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RESTRICTED LIMITS OF MINIMAL AFFINIZATIONS

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We obtain character formulas of minimal affinizations of representations of quantum groups when the underlying simple Lie algebra is orthogonal and the support of the highest weight is contained in the first three nodes of the Dynkin diagram. We also give a framework for extending our techniques to a more general situation. In particular, for the orthogonal algebras and a highest weight supported in at most one spin node, we realize the restricted classical limit of the corresponding minimal affinizations as a quotient of a module given by generators and relations and, furthermore, show that it projects onto the submodule generated by the top weight space of the tensor product of appropriate restricted Kirillov–Reshetikhin modules. We also prove a conjecture of Chari and Pressley regarding the equivalence of certain minimal affinizations in type D_4 .

Introduction

The representation theory of affine Kac–Moody algebras and their quantum groups has been intensively studied from a broad range of perspectives in the last two decades. In this paper, we focus on nontwisted quantum affine Kac–Moody algebras and their finite-dimensional representations. Let \mathfrak{g} be a finite-dimensional simple Lie algebra over the complex numbers, let $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ be the associated loop algebra, and let $U_q(\mathfrak{g})$ and $U_q(\tilde{\mathfrak{g}})$ be their Drinfeld–Jimbo quantum groups over $\mathbb{C}(q)$, where q is an indeterminate. The affine Kac–Moody algebra is a one-dimensional central extension of $\tilde{\mathfrak{g}}$, but since the center acts trivially on finite-dimensional modules, it suffices to consider the loop algebra. It turns out that the finite-dimensional representations of $U_q(\tilde{\mathfrak{g}})$ are ℓ -weight modules, that is, every vector is a linear combination of common generalized eigenvectors for $U_q(\tilde{\mathfrak{h}})$, where \mathfrak{h} is a fixed Cartan subalgebra of \mathfrak{g} and $\tilde{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}]$. Moreover, the simple modules are highest- ℓ -weight and the set of all dominant ℓ -weights is in bijection with the monoid \mathcal{P}_q^+ of n -tuples of polynomials in one variable with constant term 1, where n is the rank of \mathfrak{g} . The set of all ℓ -weights corresponds

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to the group \mathcal{P}_q associated to \mathcal{P}_q^+ . By specializing q at 1, one recovers the finite-dimensional representation theory of $\tilde{\mathfrak{g}}$.

Given a nonzero complex number a , let $\text{ev}_a : \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$ be the evaluation map $x \otimes f(t) \mapsto f(a)x$. If V is a \mathfrak{g} -module, one can consider the pullback $V(a)$ of V by ev_a . In particular, every irreducible \mathfrak{g} -module can be turned into a $\tilde{\mathfrak{g}}$ -module. In the quantum case, unless \mathfrak{g} is of type A , there is no analogue of the evaluation map, and in fact, most often an irreducible $U_q(\mathfrak{g})$ -module cannot be turned into a $U_q(\tilde{\mathfrak{g}})$ -module. By allowing the underlying vector space to be enlarged in a “controlled” way, a concept of quantum affinization of an irreducible $U_q(\mathfrak{g})$ -module was introduced in [Chari 1995]. Two affinizations are said to be equivalent if they have isomorphic $U_q(\mathfrak{g})$ -structures. It follows from the classification of the finite-dimensional irreducible $U_q(\tilde{\mathfrak{g}})$ -modules that every finite-dimensional irreducible $U_q(\mathfrak{g})$ -module has at least one equivalence class of affinizations. Moreover, there are finitely many equivalence classes of affinizations, and the usual partial order on the weight lattice P of \mathfrak{g} induces a partial order on the set of equivalence classes of affinizations of a given irreducible $U_q(\mathfrak{g})$ -module. Representatives of the minimal elements with respect to this partial order are called minimal affinizations. Although an almost complete classification of the highest ℓ -weights of equivalence classes of minimal affinizations was obtained by Chari [1995] and by Chari and Pressley [CP 1996b; 1995; 1996a], their structure remained essentially unknown except when \mathfrak{g} is of type A or B_2 . Further progress was made after the introduction of the concept of q -characters in [Frenkel and Reshetikhin 1999], which we prefer to call ℓ -characters as explained in Section 4.8.

The ℓ -character of a finite-dimensional $U_q(\tilde{\mathfrak{g}})$ -module V is the associated element $\text{char}_\ell(V)$ of the integral group ring $\mathbb{Z}[\mathcal{P}_q]$ that records the dimensions of the ℓ -weight spaces of V . Given $\lambda \in \mathcal{P}_q^+$, let us denote by $V_q(\lambda)$ the irreducible $U_q(\tilde{\mathfrak{g}})$ -module with highest ℓ -weight λ . Finding formulas for the ℓ -character of $V_q(\lambda)$ is still an open problem in general. In [2001], E. Frenkel and E. Mukhin defined an algorithm, now widely known as the Frenkel–Mukhin algorithm, which for a given $\lambda \in \mathcal{P}_q^+$ returns an element of $\mathbb{Z}[\mathcal{P}_q]$ that was conjectured and proved in certain situations to be $\text{char}_\ell(V_q(\lambda))$. Nakai and Nakanishi [2008] recently showed this is not always the case. However, even in the situations for which the conjecture holds, translating the information given by the algorithm into general closed formulas remains a challenge. For further details on the theory of ℓ -characters, beside the aforementioned literature, see the survey [Chari and Hernandez 2008] and its references. We remark that Nakai and Nakanishi [2007b; 2007a] give path-tableaux descriptions of Jacobi–Trudi determinants that, conjecturally, coincide with the ℓ -characters if \mathfrak{g} is of classical type. Hernandez [2007] partially proved this conjecture if \mathfrak{g} is of type B ; see also [Chari and Hernandez 2008].

Another approach for studying minimal affinizations is to consider their classical limit. Even though most of the ℓ -character information is lost, this process provides an effective tool for studying their $U_q(\mathfrak{g})$ -structure, that is, their characters. The $U_q(\mathfrak{g})$ -structure of the minimal affinizations belonging to the family of Kirillov–Reshetikhin modules was obtained in [Chari 2001] partially using this approach. The proof consisted in showing that the conjectural character was both a lower and an upper bound for the character of the given Kirillov–Reshetikhin module. While the latter was proved by working with the classical limit, the proof of the former was done in the quantum context. Later on, Chari and Moura [CM 2006; 2007] showed that both “upper and lower bound” parts of the proofs of the results of [Chari 2001] could be performed by working with the current algebra $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$; they also obtained in this manner characters of Kirillov–Reshetikhin modules for twisted affine algebras. These modules were introduced in [Kirillov and Reshetikhin 1987] (in the context of Yangians rather than quantum affine algebras) in connection with the Bethe Ansatz. They are the minimal affinizations of the irreducible $U_q(\mathfrak{g})$ -modules whose highest weights are multiples of the fundamental weights of \mathfrak{g} .

The main goal of this paper is to initiate a program for extending the approach of [CM 2006; 2007] to more general minimal affinizations other than Kirillov–Reshetikhin modules. We prove several partial results in this direction and carry out the whole program in the simplest cases. In particular, we obtain character formulas for minimal affinizations in the case that \mathfrak{g} is orthogonal and the support of the highest weight is contained in the first three nodes of the Dynkin diagram of \mathfrak{g} . We now give a summary of our results.

Given a dominant integral weight $\lambda = \sum m_i \omega_i$ (where ω_i for $i = 1, \dots, n$ are the fundamental weights of \mathfrak{g}), we define restricted graded $\mathfrak{g}[t]$ -modules $M(\lambda)$ and $T(\lambda)$. The former is given by generators and relations, while the latter is the submodule generated by the top weight space of $\otimes_i M(m_i \omega_i)$. We conjecture that these modules are isomorphic. This is a generalization of one of the main results of [CM 2006; 2007]. The conjecture clearly holds for type A . The defining relations for the module $M(\lambda)$ are, roughly speaking, the intersection of the relations satisfied by the corresponding restricted Kirillov–Reshetikhin modules $M(m_i \omega_i)$. In particular, it is immediate that $T(\lambda)$ is a quotient of $M(\lambda)$. We prove this conjecture when \mathfrak{g} is orthogonal and λ is supported only in the first three nodes of the Dynkin diagram of \mathfrak{g} . If \mathfrak{g} is of type D , the proof also works in the case that both spin nodes are in the support of λ . As a byproduct of the proof, we obtain the characters of the modules $M(\lambda)$ in these cases. Namely, assume \mathfrak{g} is of type B_n and that the nodes of the Dynkin diagram of \mathfrak{g} are labeled as in [Humphreys 1972]. Given $\lambda = m_1 \omega_1 + m_2 \omega_2 + m_3 \omega_3$, consider the set

$$\mathcal{A} = \{\mathbf{r} = (r_1, r_2, r_3) \in \mathbb{Z}_{\geq 0}^3 : r_1 + r_2 \leq [a_3 m_3], r_2 \leq m_1, r_3 \leq [a_2 m_2]\},$$

where $[m]$ denotes the integer part of the rational number m , $a_n = 1/2$, and $a_i = 1$ for $i \neq n$. Then, we have an isomorphism of \mathfrak{g} -modules:

$$(1) \quad M(\lambda) \cong \bigoplus_{r \in \mathcal{A}} V((m_1+r_1-r_2)\omega_1+(m_2+r_2-a_2^{-1}r_3)\omega_2+(m_3-a_3^{-1}(r_1+r_2))\omega_3).$$

Here, $V(\mu)$ denotes the irreducible \mathfrak{g} -module of highest weight $\mu \in P^+$. If \mathfrak{g} is of type D_n with $n \geq 5$ and $\lambda = m_1\omega_1 + m_2\omega_2 + m_3\omega_3 + m_{n-1}\omega_{n-1} + m_n\omega_n$, the \mathfrak{g} -structure of $M(\lambda)$ is given by (1) as well (in this case $a_i = 1$ for all i). If $n = 4$ and $\lambda \in P^+$, then $M(\lambda) \cong \bigoplus_{r=0}^{m_2} V(\lambda - r\omega_2)$ as a \mathfrak{g} -module.

On the other hand, by regarding the classical limit of a minimal affinization $V_q(\lambda)$ as a $\mathfrak{g}[t]$ -module and then shifting the associated spectral parameter to zero, we obtain modules $L(\lambda)$, which we call the restricted limit of $V_q(\lambda)$. Let λ be the maximal weight of $V_q(\lambda)$. We prove $T(\lambda)$ is a quotient of $L(\lambda)$ in Proposition 3.21. Moreover, for orthogonal \mathfrak{g} , we prove in Proposition 3.22 that $L(\lambda)$ is a quotient of $M(\lambda)$ if λ is supported in a connected subdiagram of type A if \mathfrak{g} is of type D . Therefore, if indeed $M(\lambda)$ is isomorphic to $T(\lambda)$ as conjectured, it would follow that they are also isomorphic to $L(\lambda)$ in the above cases. In particular, Equation (1) describes the $U_q(\mathfrak{g})$ -structure of $V_q(\lambda)$ when \mathfrak{g} is orthogonal and λ is supported only on the first three nodes of the Dynkin diagram of \mathfrak{g} (and possibly on one of the spin nodes if \mathfrak{g} is of type D). For \mathfrak{g} of type B_2 , Chari [1995] obtained the same result by working purely in the quantum setting. If \mathfrak{g} is of type B_n and the value of λ on the coroot associated to the spin node is even, then the ℓ -character (and hence the character) of $V_q(\lambda)$ can be computed using the tableaux expression of the Jacobi–Trudi determinant; see [Chari and Hernandez 2008, Section 7.6]. We expect that, if the minimal connected subdiagram of the Dynkin diagram of \mathfrak{g} containing the support of λ does not contain a subdiagram of type D_4 (in which case $V_q(\lambda)$ has a unique equivalence class of minimal affinizations), Proposition 3.22 remains valid and, hence, that the modules $T(\lambda)$, $M(\lambda)$, and $L(\lambda)$ are isomorphic. We will pursue proofs of these conjectures in a more general setting in a forthcoming publication.

When $V_q(\lambda)$ has more than one equivalence class of minimal affinizations, it is certainly not true that $L(\lambda)$ is a quotient of $M(\lambda)$ (in fact, it is the other way around). Chari and Pressley [1996b] proved that if λ is supported in the triply connected node of the Dynkin diagram of \mathfrak{g} , then there are exactly three equivalence classes of minimal affinizations. We define $\mathfrak{g}[t]$ -modules $M_k(\lambda)$ for $k = 1, 2, 3$ and prove that $L(\lambda)$ is a quotient of $M_k(\lambda)$ for exactly one value of k . Naturally, we expect that $L(\lambda)$ is isomorphic to the appropriate $M_k(\lambda)$. We prove that this is so if \mathfrak{g} is of type D_4 and obtain the character of $M_k(\lambda)$ in this case. Namely, let $\lambda = m_1\omega_1 + m_2\omega_2 + m_3\omega_3 + m_4\omega_4$, where the triply connected node is labeled by 4, suppose $\{i, j, k\} = \{1, 2, 3\}$, and let

$$\mathcal{A}_k = \{r \in \mathbb{Z}_{\geq 0}^3 : r_1 \leq m_k, r_1 + r_2 \leq \min\{m_i, m_j\}, r_3 \leq m_4\}.$$

Then we have an isomorphism of \mathfrak{g} -modules:

$$(2) \quad M_k(\lambda) \cong \bigoplus_{r \in \mathcal{A}_k} V(\lambda - (r_1 - r_2)\omega_k - (r_1 + r_2)(\omega_i + \omega_j) - (r_3 - r_1)\omega_4).$$

If λ is not supported in the triply connected node, Chari and Pressley proved in [1996a, Theorem 2.2] that the number of equivalence classes of minimal affinizations of $V_q(\lambda)$ grows as λ “grows”. Although we do not have a general conjecture in this case yet, the definition of $M_k(\lambda)$ makes sense in this case as well, and its character is computed in the same way as in the previous case. Moreover, the same proof we applied to the previous case in type D_4 also proves that, if λ satisfies the conditions (a) $_{i,j}$ or (b) $_{i,j}$ of [1996a, Theorem 2.2], then $L(\lambda)$ is isomorphic to $M_k(\lambda)$ for the appropriate value of k and its character is given by Equation (2). In particular, this proves the conjecture of [1996a] saying that the modules $V_q(\lambda)$ with λ satisfying conditions (a) $_{i,j}$ of [1996a, Theorem 2.2] are equivalent to those with λ satisfying conditions (b) $_{i,j}$ of that theorem.

