Big(\frac{2c_p^2(b+1)^2}{r^2(b-1)^2}(b^{\frac{2}{2-p}} + 1)\log\frac{2b}{(1-r)(b-1)}\Big)K \\
&\qquad + p\Big(\frac{2c_p^2(b+1)^2}{r^2(b-1)^2}\Big)\Big[\log 2 + \Big((b^{\frac{2}{2-p}} + 1)(\log 3 + \log\frac{1}{b^{\frac{2}{2-p}} + 1}) + 1\Big)K \\
&\qquad + \Big((b^{\frac{2}{2-p}} + 1) + 1\Big)K\log\frac{N}{K}\Big].
\end{aligned} \tag{46}$$

We can substitute any $b > 1$ and $r \in (0,1)$, and the theorem follows. These are free parameters, which can be chosen independently for each p. To be somewhat more concrete, we choose values minimizing the constant C_1 at $p = 0$, which means minimizing

$$\frac{(b+1)^3}{r^2(b-1)^2} \log \frac{2b}{(1-r)(b-1)}. \tag{47}$$

A numerical computation yields an approximate minimum value of 52.38 at $r = 0.847$ and $b = 5.43$. Substituting gives the sufficient condition

$$\begin{aligned}16.3c_p^2(5.43^{\frac{2}{2-p}} + 1)K \\ &+p16.3c_p^2\Big[\log 2 + \Big((5.43^{\frac{p}{2-p}} + 1)(\log 3 - \log(5.43^{\frac{p}{2-p}} + 1)) + 1\Big)K + \\ &(5.43^{\frac{p}{2-p}} + 2)K \log \frac{N}{K}\Big].\end{aligned} \tag{48}$$

□

The above gives an estimate of $C_1(0) \leq 122$, which is rather far from sharp. Numerical experiments (see section 4) suggest a value less than 3. The proof has many sources of non-sharpness, from the various estimates, to the exponential-Markov inequality argument behind theorem 1.5 of [23] which is never sharp. Although much of the above can be tightened somewhat, it is doubtful that our approach can give sharp constants. However, our efforts have yielded a condition that shows clearly that decreasing p allows fewer measurements to be sufficient for (3) to successfully recover sparse signals.

We note further that our restricted isometry approach yields a condition sufficient for (3) to recover all sufficiently sparse signals, with high probability for a given choice of Φ. Such a uniform recovery probability is desirable for many applications. However, what our approach is unable to obtain is a condition sufficient for a choice of Φ to recover a single sparse signal x with high probability. In situations where such a nonuniform condition would be adequate, a substantially weaker condition should be sufficient. In the case of $p = 1$, the polytope approach of Donoho and Tanner [10] gives both sharp recovery thresholds and estimates for both uniform and nonuniform recovery. It would be valuable to extend this approach to the nonconvex setting.

4. Numerical experiments

In this section we run empirical tests checking how many random, Gaussian measurements are needed for (3) to reconstruct a sparse signal. We solve (3) using an iteratively-reweighted least squares (IRLS) method. We begin with the minimum ℓ^2-norm solution of $\Phi x = b$, $u^{(0)} = A^{+}b$, and set $\epsilon_0 = 1$. We then let $u^{(n)}$ be the solution of

$$\min_u \sum_{i=1}^{N} w_i u_i^2 \text{ subject to } \Phi u = b, \tag{49}$$

where the weights are given by

$$w_i = ((u_i^{(n)})^2 + \epsilon_j)^{p/2-1}. \tag{50}$$

The solution can be given explicitly as

$$u^{(n)} = Q_n\Phi^T(\Phi Q_n\Phi^T)^{-1}b, \tag{51}$$

where Q_n is the diagonal matrix with entries $1/w_i$. This iteration is continued until convergence, deemed to be when the relative ℓ^2-norm change from the previous iterate is less than $\sqrt{\epsilon_j}/100$. The whole process is then repeated with $\epsilon_{j+1} = \epsilon_j/10$, with $u^{(0)}$ being the solution at the previous stage, through a minimum ϵ of 10^{-13}. This is the algorithm used in [27], and uses a similar ϵ-regularization strategy as used in the projected gradient algorithm in [15]. The algorithm differs from the FOCUSS algorithm of Rao and Kreutz-Delgado [28] only in the use of ϵ. Results in [27] show this ϵ approach to give drastically better sparse recovery results than the FOCUSS algorithm.