The techniques used to prove Propositions 3.21 and 3.22 (and their analogues in the case of multiple equivalence classes of minimal affinizations) require the results of [Chari 2002]. For the proof of Proposition 3.22, we also use partial information on ℓ -characters by combining the Frenkel–Mukhin algorithm with results proved in [CM 2005; Frenkel and Mukhin 2001; Hernandez 2007].

The paper is organized as follows. In Sections 1 and 2, we review some structural results of the algebras \mathfrak{g} and $\tilde{\mathfrak{g}}$ and their quantum counterparts, as well as some basic results of the finite-dimensional representation theory of these algebras. In Section 3, after reviewing the partial classification of minimal affinizations, we define the modules $M(\lambda)$, $T(\lambda)$, and $L(\lambda)$, and state our main results and conjectures regarding them. The proofs are given in Sections 4 and 5. The case of multiple equivalence classes of minimal affinizations is treated in Sections 5.11 and 5.18.

1. Quantum and classical loop algebras

Throughout the paper, let $\mathbb{C}, \mathbb{R}, \mathbb{Z}, \mathbb{Z}_{\geq m}$ denote the sets of complex numbers, reals, integers, and integers no less than m , respectively. Given a ring \mathbb{A} , the underlying multiplicative group of units is denoted by \mathbb{A}^\times . The dual of a vector space V is denoted by V^* . The symbol \cong means “isomorphic to”.

1.1. Classical algebras. Let $I = \{1, \dots, n\}$ be the set of vertices of a finite-type connected Dynkin diagram labeled as in [Humphreys 1972], and let \mathfrak{g} be the associated simple Lie algebra over \mathbb{C} with a fixed Cartan subalgebra \mathfrak{h} . Fix a set of positive roots R^+ and let

$$\mathfrak{n}^\pm = \bigoplus_{\alpha \in R^+} \mathfrak{g}_{\pm\alpha}, \quad \text{where } \mathfrak{g}_{\pm\alpha} = \{x \in \mathfrak{g} : [h, x] = \pm\alpha(h)x, \forall h \in \mathfrak{h}\}.$$

The simple roots will be denoted by α_i , the fundamental weights by ω_i , while Q, P, Q^+, P^+ will denote the root and weight lattices with corresponding positive cones, respectively. Let also $h_i \in \mathfrak{h}$ be the coroot associated to α_i for $i \in I$. We equip \mathfrak{h}^* with the partial order $\lambda \leq \mu$ if and only if $\mu - \lambda \in Q^+$. We denote by \mathcal{W} the Weyl group of \mathfrak{g} and let w_0 be the longest element of \mathcal{W} . Given $\lambda \in P$, set

$$(1-1) \quad \lambda^* = -w_0\lambda.$$

Recall that if $\lambda \in P^+$, then $\lambda^* \in P^+$ as well. Let $C = (c_{ij})_{i,j \in I}$ be the Cartan matrix of \mathfrak{g} , that is, $c_{ij} = \alpha_j(h_i)$, and let $D = \text{diag}(d_i : i \in I)$ be such that the numbers d_i are coprime positive integers and DC is symmetric.

The subalgebras $\mathfrak{g}_{\pm\alpha}$ for $\alpha \in R^+$ are one-dimensional, and $[\mathfrak{g}_{\pm\alpha}, \mathfrak{g}_{\pm\beta}] = \mathfrak{g}_{\pm\alpha\pm\beta}$ for every $\alpha, \beta \in R^+$. We denote by x_α^\pm any generator of $\mathfrak{g}_{\pm\alpha}$. In particular, if $\alpha + \beta \in R^+$, then $[x_\alpha^\pm, x_\beta^\pm]$ is a nonzero generator of $\mathfrak{g}_{\pm\alpha\pm\beta}$, and we simply write $[x_\alpha^\pm, x_\beta^\pm] = x_{\alpha+\beta}^\pm$. For each subset J of I , let \mathfrak{g}_J be the Lie subalgebra of \mathfrak{g} generated by $x_{\alpha_j}^\pm$ for $j \in J$, and define \mathfrak{n}_J^\pm and \mathfrak{h}_J in the obvious way. Let also Q_J be the subgroup of Q generated by α_j for $j \in J$, and let $R_J^+ = R^+ \cap Q_J$. Given $\lambda \in P$, let λ_J be the restriction of λ to \mathfrak{h}_J^* , and let $\lambda^J \in P$ be such that $\lambda^J(h_j) = \lambda(h_j)$ if $j \in J$ and $\lambda^J(h_j) = 0$ otherwise. By abuse of language, we will refer to any subset J of I as a subdiagram of the Dynkin diagram of \mathfrak{g} . The support of $\mu \in P$ is defined to be the subdiagram $\text{supp}(\mu) \subseteq I$ given by $\text{supp}(\mu) = \{i \in I : \mu(h_i) \neq 0\}$. Let also $\overline{\text{supp}}(\mu)$ be the minimal connected subdiagram of I containing $\text{supp}(\mu)$.

If \mathfrak{a} is a Lie algebra over \mathbb{C} , define its loop algebra to be $\tilde{\mathfrak{a}} = \mathfrak{a} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$ with bracket given by $[x \otimes t^r, y \otimes t^s] = [x, y] \otimes t^{r+s}$. Clearly $\mathfrak{a} \otimes 1$ is a subalgebra of $\tilde{\mathfrak{a}}$ isomorphic to \mathfrak{a} and, by abuse of notation, we will continue denoting its elements by x instead of $x \otimes 1$. We also consider the current algebra $\mathfrak{a}[t]$, the subalgebra of $\tilde{\mathfrak{a}}$ given by $\mathfrak{a}[t] = \mathfrak{a} \otimes \mathbb{C}[t]$. Then $\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}^- \oplus \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{n}}^+$ and $\tilde{\mathfrak{h}}$ is an abelian subalgebra and similarly for $\mathfrak{g}[t]$. The elements $x_\alpha^\pm \otimes t^r$ and $h_i \otimes t^r$ will be denoted by $x_{\alpha,r}^\pm$ and $h_{\alpha_i,r}$, respectively. Diagram subalgebras $\tilde{\mathfrak{g}}_J$ are defined in the obvious way.

Let $U(\mathfrak{a})$ denote the universal enveloping algebra of a Lie algebra \mathfrak{a} . Then $U(\mathfrak{a})$ is a subalgebra of $U(\tilde{\mathfrak{a}})$ and multiplication establishes isomorphisms of vector spaces

$$U(\mathfrak{g}) \cong U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}^+) \quad \text{and} \quad U(\tilde{\mathfrak{g}}) \cong U(\tilde{\mathfrak{n}}^-) \otimes U(\tilde{\mathfrak{h}}) \otimes U(\tilde{\mathfrak{n}}^+).$$

We can uniquely extend the assignments

$$\Delta : \mathfrak{a} \rightarrow U(\mathfrak{a}) \otimes U(\mathfrak{a}), \quad x \mapsto x \otimes 1 + 1 \otimes x, \quad S : \mathfrak{a} \rightarrow \mathfrak{a}, \quad x \mapsto -x, \quad \epsilon : \mathfrak{a} \rightarrow \mathbb{C}, \quad x \mapsto 0$$

so that $U(\mathfrak{a})$ becomes a Hopf algebra with comultiplication Δ , antipode S , and counit ϵ .

Given $a \in \mathbb{C}$, let τ_a be the Lie algebra automorphism of $\mathfrak{a}[t]$ defined so that $\tau_a(x \otimes f(t)) = x \otimes f(t - a)$ for every $x \in \mathfrak{a}$ and every $f(t) \in \mathbb{C}[t]$. If $a \neq 0$, let

$ev_a : \tilde{\mathfrak{a}} \rightarrow \mathfrak{a}$ be the evaluation map $x \otimes f(t) \mapsto f(a)x$. We also denote by τ_a and ev_a the induced maps $U(\mathfrak{a}[t]) \rightarrow U(\mathfrak{a}[t])$ and $U(\tilde{\mathfrak{a}}) \rightarrow U(\mathfrak{a})$, respectively.

For each $i \in I$ and $r \in \mathbb{Z}$, define elements $\Lambda_{i,r} \in U(\tilde{\mathfrak{h}})$ by the following equality of formal power series in the variable u :

$$(1-2) \quad \sum_{r=0}^{\infty} \Lambda_{i,\pm r} u^r = \exp\left(-\sum_{s=1}^{\infty} \frac{h_{a_i,\pm s}}{s} u^s\right).$$

1.2. Quantum algebras. Let $\mathbb{C}(q)$ be the ring of rational functions on an indeterminate q , and let $\mathbb{A} = \mathbb{C}[q, q^{-1}]$. Given $p = q^k$ for some $k \in \mathbb{Z} \setminus \{0\}$, define

$$[m]_p = \frac{p^m - p^{-m}}{p - p^{-1}}, \quad [m]_p! = [m]_p [m-1]_p \dots [2]_p [1]_p, \quad \begin{bmatrix} m \\ r \end{bmatrix}_p = \frac{[m]_p!}{[r]_p! [m-r]_p!},$$

for $r, m \in \mathbb{Z}_{\geq 0}$ with $m \geq r$. Notice that $\begin{bmatrix} m \\ r \end{bmatrix}_p \in \mathbb{A}$.

Set $q_i = q^{d_i}$. The quantum loop algebra $U_q(\tilde{\mathfrak{g}})$ of \mathfrak{g} is the algebra with generators

$$x_{i,r}^{\pm} \text{ for } i \in I, r \in \mathbb{Z}, \quad k_i^{\pm 1} \text{ for } i \in I, \quad h_{i,r} \text{ for } i \in I, r \in \mathbb{Z} \setminus \{0\},$$

and the defining relations

$$\begin{aligned} k_i k_i^{-1} &= k_i^{-1} k_i = 1, & k_i k_j &= k_j k_i, \\ k_i h_{j,r} &= h_{j,r} k_i, & k_i x_{j,s}^{\pm} k_i^{-1} &= q_i^{\pm c_{ij}} x_{j,s}^{\pm}, \\ [h_{i,r}, h_{j,s}] &= 0, & [h_{i,r}, x_{j,s}^{\pm}] &= \pm \frac{1}{r} [rc_{ij}]_q x_{j,r+s}^{\pm}, \\ x_{i,r+1}^{\pm} x_{j,s}^{\pm} - q_i^{\pm c_{ij}} x_{j,s}^{\pm} x_{i,r+1}^{\pm} &= q_i^{\pm c_{ij}} x_{i,r}^{\pm} x_{j,s+1}^{\pm} - x_{j,s+1}^{\pm} x_{i,r}^{\pm}, \\ [x_{i,r}^+, x_{j,s}^-] &= \delta_{i,j} \frac{\psi_{i,r+s}^+ - \psi_{i,r+s}^-}{q_i - q_i^{-1}}, \end{aligned}$$

$$\sum_{\sigma \in S_m} \sum_{k=0}^m (-1)^k \begin{bmatrix} m \\ k \end{bmatrix}_{q_i} x_{i,r_{\sigma(1)}}^{\pm} \dots x_{i,r_{\sigma(k)}}^{\pm} x_{j,s}^{\pm} x_{i,r_{\sigma(k+1)}}^{\pm} \dots x_{i,r_{\sigma(m)}}^{\pm} = 0 \quad \text{if } i \neq j,$$

for all sequences of integers r_1, \dots, r_m , where $m = 1 - c_{ij}$, S_m is the symmetric group on m letters, and the $\psi_{i,r}^{\pm}$ are determined by equating powers of u in the formal power series

$$\Psi_i^{\pm}(u) = \sum_{r=0}^{\infty} \psi_{i,\pm r}^{\pm} u^r = k_i^{\pm 1} \exp\left(\pm(q_i - q_i^{-1}) \sum_{s=1}^{\infty} h_{i,\pm s} u^s\right).$$

Denote by $U_q(\tilde{\mathfrak{n}}^{\pm})$ and $U_q(\tilde{\mathfrak{h}})$ the subalgebras of $U_q(\tilde{\mathfrak{g}})$ generated by $\{x_{i,r}^{\pm}\}$ and $\{k_i^{\pm 1}, h_{i,s}\}$, respectively. Let $U_q(\mathfrak{g})$ be the subalgebra generated by $x_i^{\pm} := x_{i,0}^{\pm}$ and $k_i^{\pm 1}$ for $i \in I$, and define $U_q(\mathfrak{n}^{\pm})$ and $U_q(\mathfrak{h})$ in the obvious way. $U_q(\mathfrak{g})$ is a

subalgebra of $U_q(\tilde{\mathfrak{g}})$, and multiplication establishes isomorphisms

$$U_q(\mathfrak{g}) \cong U_q(\mathfrak{n}^-) \otimes U_q(\mathfrak{h}) \otimes U_q(\mathfrak{n}^+) \quad \text{and} \quad U_q(\tilde{\mathfrak{g}}) \cong U_q(\tilde{\mathfrak{n}}^-) \otimes U_q(\tilde{\mathfrak{h}}) \otimes U_q(\tilde{\mathfrak{n}}^+)$$

of $\mathbb{C}(q)$ -vectors spaces.

Let $J \subseteq I$ and consider the subalgebra $U_q(\tilde{\mathfrak{g}}_J)$ generated by $k_j^{\pm 1}, h_{j,r}, x_{j,s}^{\pm}$ for all $j \in J, r, s \in \mathbb{Z}$ with $r \neq 0$. If $J = \{j\}$, the algebra $U_q(\tilde{\mathfrak{g}}_j) := U_q(\tilde{\mathfrak{g}}_J)$ is isomorphic to $U_{q_j}(\tilde{\mathfrak{sl}}_2)$. Similarly we define the subalgebra $U_q(\mathfrak{g}_J)$, and so on.

For $i \in I, r \in \mathbb{Z}$, and $k \in \mathbb{Z}_{\geq 0}$, define $(x_{i,r}^{\pm})^{(k)} = (x_{i,r}^{\pm})^k / [k]_{q_i}!$. Define also elements $\Lambda_{i,r}$ for $i \in I$ and $r \in \mathbb{Z}$ by

$$(1-3) \quad \sum_{r=0}^{\infty} \Lambda_{i,\pm r} u^r = \exp\left(-\sum_{s=1}^{\infty} \frac{h_{i,\pm s}}{[s]_{q_i}} u^s\right).$$

Note that

$$(1-4) \quad \Psi_i^{\pm}(u) = k_i^{\pm 1} \frac{\Lambda_i^{\pm}(q_i^{\mp 1} u)}{\Lambda_i^{\pm}(q_i^{\pm 1} u)},$$

where the division is that of formal power series in u . Although we are denoting the elements $\Lambda_{i,r}$ above by the same symbol as their classical counterparts, this will not create confusion as it will be clear from the context.

Let $U_{\mathbb{A}}(\tilde{\mathfrak{g}})$ be the \mathbb{A} -subalgebra of $U_q(\tilde{\mathfrak{g}})$ generated by the elements $(x_{i,r}^{\pm})^{(k)}$ and $k_i^{\pm 1}$ for $i \in I, r \in \mathbb{Z}$, and $k \in \mathbb{Z}_{\geq 0}$. Define $U_{\mathbb{A}}(\mathfrak{g})$ similarly and notice that $U_{\mathbb{A}}(\mathfrak{g}) = U_{\mathbb{A}}(\tilde{\mathfrak{g}}) \cap U_q(\mathfrak{g})$. For the proof of the next proposition see [Chari 2001, Lemma 2.1] and the locally cited references.

Proposition 1.3. $U_q(\tilde{\mathfrak{g}}) = \mathbb{C}(q) \otimes_{\mathbb{A}} U_{\mathbb{A}}(\tilde{\mathfrak{g}})$ and $U_q(\mathfrak{g}) = \mathbb{C}(q) \otimes_{\mathbb{A}} U_{\mathbb{A}}(\mathfrak{g})$.

Regard \mathbb{C} as an \mathbb{A} -module by letting q act as 1, and set

$$(1-5) \quad \overline{U_q(\tilde{\mathfrak{g}})} = \mathbb{C} \otimes_{\mathbb{A}} U_{\mathbb{A}}(\tilde{\mathfrak{g}}) \quad \text{and} \quad \overline{U_q(\mathfrak{g})} = \mathbb{C} \otimes_{\mathbb{A}} U_{\mathbb{A}}(\mathfrak{g}).$$

Denote by $\bar{\eta}$ the image of $\eta \in U_{\mathbb{A}}(\tilde{\mathfrak{g}})$ in $\overline{U_q(\tilde{\mathfrak{g}})}$. The proof of the next proposition can be found in [CP 1994a, Proposition 9.2.3] and [Lusztig 1993].

Proposition 1.4. $U(\tilde{\mathfrak{g}})$ is isomorphic to the quotient of $\overline{U_q(\tilde{\mathfrak{g}})}$ by the ideal generated by $\bar{k}_i - 1$. In particular, the category of $\overline{U_q(\tilde{\mathfrak{g}})}$ -modules on which the k_i act as the identity operator for all $i \in I$ is equivalent to the category of all $\tilde{\mathfrak{g}}$ -modules.

The algebra $U_q(\tilde{\mathfrak{g}})$ is a Hopf algebra and induces a Hopf algebra structure (over \mathbb{A}) on $U_{\mathbb{A}}(\tilde{\mathfrak{g}})$; see [CP 1994a; Lusztig 1993]. The induced Hopf algebra structure on $U(\tilde{\mathfrak{g}})$ coincides with the usual one. On $U_q(\mathfrak{g})$ we have, for all $i \in I$,

$$(1-6) \quad \Delta(x_i^+) = x_i^+ \otimes 1 + k_i \otimes x_i^+, \quad \Delta(x_i^-) = x_i^- \otimes k_i^{-1} + 1 \otimes x_i^-, \quad \Delta(k_i) = k_i \otimes k_i.$$

Lemma 1.5. *Suppose $x = [x_{i_1}^-, [x_{i_2}^-, \dots [x_{i_{l-1}}^-, x_{i_l}^-] \dots]]$. Then $X \in U_{\mathbb{A}}(\mathfrak{n}^-)$ and*

$$\Delta(x) \in x \otimes \left(\prod_{j=1}^l k_{i_j}^{-1} \right) + 1 \otimes x + f(q)y$$

for some $y \in U_{\mathbb{A}}(\mathfrak{g}) \otimes U_{\mathbb{A}}(\mathfrak{g})$ and some $f(q) \in \mathbb{A}$ such that $f(1) = 0$.

Proof. When $l = 1$, this is immediate from (1-6). A straightforward induction on l using the relations $k_i x_j^- = q_i^{-c_{i,j}} x_j^- k_i$ completes the proof. \square

An expression for the comultiplication Δ of $U_q(\tilde{\mathfrak{g}})$ in terms of the generators $x_{i,r}^{\pm}, h_i, r, k_i^{\pm 1}$ is not known. The following partial information will suffice for our purposes. Let X^{\pm} be the subspace of $U_{\mathbb{A}}(\tilde{\mathfrak{n}}^{\pm})$ spanned by $\{x_{j,r}^{\pm} : j \in I, r \in \mathbb{Z}\}$.

Lemma 1.6. $\Delta(x_{i,1}^-) = x_{i,1}^- \otimes k_i + 1 \otimes x_{i,1}^- + x$ for some $x \in U_{\mathbb{A}}(\mathfrak{g}) \otimes U_{\mathbb{A}}(\mathfrak{g})$ such that $\bar{x} = 0$.

Proof. It was proved in [Beck 1994; Beck et al. 1999; Damiani 1998]—see also [CP 1997, Lemma 7.5]—that $\Delta(x_{i,1}^-) = x_{i,1}^- \otimes k_i + 1 \otimes x_{i,1}^- + x$, where $x \in U_{\mathbb{A}}(\tilde{\mathfrak{g}})X^- \otimes U_{\mathbb{A}}(\tilde{\mathfrak{g}})X^+$. Since the image $\overline{\Delta(x_{i,1}^-)}$ of $\Delta(x_{i,1}^-)$ in $U(\tilde{\mathfrak{g}}) \otimes U(\tilde{\mathfrak{g}})$ is $x_{a_i,1}^- \otimes 1 + 1 \otimes x_{a_i,1}^-$, the image of x in $U(\tilde{\mathfrak{g}}) \otimes U(\tilde{\mathfrak{g}})$ must be zero. \square

The following was also proved in [Beck 1994; Beck et al. 1999; Damiani 1998]. Modulo $U_q(\tilde{\mathfrak{g}})X^- \otimes U_q(\tilde{\mathfrak{g}})X^+$, we have, for all $r \in \mathbb{Z}_{\geq 1}$,

$$(1-7) \quad \Delta(h_{i,r}) = h_{i,s} \otimes 1 + 1 \otimes h_{i,r} \quad \text{and} \quad \Delta(\Lambda_{i,r}) = \sum_{s=0}^r \Lambda_{i,r-s} \otimes \Lambda_{i,s}$$

The following general result on the dual representation of a tensor product of representations of a Hopf algebra is proved, for instance, in [Kassel 1995].

Proposition 1.7. *Let H be a Hopf algebra and V and W be finite-dimensional H -modules. Then $(V \otimes W)^* \cong W^* \otimes V^*$.*

1.8. The ℓ -weight lattice. Given a field \mathbb{F} consider the multiplicative group $\mathcal{P}_{\mathbb{F}}$ of n -tuples of rational functions $\boldsymbol{\mu} = (\boldsymbol{\mu}_1(u), \dots, \boldsymbol{\mu}_n(u))$ with values in \mathbb{F} such that $\boldsymbol{\mu}_i(0) = 1$ for all $i \in I$. We shall often think of $\boldsymbol{\mu}_i(u)$ as a formal power series in u with coefficients in \mathbb{F} . Given $a \in \mathbb{F}^{\times}$ and $i \in I$, let $\boldsymbol{\omega}_{i,a}$ be defined by

$$(\boldsymbol{\omega}_{i,a})_j(u) = 1 - \delta_{i,j} a u.$$

Clearly, if \mathbb{F} is algebraically closed, $\mathcal{P}_{\mathbb{F}}$ is the free abelian group generated by these elements, which are called fundamental ℓ -weights. It is also convenient to introduce elements $\boldsymbol{\omega}_{\lambda,a}$ for $\lambda \in P$ and $a \in \mathbb{C}(q)$ defined by

$$(1-8) \quad \boldsymbol{\omega}_{\lambda,a} = \prod_{i \in I} (\boldsymbol{\omega}_{i,a})^{\lambda(h_i)}.$$

If \mathbb{F} is algebraically closed, introduce the group homomorphism (weight map) $\text{wt} : \mathcal{P}_{\mathbb{F}} \rightarrow P$ by setting $\text{wt}(\omega_{i,a}) = \omega_i$, where ω_i is the i -th fundamental weight of \mathfrak{g} . Otherwise, let \mathbb{K} be an algebraically closed extension of \mathbb{F} , so that $\mathcal{P}_{\mathbb{F}}$ can be regarded as a subgroup of $\mathcal{P}_{\mathbb{K}}$, and define the weight map on $\mathcal{P}_{\mathbb{F}}$ by restricting the one on $\mathcal{P}_{\mathbb{K}}$ (this clearly does not depend on the choice of \mathbb{K}). Define the ℓ -weight lattice of $U_q(\tilde{\mathfrak{g}})$ to be $\mathcal{P}_q := \mathcal{P}_{\mathbb{C}(q)}$. The submonoid \mathcal{P}_q^+ of \mathcal{P}_q consisting of n -tuples of polynomials is called the set of dominant ℓ -weights of $U_q(\tilde{\mathfrak{g}})$.

Given $\lambda \in \mathcal{P}_q^+$ with $\lambda_i(u) = \prod_j (1 - a_{i,j}u)$, where $a_{i,j}$ belongs to some algebraic closure of $\mathbb{C}(q)$, let $\lambda^- \in \mathcal{P}_q^+$ be defined by $\lambda_i^-(u) = \prod_j (1 - a_{i,j}^{-1}u)$. We will also use the notation $\lambda^+ = \lambda$. Two elements λ, μ of \mathcal{P}_q^+ are said to be relatively prime if $\lambda_i(u)$ is relatively prime to $\mu_j(u)$ in $\mathbb{C}(q)[u]$ for all $i, j \in I$. Every $\nu \in \mathcal{P}_q$ can be uniquely written in the form

$$(1-9) \quad \nu = \lambda \mu^{-1} \quad \text{with } \lambda, \mu \in \mathcal{P}_q^+ \text{ relatively prime.}$$

Given this $\nu = \lambda \mu^{-1}$, define a $\mathbb{C}(q)$ -algebra homomorphism $\Psi_{\nu} : U_q(\tilde{\mathfrak{h}}) \rightarrow \mathbb{C}(q)$ by setting

$$(1-10) \quad \Psi_{\nu}(k_i^{\pm 1}) = q_i^{\pm \text{wt}(\nu)(h_i)} \quad \text{and} \quad \sum_{r \geq 0} \Psi_{\nu}(\Lambda_{i, \pm r}) u^r = \frac{(\lambda^{\pm})_i(u)}{(\mu^{\pm})_i(u)},$$

where the division is that of formal power series in u . The next proposition is easy.

Proposition 1.9. *The map $\Psi : \mathcal{P}_q \rightarrow (U_q(\tilde{\mathfrak{h}}))^*$ given by $\nu \mapsto \Psi_{\nu}$ is injective.*

Define the ℓ -weight lattice \mathcal{P} of $\tilde{\mathfrak{g}}$ to be the subgroup of \mathcal{P}_q generated by $\omega_{i,a}$ for all $i \in I$ and all $a \in \mathbb{C}^{\times}$ or, equivalently, $\mathcal{P} = \mathcal{P}_{\mathbb{C}}$. Observe that every element $\lambda \in \mathcal{P}$ can be uniquely decomposed as

$$(1-11) \quad \lambda = \prod_j \omega_{\lambda_j, a_j} \quad \text{for some } \lambda_j \in P \text{ and } a_i \neq a_j \in \mathbb{C}.$$

Set also $\mathcal{P}^+ = \mathcal{P} \cap \mathcal{P}_q^+$.