We fix $N = 256$ and $K = 40$. For each of 100 trials, we randomly select the entries of an 140×256 matrix A from a Gaussian distribution with mean zero and unit variance, randomly select which 40 components of x will be nonzero, and randomly select their values from a mean-zero, unit-variance Gaussian distribution. We then use the above algorithm to solve (3), with Φ consisting of the first M rows of A, for each M from 60 to 140. This is all done for each $p \in \{0.01, 0.02, \dots, 1\}$, with the same matrices A and signals x used for each p. We also do it for $p = 0$, which amounts to minimizing the objective $\sum_i \log(u_i^2 + 10^{-13})$.

The results are in figures 1 and 2. On the one hand, reducing p below 1 clearly reduces the number of measurements needed for perfect recovery, and the improvement is nearly monotonic in p. On the other hand, there is almost no improvement for p much below $1/2$. This is in contrast with the form of our theoretical results; this suggests the possibility that for small p, more measurements may be needed for the algorithm to converge to the global solution than are needed for the global minimizer to equal the sparse signal.

We consider the signal successfully recovered when the sup-norm error is below 10^{-4}. We always find in such instances that further iteration of the algorithm through smaller values of ϵ results in still smaller sup-norm errors, generally below 10^{-13}. For the number of outer iterations chosen, figure 3 shows that when the signal x is recovered, smaller values of p give much smaller reconstruction errors. Thus, using smaller p results in either a more accurate solution, or a solution of specified accuracy obtained more quickly. We also note that the number of iterations needed for convergence for each value of ϵ also increases as p increases.

Of course, the algorithm can only be expected to produce a local minimum of (3). However, when our solution is exactly x, that we have results that give circumstances under which the global solution of (3) is x strongly suggests that we are computing global solutions, at least under a reasonable set of circumstances.

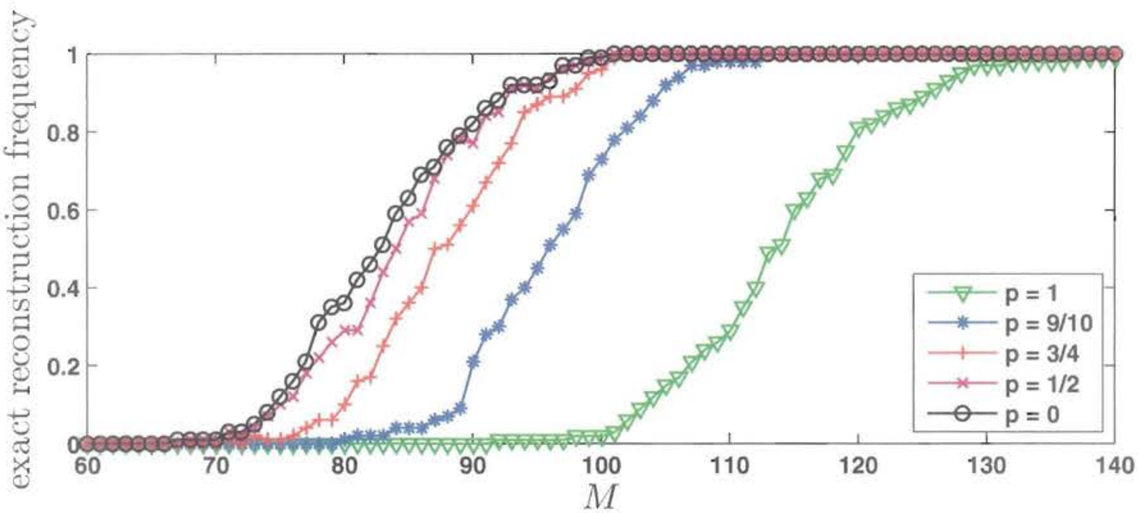

Figure 1. Plots of exact recovery frequency versus the number of measurements, for a few values of p. The signals have $N = 256$ components, $K = 40$ of them being nonzero. Compared with $p = 1$, we see a dramatic decrease in the number of measurements needed for p even slightly less than 1. Reducing p much below $1/2$ gives only a slight increase in the recovery frequency, and does not reduce the number of measurements needed for recovery to always be observed.