From now on we will identify \mathcal{P}_q with its image in $(U_q(\tilde{\mathfrak{h}}))^*$ under Ψ . Similarly, \mathcal{P} will be identified with a subset of $U(\tilde{\mathfrak{h}})^*$ via the homomorphism $\Psi_{\nu} : U(\tilde{\mathfrak{h}}) \rightarrow \mathbb{C}$ determined by

$$(1-12) \quad \Psi_{\nu}(h_i) = \text{wt}(\nu)(h_i) \quad \text{and} \quad \sum_{r \geq 0} \Psi_{\nu}(\Lambda_{i, \pm r}) u^r = \frac{(\lambda^{\pm})_i(u)}{(\mu^{\pm})_i(u)}.$$

It will be convenient to introduce the following notation. Given $i \in I$, $a \in \mathbb{C}(q)^{\times}$, and $r \in \mathbb{Z}_{\geq 0}$, define

$$(1-13) \quad \omega_{i,a,r} = \prod_{j=0}^{r-1} \omega_{i, aq_i^{r-1-2j}}.$$

Define also the polynomial

$$(1-14) \quad f_{i,a,r}(u) = \prod_{j=0}^{r-1} (1 - aq_i^{r-1-2j}u).$$

Observe that given $f(u) \in \mathbb{C}(q)[u]$ having all its roots in $\mathbb{C}(q)$ and such that $f(1) = 0$, there exist unique $m \in \mathbb{Z}_{\geq 0}$, $a_1, \dots, a_m \in \mathbb{C}(q)^\times$, and $r_1, \dots, r_m \in \mathbb{Z}_{\geq 1}$ such that

$$(1-15) \quad f(u) = \prod_{k=1}^m f_{i,a_k,r_k}(u) \quad \text{with} \quad \frac{a_l}{a_j} \neq q^{\pm(r_l+r_j-2p)} \text{ for } 0 \leq p < \min\{r_l, r_j\}.$$

In particular, given $\lambda \in \mathcal{P}_q^+$ such that $\lambda_i(u)$ splits in $\mathbb{C}(q)[u]$ for all $i \in I$, there exist unique $m_i \in \mathbb{Z}_{\geq 0}$, $a_{i,k} \in \mathbb{C}(q)^\times$, and $r_{i,k} \in \mathbb{Z}_{\geq 1}$ such that

$$(1-16) \quad \lambda = \prod_{i \in I} \prod_{k=1}^{m_i} \omega_{i,a_{i,k},r_{i,k}},$$

with $\frac{a_{i,j}}{a_{i,l}} \neq q_i^{\pm(r_{i,j}+r_{i,l}-2p)}$ and $\sum_{k=1}^{m_i} r_{i,k} = \text{wt}(\lambda)(h_i)$

for all $i \in I$, $j \neq l$, and $0 \leq p < \min\{r_{i,j}, r_{i,l}\}$.

If $J \subseteq I$ and $\lambda \in \mathcal{P}_q$, let λ_J be the associated J -tuple of rational functions. Note that if $\lambda_j(u) \in \mathbb{C}(q_j)(u)$ for all $j \in J$, then λ_J can be regarded as an element of the ℓ -weight lattice of $U_q(\tilde{\mathfrak{g}}_J)$. Let also $\lambda^J \in \mathcal{P}_q$ be such that $(\lambda^J)_j(u) = \lambda_j(u)$ for every $j \in J$ and $(\lambda^J)_j(u) = 1$ otherwise.

Recall that w_0 defines a Dynkin diagram automorphism such that $w_0 \cdot i = j$ and only if $w_0\omega_i = -\omega_j$ for $i, j \in I$. Given $\lambda \in \mathcal{P}_q^+$, let $\lambda^* \in \mathcal{P}_q^+$ be the element defined by

$$(1-17) \quad (\lambda^*)_i(u) = \lambda_{w_0 \cdot i}(q^{r^\vee h^\vee} u),$$

where h^\vee is the dual Coxeter number of \mathfrak{g} and $r^\vee = \max\{c_{ij}c_{ji} : i, j \in I, i \neq j\}$ is the lacing number of \mathfrak{g} . Define also the element ${}^*\lambda$ by requiring $({}^*\lambda)^\pm = (\lambda^*)^\mp$.

Given $i \in I$ and $a \in \mathbb{C}(q)^\times$, define the simple ℓ -root $\alpha_{i,a}$ by

$$\alpha_{i,a} = (\omega_{i,aq_i,2})^{-1} \prod_{j \neq i} \omega_{j,aq_i,-c_{j,i}}.$$

The subgroup of \mathcal{P}_q generated by the simple ℓ -roots is called the ℓ -root lattice of $U_q(\tilde{\mathfrak{g}})$ and will be denoted by \mathcal{Q}_q . Let also \mathcal{Q}_q^+ be the submonoid generated by the simple ℓ -roots. Clearly $\text{wt}(\alpha_{i,a}) = \alpha_i$. Define a partial order on \mathcal{P}_q by $\mu \leq \lambda$ if $\lambda\mu^{-1} \in \mathcal{Q}_q^+$.

2. Finite-dimensional representations

2.1. Simple Lie algebras. We now review some of basic facts of the representation theory of \mathfrak{g} and $U_q(\mathfrak{g})$. For details, see for instance [Humphreys 1972] or CP 1994a].

Given a \mathfrak{g} -module V and $\mu \in \mathfrak{h}^*$, let $V_\mu = \{v \in V : hv = \mu(h)v \text{ for all } h \in \mathfrak{h}\}$. A nonzero vector $v \in V_\mu$ is called a weight vector of weight μ . If v is a weight vector such that $\mathfrak{n}^+v = 0$, then v is called a highest-weight vector. If V is generated by a highest-weight vector of weight λ , then V is said to be a highest-weight module of highest weight λ .

We summarize the basic facts about finite-dimensional \mathfrak{g} -modules:

Theorem 2.2. *Let V be a finite-dimensional \mathfrak{g} -module.*

- (a) $V = \bigoplus_{\mu \in P} V_\mu$ and $\dim V_\mu = \dim V_{w\mu}$ for all $w \in \mathfrak{W}$.
- (b) V is completely reducible.
- (c) For each $\lambda \in P^+$, the $U(\mathfrak{g})$ -module $V(\lambda)$ generated by a vector v satisfying

$$x_{\alpha_i}^+v = 0, \quad h_i v = \lambda(h_i)v, \quad (x_{\alpha_i}^-)^{\lambda(h_i)+1}v = 0 \quad \text{for all } i \in I$$

is irreducible and finite-dimensional. If V is irreducible, then V is isomorphic to $V(\lambda)$ for some $\lambda \in P^+$.

- (d) If $\lambda \in P^+$ and $V \cong V(\lambda)$, then $V_\mu \neq 0$ if and only if $w\mu \leq \lambda$ for all $w \in \mathfrak{W}$. Furthermore, the lowest weight of $V(\lambda)$ is $-\lambda^*$. In particular, $V(\lambda)^* \cong V(\lambda^*)$.

Lemma 2.3. *Let V be a finite-dimensional \mathfrak{g} -module, and suppose $l \in \mathbb{Z}_{\geq 1}$, $v_k \in P$, and $v_k \in V_{v_k}$ for $k = 1, \dots, l$ are such that $V = \sum_{k=1}^l U(\mathfrak{n}^-)v_k$. Fix a decomposition $V = \bigoplus_{j=1}^m V_j$ where $m \in \mathbb{Z}_{\geq 1}$ and $V_j \cong V(\mu_j)$ for some $\mu_j \in P^+$, and let $\pi_j : V \rightarrow V_j$ be the associated projection for $j = 1, \dots, m$. Then, there exist distinct $k_1, \dots, k_m \in \{1, \dots, l\}$ such that $v_{k_j} = \mu_j$ and $\pi_j(v_{k_j}) \neq 0$.*

Proof. Proceed by induction on m . If $m = 1$, the lemma is immediate. Otherwise, suppose, without loss of generality, that μ_m is a maximal weight of V . In that case, there must exist k_m such that $v_{k_m} = \mu_m$ and v_{k_m} generates an irreducible submodule of V isomorphic to $V(\mu_m)$. In particular, there exists j such that $\mu_j = \mu_m$ and $\pi_j(v_{k_m}) \neq 0$. Up to reordering, we can assume $j = m$. The lemma now easily follows from the induction hypothesis applied to $\bar{V} := V/U(\mathfrak{g})v_{k_m}$ and the induced decomposition $\bar{V} = \bigoplus_{j=1}^{m-1} \bar{V}_j$, where \bar{V}_j is the image of V_j in \bar{V} . \square

Let $\mathbb{Z}[P]$ be the integral group ring over P and denote by $e : P \rightarrow \mathbb{Z}[P]$, $\lambda \mapsto e^\lambda$, the inclusion of P in $\mathbb{Z}[P]$, so that $e^\lambda e^\mu = e^{\lambda+\mu}$. Given a finite-dimensional \mathfrak{g} -module V , the character of V is defined to be

$$(2-1) \quad \text{ch}(V) = \sum_{\mu \in P} \dim(V_\mu)e^\mu.$$

Given a $U_q(\mathfrak{g})$ -module V and $\mu \in P$, let

$$V_\mu = \{v \in V : k_i v = q_i^{\mu(h_i)} v \text{ for all } i \in I\}.$$

A nonzero vector $v \in V_\mu$ is called a weight vector of weight μ . If v is a weight vector such that $x_i^+ v = 0$ for all $i \in I$, then v is called a highest-weight vector. If V is generated by a highest-weight vector of weight λ , then V is said to be a highest-weight module of highest weight λ . A $U_q(\mathfrak{g})$ -module V is said to be a weight module if $V = \bigoplus_{\mu \in P} V_\mu$. Denote by \mathcal{C}_q be the category of all finite-dimensional weight modules of $U_q(\mathfrak{g})$.

Remark. A $U_q(\mathfrak{g})$ -module V satisfying $V = \bigoplus_{\mu \in P} V_\mu$ is usually called a weight-module of type **1**. We shall not discuss what type **1** means here. For further details see [CP 1994a] for instance.

The character of an object $V \in \mathcal{C}_q$ is defined by (2-1). The following theorem is the quantum analogue of Theorem 2.2.

Theorem 2.4. *Let $V \in \mathcal{C}_q$.*

- (a) $\dim V_\mu = \dim V_{w\mu}$ for all $w \in \mathcal{W}$.
- (b) V is completely reducible.
- (c) For each $\lambda \in P^+$, the $U(\mathfrak{g})$ -module $V_q(\lambda)$ generated by a vector v satisfying

$$x_i^+ v = 0, \quad k_i v = q_i^{\lambda(h_i)} v, \quad (x_i^-)^{\lambda(h_i)+1} v = 0 \quad \text{for all } i \in I$$

is irreducible and finite-dimensional. If V is irreducible, then V is isomorphic to $V_q(\lambda)$ for some $\lambda \in P^+$.

- (d) *If $\lambda \in P^+$ and $V \cong V_q(\lambda)$, then $\text{ch}(V) = \text{ch}(V(\lambda))$. In particular, $V_q(\lambda)^*$ is isomorphic to $V_q(\lambda^*)$.*

If $J \subseteq I$, we shall denote by $V_q(\lambda_J)$ the $U_q(\mathfrak{g}_J)$ -irreducible module of highest weight λ_J . Similarly $V(\lambda_J)$ denotes the corresponding irreducible \mathfrak{g}_J -module.

2.5. Loop algebras. Let V be a $U_q(\tilde{\mathfrak{g}})$ -module. We say that a nonzero vector $v \in V$ is an ℓ -weight vector if there exists $\lambda \in \mathcal{P}_q$ and $k \in \mathbb{Z}_{>0}$ such that $(\eta - \Psi_\lambda(\eta))^k v = 0$ for all $\eta \in U_q(\tilde{\mathfrak{h}})$. In that case, we call λ the ℓ -weight of v . We say V is an ℓ -weight module if every vector of V is a linear combination of ℓ -weight vectors. In that case, let V_λ denote the subspace spanned by all ℓ -weight vectors of ℓ -weight λ . An ℓ -weight vector v is said to be a highest- ℓ -weight vector if $\eta v = \Psi_\lambda(\eta)v$ for every $\eta \in U_q(\tilde{\mathfrak{h}})$ and $x_{i,r}^+ v = 0$ for all $i \in I$ and all $r \in \mathbb{Z}$. We say V is a highest- ℓ -weight module if it is generated by a highest- ℓ -weight vector. The notion of lowest- ℓ -weight module is defined similarly. Denote by $\tilde{\mathcal{C}}_q$ the category of all finite-dimensional ℓ -weight modules of $U_q(\tilde{\mathfrak{g}})$. Clearly $\tilde{\mathcal{C}}_q$ is an abelian category.

Observe that if $V \in \tilde{\mathcal{C}}_q$, then $V \in \mathcal{C}_q$ and

$$(2-2) \quad V_\lambda = \bigoplus_{\lambda: \text{wt}(\lambda)=\lambda} V_\lambda.$$

Moreover, if V is a highest- ℓ -weight module of highest ℓ -weight λ , then

$$(2-3) \quad \dim(V_{\text{wt}(\lambda)}) = 1 \quad \text{and} \quad V_\mu \neq 0 \text{ implies } \mu \leq \text{wt}(\lambda).$$

Define the concepts of ℓ -weight vector and so on for $\tilde{\mathfrak{g}}$ in a similar way, and denote by $\tilde{\mathcal{C}}$ the category of all finite-dimensional $\tilde{\mathfrak{g}}$ -modules. The next proposition is easily established using (2-3).

Proposition 2.6. *If V is a highest- ℓ -weight module, then it has a unique proper submodule and hence a unique irreducible quotient.*

Definition 2.7. Let $\lambda \in \mathcal{P}_q^+$ and $\lambda = \text{wt}(\lambda)$. The Weyl module $W_q(\lambda)$ of highest ℓ -weight λ is the $U_q(\tilde{\mathfrak{g}})$ -module defined by the quotient of $U_q(\tilde{\mathfrak{g}})$ by the left ideal generated by the elements $x_{i,r}^+$, $(x_{i,r}^-)^{\lambda(h_i)+1}$, and $\eta - \Psi_\lambda(\eta)$ for every $i \in I$, $r \in \mathbb{Z}$, and $\eta \in U_q(\tilde{\mathfrak{h}})$. Denote by $V_q(\lambda)$ the irreducible quotient of $W_q(\lambda)$. For $\lambda \in \mathcal{P}^+$, the Weyl module $W(\lambda)$ of $\tilde{\mathfrak{g}}$ and its irreducible quotient $V(\lambda)$ are defined similarly.

The next theorem was proved in [CP 2001, Lemmas 4.6 and 4.7] for simply laced \mathfrak{g} and in [Chari 2001, Proposition 2.2] for \mathfrak{g} with lacing number $r^\vee = 2$. For the sake of completeness, a proof for \mathfrak{g} of type G_2 will appear in [Jakelić and Moura 2009b].

Theorem 2.8. *For each $\lambda \in \mathcal{P}_q^+$, the module $W_q(\lambda)$ is the universal finite-dimensional $U_q(\tilde{\mathfrak{g}})$ -module with highest ℓ -weight λ ; every simple object of $\tilde{\mathcal{C}}_q$ is highest- ℓ -weight. For every $\lambda \in \mathcal{P}^+$, the module $W(\lambda)$ is the universal finite-dimensional $\tilde{\mathfrak{g}}$ -module with highest ℓ -weight λ ; every simple object of $\tilde{\mathcal{C}}$ is highest- ℓ -weight.*

Remark. It is not true that the module $V_q(\lambda)$ belongs to $\tilde{\mathcal{C}}_q$ for every $\lambda \in \mathcal{P}_q^+$. This is so because $\mathbb{C}(q)$ is not algebraically closed. In fact, one can prove, using some results of Section 4.8 below, that $V_q(\lambda)$ is in $\tilde{\mathcal{C}}_q$ if and only if $\lambda_i(u)$ splits in $\mathbb{C}(q)[u]$ for every $i \in I$. Otherwise, $V_q(\lambda)$ is quasi- ℓ -weight in a sense analogous to that defined in [Jakelić and Moura 2009a] in the context of hyper loop algebras.

The following lemma is a consequence of the proof of Theorem 2.8.

Lemma 2.9. *Let V be a highest- ℓ -weight module of $\tilde{\mathfrak{g}}$ and v be a highest- ℓ -weight vector. Then $V = U(\mathfrak{g}[t])v$.*

If $J \subseteq I$, we shall denote by $V_q(\lambda_J)$ the $U_q(\tilde{\mathfrak{g}}_J)$ -irreducible module of highest ℓ -weight λ_J . Similarly $V(\lambda_J)$ denotes the corresponding irreducible $\tilde{\mathfrak{g}}_J$ -module. Similar notations for the Weyl modules are defined in the obvious way.

Proposition 2.10 [Frenkel and Mukhin 2001]. *For every $\lambda \in \mathcal{P}_q^+$, $V_q(\lambda)$ is a lowest- ℓ -weight module with lowest ℓ -weight $(\lambda^*)^{-1}$. In particular, $V_q(\lambda)^* \cong V_q(\lambda^*)$.*

2.11. Evaluation modules and Cartan involution. Given a \mathfrak{g} -module V , let $V(a)$ be the $\tilde{\mathfrak{g}}$ -module obtained by pulling back the evaluation map ev_a . Such modules are called evaluation modules. If $V = V(\lambda)$, we use the notation $V(\lambda, a)$ for the corresponding evaluation module.

Theorem 2.12 [Chari 1986; CP 1986; 2001]. *Let $\lambda \in \mathcal{P}^+$.*

- (a) *If $\lambda = \omega_{\lambda,a}$ for some $\lambda \in P^+$ and some $a \in \mathbb{C}^\times$, then $V(\lambda) \cong V(\lambda, a)$.*
- (b) *If $\lambda = \prod_j \omega_{\lambda_j, a_j}$ as in (1-11), then*

$$V(\lambda) \cong \bigoplus_j V(\lambda_j, a_j) \quad \text{and} \quad W(\lambda) \cong \bigoplus_j W(\omega_{\lambda_j, a_j}).$$

Corollary 2.13. *Every object in $\tilde{\mathcal{C}}$ is an ℓ -weight module.*

Assume \mathfrak{g} is of type A and consider the $\mathbb{C}(q)$ -algebra $U'_q(\mathfrak{g})$ given by generators x_i^\pm and $k_\mu^{\pm 1}$ with $i \in I$, $\mu \in P$, and the defining relations

$$k_\mu k_\mu^{-1} = k_\mu^{-1} k_\mu = 1, \quad k_\mu k_\nu = k_{\mu+\nu}, \quad k_\mu x_j^\pm k_\mu^{-1} = q^{\mu(h_j)} x_j^\pm,$$

$$[x_i^+, x_j^-] = \delta_{i,j} \frac{k_{\alpha_i} - k_{\alpha_i}^{-1}}{q - q^{-1}}, \quad \sum_{k=0}^{1-c_{ij}} (-1)^k (x_i^\pm)^{(1-c_{ij}-k)} x_j^\pm (x_i^\pm)^{(k)} = 0 \quad \text{if } i \neq j,$$

There is an obvious monomorphism of algebras $U_q(\mathfrak{g}) \rightarrow U'_q(\mathfrak{g})$ taking $k_i \mapsto k_{\alpha_i}$. A $U'_q(\mathfrak{g})$ -module is said to be a weight module if the generators k_ν for $\nu \in P$ act diagonally with eigenvalues of the form $q^{(\nu, \mu)}$ for some $\mu \in P$, where (\cdot, \cdot) is the bilinear form such that $(\alpha_i, \alpha_j) = c_{ij}$. It is not hard to see that restriction establishes an equivalence of categories from that of $U'_q(\mathfrak{g})$ -weight modules to \mathcal{C}_q . Henceforth we identify these two categories using this equivalence. The next proposition was proved in [Jimbo 1986, Section 2] and [CP 1994b, Proposition 3.4].

Proposition 2.14. *Let \mathfrak{g} be of type A . Then there exists an algebra homomorphism $\text{qev} : U_q(\tilde{\mathfrak{g}}) \rightarrow U'_q(\mathfrak{g})$ such that if $\lambda \in P^+$ and V is the pullback of $V_q(\lambda)$ by qev , then there exists $l(\lambda) \in \mathbb{Z}$ such that V is isomorphic to $V_q(\lambda)$, where*

$$\lambda = \prod_{i \in I} \omega_{i, a_i, \lambda(h_i)} \quad \text{with } a_1 = q^{l(\lambda)} \text{ and } \frac{a_{i+1}}{a_i} = q^{\lambda(h_i) + \lambda(h_{i+1}) + 1} \text{ for } i < n.$$

Given $a \in \mathbb{C}(q)^\times$, there exists a unique $\mathbb{C}(q)$ -algebra automorphism ϱ_a of $U_q(\tilde{\mathfrak{g}})$ such that ϱ_a is the identity on $U_q(\mathfrak{g})$ and $\varrho_a(x_{i,r}^\pm) = a^r x_{i,r}^\pm$. Let $\text{qev}_a = \text{qev} \circ \varrho_a$.

Denote by $V_q(\lambda, a)$ the pullback of $V_q(\lambda)$ by the evaluation map qev_a . It is easy to see that $V_q(\lambda, a) \cong V_q(\lambda)$ where

$$\lambda = \prod_{i \in I} \omega_{i, a_i, \lambda(h_i)} \quad \text{with } a_1 = aq^{l(\lambda)} \text{ and } \frac{a_{i+1}}{a_i} = q^{\lambda(h_i) + \lambda(h_{i+1}) + 1} \text{ for } i < n.$$

It turns out that, for \mathfrak{g} not of type A , there is no analogue of the map qev . In fact, it is known (see [Chari 2001] for instance) that there exists $i \in I$ and $m \in \mathbb{Z}_{\geq 0}$ such that the action of $U_q(\mathfrak{g})$ on $V_q(m\omega_i)$ cannot be extended to one of $U_q(\tilde{\mathfrak{g}})$.

One easily checks that there exists a unique algebra involution $\tilde{\sigma}$ of $U_q(\tilde{\mathfrak{g}})$ such that $\tilde{\sigma}(x_{i,r}^{\pm}) = x_{i,-r}^{\mp}$, $\tilde{\sigma}(k_i) = k_i^{-1}$, and $\tilde{\sigma}(h_{i,s}) = -h_{i,-s}$ for all $i \in I$, $r, s \in \mathbb{Z}$ and $s \neq 0$. The involution $\tilde{\sigma}$ is called the Cartan involution and it is also a coalgebra antiinvolution. The restriction of $\tilde{\sigma}$ to $U_q(\mathfrak{g})$ defines an involution σ of $U_q(\mathfrak{g})$ also called the Cartan involution. Given a $U_q(\tilde{\mathfrak{g}})$ -module V , let $V^{\tilde{\sigma}}$ be the pullback of V by $\tilde{\sigma}$. Similarly, V^{σ} will denote the pullback of a $U_q(\mathfrak{g})$ -module V by σ . It is not difficult to see that a highest- ℓ -weight vector of $V_q(\lambda)$ is a lowest- ℓ -weight vector of $V_q(\lambda)^{\tilde{\sigma}}$. Moreover, it follows from (1-3) that

$$\tilde{\sigma}(\Lambda_i^{\pm}(u)) = (\Lambda_i^{\mp}(u))^{-1}, \quad \text{where } \Lambda_i^{\pm}(u) = \sum_{r=0}^{\infty} \Lambda_{i, \pm r} u^r$$

and the inverse is that of formal power series in u . It is now not difficult to complete the proof of the next proposition.

Proposition 2.15. *Let $\lambda \in P^+$ and $\lambda \in \mathcal{P}_q^+$. Then*

$$V_q(\lambda)^{\sigma} \cong V_q(\lambda^*) \quad \text{and} \quad V_q(\lambda)^{\tilde{\sigma}} \cong V_q(*\lambda).$$

The analogous result in the classical case is established similarly.

We end this subsection with a remark. Let \mathfrak{g} be of type A , let $\lambda \in \mathcal{P}_q^+$ be such that $V_q(\lambda) \cong V_q(\lambda, a)$ for some $a \in \mathbb{C}(q)^\times$, and set $b_n = (aq^{l(\lambda) + n + 1})^{-1}$. Then

$$(2-4) \quad *\lambda = \prod_{i \in I} \omega_{i, b_i, \lambda^*(h_i)} \quad \text{with } \frac{b_i}{b_{i-1}} = q^{-(\lambda^*(h_i) + \lambda^*(h_{i-1}) + 1)} \text{ for } i > 1.$$

2.16. Classical limits.

Definition 2.17. Denote by $\mathcal{P}_{\mathbb{A}}^+$ the subset of \mathcal{P}_q consisting of n -tuples of polynomials with coefficients in \mathbb{A} . Let also $\mathcal{P}_{\mathbb{A}}^{++}$ be the subset of $\mathcal{P}_{\mathbb{A}}^+$ consisting of n -tuples of polynomials whose leading terms are in $\mathbb{C}q^{\mathbb{Z}} \setminus \{0\} = \mathbb{A}^\times$. Given $\lambda \in \mathcal{P}_{\mathbb{A}}^+$, let $\bar{\lambda}$ be the element of \mathcal{P}^+ obtained from λ by evaluating q at 1.

Recall that an \mathbb{A} -lattice (or form) of a $\mathbb{C}(q)$ -vector space V is a free \mathbb{A} -submodule L of V such that $\mathbb{C}(q) \otimes_{\mathbb{A}} L = V$. If V is a $U_q(\tilde{\mathfrak{g}})$ -module, a $U_{\mathbb{A}}(\tilde{\mathfrak{g}})$ -admissible

lattice of V is an \mathbb{A} -lattice of V that is also a $U_{\mathbb{A}}(\tilde{\mathfrak{g}})$ -submodule of V . Given a $U_{\mathbb{A}}(\tilde{\mathfrak{g}})$ -admissible lattice of a $U_q(\tilde{\mathfrak{g}})$ -module V , define

$$(2-5) \quad \bar{L} = \mathbb{C} \otimes_{\mathbb{A}} L,$$

where \mathbb{C} is regarded as an \mathbb{A} -module by letting q act as 1. Then \bar{L} is a $\tilde{\mathfrak{g}}$ -module by Proposition 1.4 and $\dim(\bar{L}) = \dim(V)$. The next theorem is essentially a corollary of the proof of Theorem 2.8.

Theorem 2.18. *Let V be a nontrivial quotient of $W_q(\lambda)$ for some $\lambda \in \mathcal{P}_{\mathbb{A}}^{++}$, let v be a highest- ℓ -weight vector of V , and let $L = U_{\mathbb{A}}(\tilde{\mathfrak{g}})v$. Then L is a $U_{\mathbb{A}}(\tilde{\mathfrak{g}})$ -admissible lattice of V and $\text{ch}(\bar{L}) = \text{ch}(V)$. In particular, \bar{L} is a quotient of $W(\bar{\lambda})$.*

Definition 2.19. Let $\lambda \in \mathcal{P}_{\mathbb{A}}^{++}$, let v be a highest- ℓ -weight vector of $V_q(\lambda)$, and let $L = U_{\mathbb{A}}(\tilde{\mathfrak{g}})v$. We denote by $\overline{V_q(\lambda)}$ the $\tilde{\mathfrak{g}}$ -module \bar{L} .