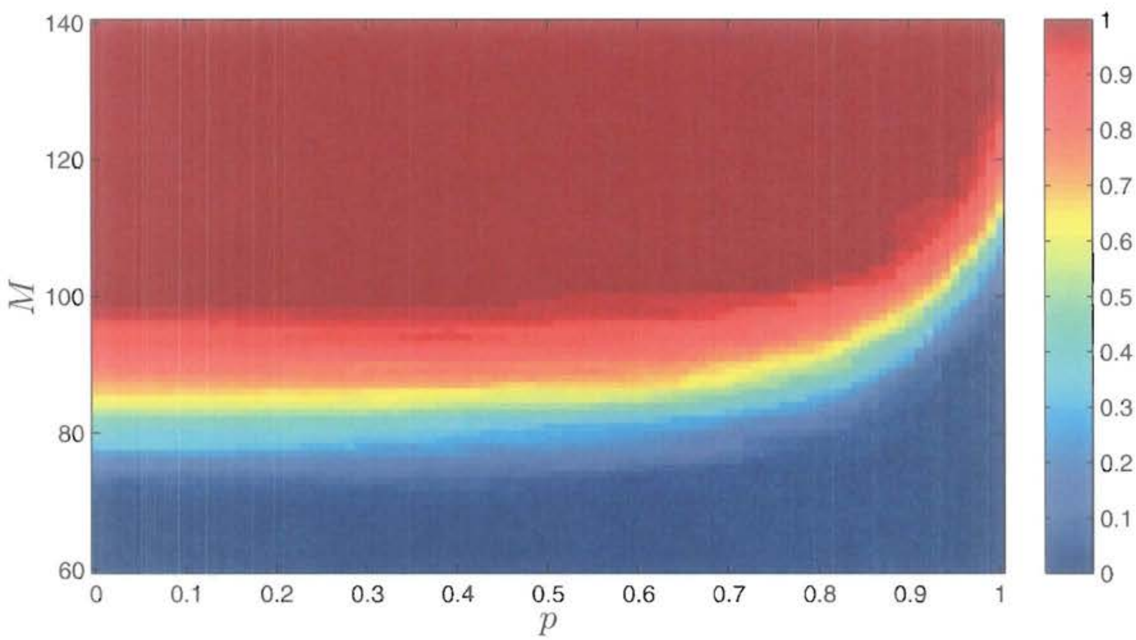

Figure 2. Same data as in figure 1, but with every p shown. The perfect recovery threshold is surprisingly flat for $p < 1/2$; this suggests that the algorithm may not be converging to the global minimum as often for smaller p.

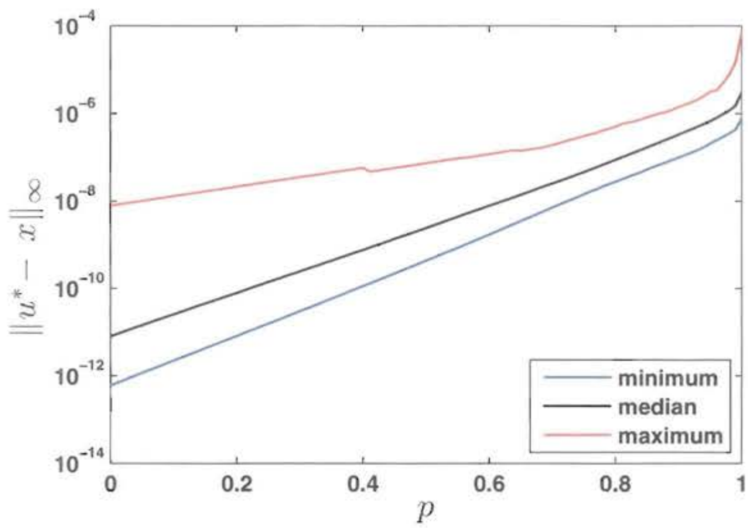

Figure 3. The smallest, median, and largest sup-norm error among successfully recovered signals x. Decreasing p results in a much more accurate solution for the same number of outer iterations.

5. Conclusions

The results in this paper and in [15] give several ways in which ℓ^p minimization can be seen to allow recovery of sparse signals using fewer measurements than ℓ^1 minimization. Our condition for how many random, Gaussian measurements are sufficient with a combinatorially-small probability of failure, though not sharp, shows a clear structural dependence of the number of measurements on the value of p. These findings are partially supported by our numerical experiments, in that reducing p reduces the number of measurements needed for perfect recovery, but seemingly by less than expected for small p. We also find that when sparse recovery is successful, fewer iterations of our process are required to give very complete convergence when p is small.

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