We shall also use the following straightforward lemma.

Lemma 2.20. *Suppose V and V' are $U_q(\tilde{\mathfrak{g}})$ -modules, and L and L' are $U_{\mathbb{A}}(\tilde{\mathfrak{g}})$ -submodules. Suppose $\phi : V \rightarrow V'$ is a $U_q(\tilde{\mathfrak{g}})$ -module map such that $\phi(L) \subseteq L'$. Then $\bar{\phi} := 1 \otimes \phi : \bar{L} \rightarrow \bar{L}'$ is a $\tilde{\mathfrak{g}}$ -module map.*

3. Minimal affinizations

3.1. Classification. Chari [1995] introduced the notion of minimal affinizations of an irreducible $U_q(\mathfrak{g})$ -module, as follows.

Given $\lambda \in P^+$, an object $V \in \tilde{\mathcal{C}}_q$ is said to be an affinization of $V_q(\lambda)$ if, as a $U_q(\mathfrak{g})$ -module, $V \cong V_q(\lambda) \oplus \bigoplus_{\mu < \lambda} V_q(\mu)^{\oplus m_{\mu}(V)}$ for some $m_{\mu}(V) \in \mathbb{Z}_{\geq 0}$. We say two affinizations of $V_q(\lambda)$ are equivalent if they are isomorphic as $U_q(\mathfrak{g})$ -modules. If $\lambda \in \mathcal{P}_q^+$ is such that $\text{wt}(\lambda) = \lambda$, then $V_q(\lambda)$ is clearly an affinization of $V_q(\lambda)$. The partial order on P^+ induces a natural partial order on the set of (equivalence classes of) affinizations of $V_q(\lambda)$. Namely, if V and W are affinizations of $V_q(\lambda)$, we say that $V \leq W$ if either $m_{\mu}(V) \leq m_{\mu}(W)$ for all $\mu \in P^+$ or if for all $\mu \in P^+$ such that $m_{\mu}(V) > m_{\mu}(W)$, there exists $\nu > \mu$ such that $m_{\nu}(V) < m_{\nu}(W)$. A minimal element of this partial order is said to be a minimal affinization.

Suppose \mathfrak{g} is not of types D or E . Given $\lambda \in \mathcal{P}_q^+$, set

$$\lambda^{\circ} = \lambda^* \quad \text{if } \mathfrak{g} = \mathfrak{sl}_{n+1} \text{ and } \lambda^{\circ} = *\lambda \text{ otherwise.}$$

Recall that, in these cases, $\lambda^* = \lambda$ for all $\lambda \in P^+$ except if \mathfrak{g} is of type A .

The following main result of [Chari 1995; CP 1996b; 1995] gives a partial classification of the highest ℓ -weights of the minimal affinizations, and gives the complete classification when \mathfrak{g} is not of types D or E .

Theorem 3.2. *Let $\lambda \in \mathcal{P}_q^+$, $\lambda = \text{wt}(\lambda)$, and $V = V_q(\lambda)$. Suppose \mathfrak{g} is not of types D or E . Then V is a minimal affinization of $V_q(\lambda)$ if and only if V^* and $V^{\tilde{\sigma}}$ are*

minimal affinizations of $V_q(\lambda^*)$. In that case, there exist $a \in \mathbb{C}(q)^\times$ and $\mu \in \{\lambda, \lambda^*\}$ such that either λ or λ° is equal to

$$\prod_{i=1}^n \omega_{i, a_i, \mu(h_i)} \quad \text{with } a_1 = a \quad \text{and} \quad \frac{a_{i+1}}{a_i} = q^{d_i \mu(h_i) + d_{i+1} \mu(h_{i+1}) + r_i^\vee}$$

for all $i \in I, i < n$, where $r_i^\vee = d_i - 1 - c_{i, i+1}$. Equivalently, V is a minimal affinization of $V_q(\lambda)$ if and only if there exist $a \in \mathbb{C}(q)^\times$ and $\epsilon \in \{1, -1\}$ such that

$$\lambda = \prod_{i=1}^n \omega_{i, a_i, \lambda(h_i)} \quad \text{with } a_1 = a \quad \text{and} \quad \frac{a_{i+1}}{a_i} = q^{\epsilon(d_i \lambda(h_i) + d_{i+1} \lambda(h_{i+1}) + r_i^\vee)}$$

for all $i \in I$ with $i < n$. If \mathfrak{g} is of type D or E , suppose the support of λ is contained in a connected subdiagram $J \subseteq I$ of type A . Then V is a minimal affinization of $V_q(\lambda)$ if and only if $V_q(\lambda_J)$ is a minimal affinization of $V_q(\lambda_J)$.

The next corollary is immediate (recall from Section 1.1 that $\overline{\text{supp}}(\lambda)$ is the minimal connected subdiagram of I containing $\text{supp}(\lambda)$).

Corollary 3.3. *If $\lambda \in P^+$ is such that $\overline{\text{supp}}(\lambda)$ does not contain a subdiagram of type D_4 , then $V_q(\lambda)$ has a unique equivalence class of minimal affinizations.*

Remark. We warn that the conditions we give in Theorem 3.2 do not match the ones given in [Chari 1995; CP 1996b; 1995], due to different normalizations in some definitions. Our notation follows more closely that of [Hernandez 2007], which is more uniform. We also note that $r_i^\vee = d_{i+1} - c_{i+1, i}$ and $r_i^\vee \in \{r^\vee - 1, r^\vee\}$ for all $i \in I$ with $i < n$. It is easy to check that $r_i^\vee = r^\vee$ for all $i < n$ if \mathfrak{g} is of types A, B , or G . If \mathfrak{g} is of type C , then $r_i^\vee = r^\vee - 1$ if and only if $i < n - 1$. Finally, if \mathfrak{g} is of type F , then $r_i^\vee = r^\vee$ if and only if α_i is a long root.

Corollary 3.4. *For every $a \in \mathbb{C}(q)^\times, i \in I$ and $m \in \mathbb{Z}_{\geq 0}$, the module $V_q(\omega_{i, a, m})$ is a minimal affinization of $V_q(m\omega_i)$.*

The modules $V_q(\omega_{i, a, m})$ are known as Kirillov–Reshetikhin modules.

In the cases not covered by Theorem 3.2, that is, when $\overline{\text{supp}}(\lambda)$ contains a subdiagram of type D_4 , then $V_q(\lambda)$ may have more than one equivalence class of minimal affinizations; see [CP 1996b; 1996a]. We shall briefly discuss these cases in Sections 5.11 and 5.18.

We now state a few results that were used in the proof of Theorem 3.2 and will be useful for us as well. The proofs can be found in [CP 1996b].

Lemma 3.5. *Suppose $\emptyset \neq J \subseteq I$ is a connected subdiagram of the Dynkin diagram of \mathfrak{g} . Let $V = V_q(\lambda)$, let v be a highest- ℓ -weight vector of V , and let $V_J = U_q(\mathfrak{g}_J)v$. Then $V_J \cong V_q(\lambda_J)$.*

Definition 3.6. Suppose \mathfrak{g} is not of type D or E . A connected subdiagram $J \subseteq I$ is said to be admissible if J is of type A . If \mathfrak{g} is of type D or E , let $i_0 \in I$ be the unique element connected to three other nodes. A connected subdiagram $J \subseteq I$ is said to be admissible if J is of type A and $J \setminus \{i_0\}$ is connected.

Proposition 3.7. Suppose $J \subseteq I$ is admissible and that $\lambda \in \mathcal{P}_q^+$ is such that $V_q(\lambda)$ is a minimal affinization of $V_q(\lambda)$, where $\lambda = \text{wt}(\lambda)$. Then $V_q(\lambda_J)$ is a minimal affinization of $V_q(\lambda_J)$.

Proposition 3.8. Let $\lambda \in P_q^+$ and $\lambda = \text{wt}(\lambda)$. If $V_q(\lambda)$ is a minimal affinization of $V_q(\lambda)$, then there exist $a_i \in \mathbb{C}(q)^\times$ for $i \in I$ such that $\lambda = \prod_{i \in I} \omega_{i, a_i, \lambda(h_i)}$ and $a_i/a_j \in q^{\mathbb{Z}}$ for all $i, j \in I$.

Proof. The existence of $a_i \in \mathbb{C}(q)^\times$ for $i \in I$ such that $\lambda = \prod_{i \in I} \omega_{i, a_i, \lambda(h_i)}$ follows from Proposition 3.7 and the classification of minimal affinizations for \mathfrak{sl}_2 . The condition $a_i/a_j \in q^{\mathbb{Z}}$ for all $i, j \in I$ can be proved from results of [Chari 2002] (see Section 4.1 below). Alternatively, the proposition is immediate from Theorems 3.2 and 5.12 in the cases covered by them. \square

Corollary 3.9. For every $\lambda \in P^+$, there exist $\lambda \in \mathcal{P}_{\mathbb{A}}^{++}$ such that $V_q(\lambda)$ is a minimal affinization of $V_q(\lambda)$.

3.10. Restricted limits.

Definition 3.11. Let V be a $\mathbb{Z}_{\geq 0}$ -graded vector space and denote its s -th graded piece by $V[s]$. A $\mathfrak{g}[t]$ -module V is said to be $\mathbb{Z}_{\geq 0}$ -graded if V is a $\mathbb{Z}_{\geq 0}$ -graded vector space and $x \otimes t^r v \in V[r + s]$ for every $v \in V[s]$, $x \in \mathfrak{g}$, and $r, s \in \mathbb{Z}_{\geq 0}$. A $\mathbb{Z}_{\geq 0}$ -graded $\mathfrak{g}[t]$ -module V satisfying $V[r] = 0$ for $r \gg 0$ is said to be a restricted $\mathfrak{g}[t]$ -module. If V is a $\mathbb{Z}_{\geq 0}$ -graded $\mathfrak{g}[t]$ -module, denote by $V(s)$ the quotient of V by its $\mathfrak{g}[t]$ -submodule $\bigoplus_{k > s} V[k]$.

The next lemma follows immediately from Proposition 3.8.

Lemma 3.12. Suppose $\lambda \in \mathcal{P}_{\mathbb{A}}^{++}$ is such that $V_q(\lambda)$ is a minimal affinization. Then $\bar{\lambda} = \omega_{\lambda, a}$ for some $a \in \mathbb{C}^\times$, where $\lambda = \text{wt}(\lambda)$.

Proposition 3.13. Suppose $\lambda \in \mathcal{P}_{\mathbb{A}}^{++}$ is such that $V_q(\lambda)$ is a minimal affinization and that $J \subseteq I$ is an admissible subdiagram. Let v be a highest- ℓ -weight vector of $V = \overline{V_q(\lambda)}$, let $\lambda = \text{wt}(\lambda)$, and let $a \in \mathbb{C}^\times$ be such that $\bar{\lambda} = \omega_{\lambda, a}$. Then $x_{\alpha, r}^- v = a^r x_{\alpha}^- v$ for every $\alpha \in R_J^+$.

Proof. Let J be admissible, let $\alpha \in R_J^+$, and let $V_J = U_q(\tilde{\mathfrak{g}}_J)v'$, where $v' \in V_q(\lambda)$ is such that $\bar{v}' = v$. Then V_J is a minimal affinization by Proposition 3.7 and, since J is of type A , V_J is irreducible as a $U_q(\mathfrak{g}_J)$ -module by Theorem 3.2. Hence, the $\tilde{\mathfrak{g}}_J$ -submodule of V generated by v is isomorphic to $V(\lambda_J, a)$. \square

Recall the definition of the maps $\tau_a : \mathfrak{g}[t] \rightarrow \mathfrak{g}[t]$ from Section 1.1.

Definition 3.14. Let $\lambda \in \mathcal{P}_{\mathbb{A}}^{+++}$, let $\lambda = \text{wt}(\lambda)$, and let $a \in \mathbb{C}^\times$ be such that $\bar{\lambda} = \omega_{\lambda,a}$. The $\mathfrak{g}[t]$ -module $L(\lambda)$ is defined to be the pullback of $\overline{V_q(\bar{\lambda})}$ by τ_a . Define also the module $A(\lambda)$ to be the $\mathfrak{g}[t]$ -module given by the quotient of $U(\mathfrak{g}[t])$ by the left ideal generated by

$$\mathfrak{n}^+[t], \quad \mathfrak{h} \otimes t\mathbb{C}[t], \quad h_i - \lambda(h_i), \quad (x_{\alpha_i}^-)^{\lambda(h_i)+1}, \quad x_{\alpha_i}^-$$

for all $i \in I$ and all $\alpha \in R_J^+$ for some admissible subdiagram $J \subseteq I$. Denote by v_λ the image of 1 in $A(\lambda)$, so that $A(\lambda) = U(\mathfrak{n}^-[t])v_\lambda$.

It immediately follows from [Theorem 2.12](#), [Proposition 3.13](#), and [Lemma 2.9](#) that $L(\lambda)$ is a quotient of $A(\lambda)$. It is also clear that $A(\lambda)$ is a $\mathbb{Z}_{\geq 0}$ -graded $\mathfrak{g}[t]$ -module. We call the module $L(\lambda)$ the restricted limit of $V_q(\lambda)$. It is immediate from [Theorem 2.18](#) that

$$(3-1) \quad \text{ch}(L(\lambda)) = \text{ch}(V_q(\lambda)).$$

When $\lambda = m\omega_i$ for some $m \in \mathbb{Z}_{\geq 0}$ and some $i \in I$, the modules $L(\omega_{i,a,m})$ are the restricted Kirillov–Reshetikhin modules of highest-weight $m\omega_i$. Chari and Moura studied these for \mathfrak{g} of classical type in [\[2006\]](#) and for \mathfrak{g} of type G_2 in [\[2007\]](#).

Proposition 3.15. *For every $\lambda \in P^+$, the module $A(\lambda)$ is finite-dimensional. In particular, $A(\lambda)$ is restricted.*

Proof. Since $A(\lambda) = U(\mathfrak{n}^-[t])v_\lambda$, it immediately follows that $(A(\lambda)[r])_\mu$ is finite-dimensional for every $r \in \mathbb{Z}_{\geq 0}$ and every $\mu \in P$. The relations $(x_{\alpha_i}^-)^{\lambda(h_i)+1}v_\lambda = 0$ for all $i \in I$ implies, as usual, that the elements $x_{\alpha_i}^\pm$ act locally nilpotently on $A(\lambda)$ and hence $\dim(A(\lambda)_\mu) = \dim(A(\lambda)_{w\mu})$ for every $\mu \in P$ and $w \in {}^q\mathcal{W}$. This in turn implies that $A(\lambda)_\mu \neq 0$ if and only if $w_0\lambda \leq \mu \leq \lambda$. Hence $A(\lambda)$ has only finitely-many nontrivial weight spaces. Using the defining relations of $A(\lambda)$ together with basic commutation relations in $\mathfrak{g}[t]$, it is trivial to see that $x_{\alpha,r}^-v_\lambda = 0$ for all $\alpha \in R^+$ if $r \gg 0$. This together with the PBW theorem then implies that $(A(\lambda)[s])_\mu = 0$ for every $\mu \in P$ if $s \gg 0$. In fact, let $r \in \mathbb{Z}_{\geq 0}$ be such that $x_{\alpha,s}^-v_\lambda = 0$ for all $\alpha \in R^+$ and all $s \geq r$. Fix a total order on $R^+ \times \mathbb{Z}_{\geq 0}$ such that $(\alpha, k) < (\beta, l)$ whenever $k < r$ and $l \geq r$. The PBW monomials for $U(\mathfrak{n}^-[t])$ are then formed such that $x_{\beta,l}^-$ occur to the right of $x_{\alpha,k}^-$ whenever $(\alpha, k) < (\beta, l)$. Hence, to get to the s -th graded piece of $A(\lambda)$ with $s \gg r$, one would have to apply elements of the form $x_{\alpha,k}^-$ with $k < r$ to v_λ “too many times”, and the maximal possible weight of $A(\lambda)[s]$ would fall out of the set of weights lying in between $w_0\lambda$ and λ . □

3.16. Relations for $L(\lambda)$. We now state our main results and conjectures.

Definition 3.17. Let $m \in \mathbb{Z}_{\geq 0}$ and $i \in I$. The $\mathfrak{g}[t]$ -module $M(m\omega_i)$ is the quotient of $U(\mathfrak{g}[t])$ by the left ideal generated by

$$\mathfrak{n}^+[t], \quad \mathfrak{h} \otimes t\mathbb{C}[t], \quad h_j, \quad h_i - m, \quad x_{\alpha_j}^-, \quad (x_{\alpha_i}^-)^{m+1}, \quad x_{\alpha_i,1}^- \quad \text{for all } j \neq i.$$

Quite clearly $M(m\omega_i)$ is a $\mathbb{Z}_{\geq 0}$ -graded $\mathfrak{g}[t]$ -module, and $A(m\omega_i)$ is a quotient of $M(m\omega_i)$. The next proposition follows from [Chari 2001; CM 2006; 2007].

Proposition 3.18. *Suppose \mathfrak{g} is not of type E or F . Let $i \in I, m \in \mathbb{Z}_{\geq 0}$, and $a \in \mathbb{C}^\times$.*

- (a) *There exists $b_i \in \{1, 2, 3\}$ such that, if $m = m_1 b_i + m_0$ with $0 \leq m_0 < b_i$ and $T(m\omega_i)$ is the $\mathfrak{g}[t]$ -submodule of $M(b_i\omega_i)^{\otimes m_1} \otimes M(m_0\omega_i)$ generated by the top weight space, then $M(m\omega_i) \cong T(m\omega_i)$.*
- (b) *$M(m\omega_i) \cong A(m\omega_i) \cong L(\omega_{i,a,m})$.*

Our goal is to generalize the proposition above for minimal affinizations other than Kirillov–Reshetikhin modules. Let us introduce notation, as follows. Given $i \in I$ and $m, r \in \mathbb{Z}_{\geq 0}$, let $v_{i,m}$ be the image of 1 in $M(m\omega_i)$ and set

$$R^+(i, m, r) = \{\alpha \in R^+ : x_{\alpha,r}^- v_{i,m} = 0\}.$$

Since $(\mathfrak{h} \otimes t\mathbb{C}[t])v_{i,m} = 0$, it follows that

$$R^+(i, m, r) \subseteq R^+(i, m, s) \quad \text{for all } s \geq r.$$

The sets $R^+(i, m, r)$ for \mathfrak{g} not of types E and F were explicitly described in [Chari 2001; CM 2006; 2007]. We will eventually write them down precisely. For the moment, let us just observe that $R^+(i, m, r) = R^+$ if $r \gg 0$ since $A(m\omega_i)$ is restricted. In fact, if \mathfrak{g} is of classical type, then $R(i, m, 2) = R^+$ for every $i \in I$ and $m \in \mathbb{Z}_{\geq 0}$. Observe also that $R^+(i, 0, 0) = R^+$ for all $i \in I$ since $L(0)$ is the trivial representation. Now, given $\lambda \in P^+$ and $r \in \mathbb{Z}_{\geq 0}$, set

$$R^+(\lambda, r) = \bigcap_{i \in I} R^+(i, \lambda(h_i), r).$$

Since $R^+(j, 0, s) = R^+$ for all $j \in I$ and $s \in \mathbb{Z}_{\geq 0}$, it follows that $R^+(m\omega_i, r) = R^+(i, m, r)$ for all $i \in I$ and $m, r \in \mathbb{Z}_{\geq 0}$ and that

$$R^+(\lambda, r) = R^+ \quad \text{if } r \gg 0.$$

Definition 3.19. Given $\lambda \in P^+$, let $M(\lambda)$ be the $\mathfrak{g}[t]$ -module given by the quotient of $U(\mathfrak{g}[t])$ by the left ideal generated by

$$(3-2) \quad n^+[t], \quad \mathfrak{h} \otimes t\mathbb{C}[t], \quad h_i - \lambda(h_i), \quad (x_{\alpha_i}^-)^{\lambda(h_i)+1}, \quad x_{\alpha,r}^-$$

for all $i \in I, r \in \mathbb{Z}_{\geq 0}$, and $\alpha \in R^+(\lambda, r)$. Let $T(\lambda)$ be the $\mathfrak{g}[t]$ -submodule of $\bigoplus_{i \in I} M(\lambda(h_i)\omega_i)$ generated by the top weight space.

Definitions 3.17 and 3.19 of $M(m\omega_i)$ coincide since $R^+(m\omega_i, r) = R^+(i, m, r)$ for all $i \in I$ and $m, r \in \mathbb{Z}_{\geq 0}$. The modules $M(\lambda)$ are clearly $\mathbb{Z}_{\geq 0}$ -graded. It follows from Proposition 3.13 that $M(\lambda)$ is a quotient of $A(\lambda)$ and hence a restricted $\mathfrak{g}[t]$ -module. Also, $T(\lambda)$ is clearly a restricted quotient of $M(\lambda)$ by Proposition 3.18.

The following is what we expect to be the generalization of [Proposition 3.18](#) when \mathfrak{g} and λ are as in [Theorem 3.2](#).

Conjecture 3.20. *Let $\lambda \in P^+$ be such that $\overline{\text{supp}}(\lambda)$ does not contain a subdiagram of type D_4 , and suppose $\lambda \in \mathcal{P}_{\mathbb{A}}^{++}$ is such that $V_q(\lambda)$ is a minimal affinization of $V_q(\lambda)$. Then, $T(\lambda) \cong M(\lambda) \cong L(\lambda)$.*

[Proposition 3.18](#) says the conjecture holds when λ is a multiple of a fundamental weight and \mathfrak{g} is not of type E or F . It is quite simple to see that the conjecture also holds when \mathfrak{g} is of type A for general $\lambda \in P^+$. We now state our main partial results in the direction of proving [Conjecture 3.20](#).

Proposition 3.21. *Let $\lambda \in \mathcal{P}_{\mathbb{A}}^{++}$ be such that $V_q(\lambda)$ is a minimal affinization of $V_q(\lambda)$, where $\lambda = \text{wt}(\lambda)$. Then, $T(\lambda)$ is a quotient of $L(\lambda)$.*

Proposition 3.22. *Let $\lambda \in P^+$ be such that $\overline{\text{supp}}(\lambda)$ does not contain a subdiagram of type D_4 , and suppose \mathfrak{g} is orthogonal. Then, $L(\lambda)$ is a quotient of $M(\lambda)$.*

Corollary 3.23. *In the conditions of [Proposition 3.22](#), the first isomorphism of [Conjecture 3.20](#) implies the second.*

Proposition 3.24. *[Conjecture 3.20](#) holds in the following cases:*

- (a) \mathfrak{g} is of type B and $\text{supp}(\lambda) \subseteq \{1, 2, 3, n\}$ with $\lambda(h_n) \leq 1$ if $n > 3$.
- (b) \mathfrak{g} is of type D and $\text{supp}(\lambda) \subseteq (\{1, 2, 3\} \cap J) \cup \{m\}$ with $m \in \{n - 1, n\}$. Here $J = I \setminus \{n - 1, n\}$.
- (c) \mathfrak{g} is of type D and $\text{supp}(\lambda) \subseteq \{n - 2, n - 1, n\}$.

In proving [Proposition 3.24](#), we obtain character formulas for $M(\lambda)$. The proofs of [Propositions 3.21](#) and [3.22](#) are given in [Sections 4.1](#) and [4.15](#), respectively. [Proposition 3.24](#) is proved in [Sections 5.3](#) and [5.8](#).

Remark. If \mathfrak{g} is of classical type, then $R^+(\lambda, 2) = R^+$ for every $\lambda \in P^+$ since $R(i, m, 2) = R^+$ for every $i \in I$ and $m \in \mathbb{Z}_{\geq 0}$, as mentioned previously. This implies that the modules $M(\lambda)$ can be regarded as modules for the truncated algebra $\mathfrak{g}[t]/(\mathfrak{g} \otimes t^2\mathbb{C}[t])$ in this case. This was what motivated [Chari and Greenstein \[2009\]](#) to study the relations between the finite-dimensional representation theory of $U_q(\tilde{\mathfrak{g}})$ and Koszul algebras. We shall leave the discussion of how our methods are related to theirs to a forthcoming publication.

4. Tensor products

4.1. Tensor products of Kirillov–Reshetikhin modules. The goal of this section is to prove [Proposition 3.21](#). The following fact is easily established from [\(1-7\)](#).

Proposition 4.2. *Let $\lambda, \mu \in \mathcal{P}_q^+$. Then, the $U_q(\tilde{\mathfrak{g}})$ -submodule of $V_q(\lambda) \otimes V_q(\mu)$ generated by the top weight space is a quotient of $W_q(\lambda, \mu)$.*

The following proposition follows from the results of [Chari 2002].

Proposition 4.3. *Let $l \in \mathbb{Z}_{\geq 1}$, $i_j \in I$, $m_j \in \mathbb{Z}_{\geq 1}$, and $a_j \in \mathbb{C}(q)^\times$ for $j = 1, \dots, l$. If $a_j/a_k \notin q^{\mathbb{Z}_{>0}}$ for $j > k$, then $V_q(\omega_{i_1, a_1, m_1}) \otimes \cdots \otimes V_q(\omega_{i_l, a_l, m_l})$ is a highest- l -weight module.*

Corollary 4.4. *Let $\lambda \in P^+$, $a_i \in \mathbb{C}(q)^\times$, $i \in I$, and $\lambda = \prod_{i \in I} \omega_{i, a_i, \lambda(h_i)}$. Then there is an ordering i_1, \dots, i_n of I such that $V_q(\lambda)$ is isomorphic to the $U_q(\tilde{\mathfrak{g}})$ -submodule of $V_q(\omega_{i_1, a_{i_1}, \lambda(h_{i_1})}) \otimes \cdots \otimes V_q(\omega_{i_n, a_{i_n}, \lambda(h_{i_n})})$ generated by the top weight space.*

Proof. Let $\omega \in \mathcal{P}_q^+$ be such that $\omega^* = \lambda$, and write $\omega_i(u) = \omega_{i, b_i, \lambda^*(h_i)}$ for some $b_i \in \mathcal{P}_q^+$. Let also $i' = w_0 \cdot i$ for all $i \in I$. It follows from Proposition 4.3 that there is an ordering i_1, \dots, i_n of I such that

$$V := V_q(\omega_{i'_n, b_{i'_n}, \lambda^*(h_{i'_n})}) \otimes \cdots \otimes V_q(\omega_{i'_1, b_{i'_1}, \lambda^*(h_{i'_1})})$$

is highest- l -weight. Let W be the proper maximal submodule of V . Thus, we have a short exact sequence $0 \rightarrow W \rightarrow V \rightarrow V_q(\omega) \rightarrow 0$. Then, by Propositions 2.10 and 1.7, we also have the short exact sequence

$$0 \rightarrow V_q(\lambda) \rightarrow V_q(\omega_{i_1, a_{i_1}, \lambda(h_{i_1})}) \otimes \cdots \otimes V_q(\omega_{i_n, a_{i_n}, \lambda(h_{i_n})}) \rightarrow W^* \rightarrow 0,$$

since $V^* \cong V_q(\omega_{i_1, a_{i_1}, \lambda(h_{i_1})}) \otimes \cdots \otimes V_q(\omega_{i_n, a_{i_n}, \lambda(h_{i_n})})$ and $V_q(\omega)^* \cong V_q(\lambda)$. □

Proof of Proposition 3.21. Given $i \in I$, let $a_i \in \mathbb{A}^\times$ be such that $\lambda = \prod_{i \in I} \omega_{i, a_i, \lambda(h_i)}$ and let v_i be a highest- l -weight vector of $V(\omega_{i, a_i, \lambda(h_i)})$. Let also i_1, \dots, i_n be an ordering of I as in Corollary 4.4, and let

$$v = v_{i_1} \otimes \cdots \otimes v_{i_n} \in V_q(\omega_{i_1, a_{i_1}, \lambda(h_{i_1})}) \otimes \cdots \otimes V_q(\omega_{i_n, a_{i_n}, \lambda(h_{i_n})}).$$

Consider $L_i = U_{\mathbb{A}}(\tilde{\mathfrak{g}})v_i$, $L = U_{\mathbb{A}}(\tilde{\mathfrak{g}})v$, and $L' = L_{i_1} \otimes \cdots \otimes L_{i_n}$. Let $a \in \mathbb{C}^\times$ be such that $\bar{\lambda} = \omega_{\lambda, a}$, and observe that $L(\lambda) \cong \tau_a^*(\bar{L})$ and $M(\lambda(h_i)\omega_i) \cong \tau_a^*(\bar{L}_i)$, where τ_a^*K denotes the pullback of a $\mathfrak{g}[t]$ -module K by τ_a . Moreover, it is easy to see that $L \subseteq L'$ and that $\bar{L}' \cong \bar{L}_{i_1} \otimes \cdots \otimes \bar{L}_{i_n}$.

Let $\bar{\phi}: \bar{L} \rightarrow \bar{L}'$ be the map given by Lemma 2.20 with ϕ being the inclusion

$$V_q(\lambda) \rightarrow V_q(\omega_{i_1, a_{i_1}, \lambda(h_{i_1})}) \otimes \cdots \otimes V_q(\omega_{i_n, a_{i_n}, \lambda(h_{i_n})}),$$

after identifying $V_q(\lambda)$ with $U_q(\tilde{\mathfrak{g}})v$. It follows that $\tau_a^*(\bar{\phi}): L(\lambda) \rightarrow M(\lambda(h_{i_1})\omega_{i_1}) \otimes \cdots \otimes M(\lambda(h_{i_n})\omega_{i_n})$ is a $\mathfrak{g}[t]$ -module map whose image is $T(\lambda)$. □

4.5. A smaller set of relations for $M(\lambda)$. In this subsection, \mathfrak{g} is orthogonal. Let

$$R_1^+ = \{\alpha \in R^+ : \alpha = \sum_{i \in I} n_i \alpha_i \text{ with } n_i \leq 1 \text{ for all } i \in I\}.$$

Proposition 4.6. *For every $\lambda \in P^+$, the module $M(\lambda)$ is isomorphic to the $\mathfrak{g}[t]$ -module $N(\lambda)$ generated by a vector v satisfying, for all $\alpha \in R_1^+$,*

$$h_i v = \lambda(h_i)v \quad \text{and} \quad n^+[t]v = \mathfrak{h} \otimes t\mathbb{C}[t]v = (x_{\alpha_i}^-)^{\lambda(h_i)+1}v = x_{\alpha_i,1}^- v = 0.$$

Proof. Since every admissible $J \subseteq I$ is of type A , it follows that $N(\lambda)$ is a quotient of $A(\lambda)$ and hence a finite-dimensional restricted $\mathfrak{g}[t]$ -module. Moreover, it is easy to see that $M(\lambda)$ is a quotient of $N(\lambda)$. For the converse, set $\alpha_{i,j} = \sum_{k=i}^j \alpha_k$ for all $i, j \in I$ and $i \leq j$ if \mathfrak{g} is of type B , and for all $i \leq j < n$ if \mathfrak{g} is of type D . If \mathfrak{g} is of type D , set also $\alpha_{i,n} = \alpha_{i,n-2} + \alpha_n$ for $i < n - 1$ or $i = n$ and $\vartheta_i = \alpha_{i,n-1} + \alpha_n$ for $i \leq n - 2$. Furthermore, given $i \leq j < n$ (or $j < n - 2$ if \mathfrak{g} is of type D) set

$$\theta_{i,j} = \begin{cases} \alpha_{i,n} + \alpha_{j+1,n} & \text{if } \mathfrak{g} \text{ is of type } B_n, \\ \alpha_{i,n-1} + \alpha_{j+1,n} & \text{if } \mathfrak{g} \text{ is of type } D_n. \end{cases}$$

Then $R_1^+ = \{\alpha_{i,j} : i, j \in I\}$ (or $R_1^+ = \{\alpha_{i,j} : i, j \in I\} \cup \{\vartheta_i : i \leq n - 2\}$ if \mathfrak{g} is of type D) and $R^+ = R_1^+ \cup \{\theta_{i,j} : i, j \in I\}$.

Denote by $v_{i,m}$ the image of 1 in $M(m\omega_i)$ for $i \in I$ and $m \in \mathbb{Z}_{\geq 0}$. Since $R^+(i, m, 0) = R^+$ if $m = 0$, we shall assume $m > 0$. Moreover, since we already know that Proposition 4.6 holds when λ is a multiple of a fundamental weight, we assume from now on that $\lambda \in P^+$ is not a multiple of a fundamental weight. From here we split the proof that $N(\lambda)$ is a quotient of $M(\lambda)$ in separate cases according to the type of \mathfrak{g} .

Case: \mathfrak{g} is of type B . It follows from [Chari 2001; CM 2006] that

$$\begin{aligned} R^+(i, m, 0) &= \{\alpha_{j,k} : i < j \text{ or } k < i\} \cup \{\theta_{j,k} : i < j\} \quad \text{and} \\ R^+(n, 1, 1) &= R^+(i, m, 2) = R^+ \quad \text{for all } i \in I \text{ and } m > 0 \end{aligned}$$

and

$$R^+(i, m, 1) = R^+(i, m, 0) \cup \{\alpha_{j,k} : j \leq i \leq k\} \cup \{\theta_{j,k} : i \leq k\} \quad \text{if } (i, m) \neq (n, 1).$$

Set

$$i_\lambda = \begin{cases} \min\{i : \lambda(h_j) = 0 \text{ for all } j > i\} & \text{if } \lambda(h_n) \neq 1, \\ \min\{i : \lambda(h_j) = 0 \text{ for all } i < j < n\} & \text{otherwise.} \end{cases}$$

It follows from the above that

$$(4-1) \quad R^+(\lambda, 1) = R^+(i_\lambda, \lambda(h_{i_\lambda}), 1) = R^+ \setminus \{\theta_{j,k} : k < i_\lambda\}.$$

Proposition 4.6 follows immediately in the case $\lambda(h_n) > 1$.

To complete the proof of Proposition 4.6, assume first that $\lambda(h_n) = 0$ and notice that $x_{\alpha_{i,j}}^- v = x_{\theta_{i,j}}^- v = 0$ if $i > i_\lambda$. It follows that

$$x_{\theta_{i,j},r}^- v = [x_{\alpha_{i,n},r}^-, x_{\alpha_{j,n}}^-]v = 0 \quad \text{for all } i, j \in I, j > i_\lambda \text{ and } r \in \mathbb{Z}_{>0}.$$

Also, if $r > 1$ and $j \leq i_\lambda$, then $x_{\theta_{i,j},r}^- v = [x_{\alpha_{i,n},r-1}^-, x_{\alpha_{j+1,n},1}^-] v = 0$. This completes the proof in this case.

If $\lambda(h_n) = 1$, then $x_{\alpha_{i,j}}^- v = 0$ if $i_\lambda < i \leq j < n$. Therefore, to conclude the proof, it suffices to show that $x_{\theta_{i,n-1},1}^- v = 0$ for all $i > i_\lambda$. We prove this inductively on $n - i$. In fact, it follows from the PBW theorem that

$$N(\lambda)[1] \subseteq \sum_{i \leq j < n} U(\mathfrak{g}) x_{\theta_{i,j},1}^- v.$$

In particular, the set of weights of $N(\lambda)[1]$ is contained in $S - Q^+$, where $S = \{\lambda - \theta_{i,j} : i \leq j < n\}$. It is easy to see that $\lambda - \theta_{n-1,n-1}$ is a maximal element of S . Hence, if $x_{\theta_{n-1,n-1},1}^- v$ were nonzero, $V(\lambda - \theta_{n-1,n-1})$ would be an irreducible constituent of $N(\lambda)[1]$. But the condition $\lambda(h_n) = 1$ implies $\lambda - \theta_{n-1,n-1} \notin P^+$. Since $N(\lambda)$ is finite-dimensional, the inductive argument starts. Now suppose $x_{\theta_{i,n-1},1}^- v = 0$ for all $i \geq j$ for some $j \leq n - 1$ and observe that $\lambda - \theta_{j-1,n-1}$ is a maximal element of $S \setminus \{\lambda - \theta_{i,n-1} : i \geq j\}$. Once more $\lambda - \theta_{j-1,n-1} \notin P^+$, and we conclude the inductive argument as before.

Case: \mathfrak{g} is of type D. In this case we have

$$\begin{aligned} R^+(i, m, 0) &= \{\alpha_{j,k} : i < j \text{ or } k < i\} \cup \{\vartheta_j, \theta_{j,k} : i < j\} \quad \text{if } i \neq n, n - 1, \\ R^+(i, m, 0) &= \{\alpha_{j,k} : k < n - 1 \text{ or } k = i'\} \quad \text{if } \{i, i'\} = \{n, n - 1\}, \\ R^+(1, m, 1) &= R^+(n - 1, m, 1) = R^+(n, m, 1) = R^+(i, m, 2) = R^+ \quad \text{for all } i \in I, \\ R^+(i, m, 1) &= R^+(i, m, 0) \cup \{\alpha_{j,k} : j \leq i \leq k\} \cup \{\theta_{j,k} : i \leq k\} \quad \text{if } i \notin \{1, n - 1, n\}. \end{aligned}$$

In particular,

$$(4-2) \quad R^+(\lambda, 1) = R^+ \quad \text{if } \lambda(h_i) = 0 \text{ for all } i \notin \{1, n - 1, n\},$$

and hence $M(\lambda)$ is an irreducible \mathfrak{g} -module. Set

$$i_\lambda = \begin{cases} 1 & \text{if } \lambda(h_i) = 0 \text{ for all } i \notin \{1, n - 1, n\}, \\ \min\{i : \lambda(h_j) = 0 \text{ for } i < j < n - 1\} & \text{otherwise.} \end{cases}$$

It follows that $R^+(\lambda, 1) = R^+(i_\lambda, \lambda(h_{i_\lambda}), 1) = R^+ \setminus \{\theta_{j,k} : k < i_\lambda\}$. We are left to show that $x_{\theta_{i,j},1}^- v = 0$ if $j > i_\lambda$. But this is clear since $x_{\alpha_{j,n-2}}^- v = 0$ if $i_\lambda < j$ and $x_{\theta_{i,j},1}^- = [x_{\alpha_{j,n-2}}^-, x_{\vartheta_{i,1}}^-]$. □

The following corollary is now immediate and proves the first isomorphism of [Conjecture 3.20](#) in some very particular cases.

Corollary 4.7. *Suppose $\lambda \in P^+$ is such that*

- (a) $\lambda(h_i) = 0$ for all $1 < i < n$ and $\lambda(h_n) \leq 1$ if \mathfrak{g} is of type B, and
- (b) $\lambda(h_i) = 0$ for all $i \notin \{1, n - 1, n\}$ if \mathfrak{g} is of type D.

Then, $M(\lambda)$ is irreducible as a \mathfrak{g} -module. In particular, $M(\lambda) \cong T(\lambda)$.

4.8. The ℓ -characters. Let $\mathbb{Z}[\mathcal{P}_q]$ be the integral group ring over \mathcal{P}_q . We define the ℓ -character of $V \in \widetilde{\mathcal{C}}_q$ to be the element $\text{ch}_\ell(V) = \sum_{\mu \in \mathcal{P}_q} \dim(V_\mu) \mu$, of $\mathbb{Z}[\mathcal{P}_q]$. The ℓ -characters are better known as q -characters, since this was the name used when they were first defined in [Frenkel and Reshetikhin 1999]. We prefer “ ℓ -characters” for two reasons: they record information about the dimension of the ℓ -weight spaces of V (which are not known as q -weight spaces), and the definition makes sense in the classical context as well. However, due to Theorem 2.12, the study of ℓ -characters in the classical case easily reduces to the study of characters, and so ℓ -characters are interesting only in the quantum case.

The proofs of the next four results can be found in [CM 2005; Frenkel and Mukhin 2001].

Proposition 4.9. *Let $\mathfrak{g} = \mathfrak{sl}_2$, $a \in \mathbb{C}(q)^\times$, and $r \in \mathbb{Z}_{\geq 0}$. Then*

$$\text{ch}_\ell(V_q(\omega_{i,a,r})) = \omega_{i,a,r} \sum_{k=0}^r \left(\prod_{j=1}^k \omega_{i,aq^{r-2j},2} \right)^{-1} = \omega_{i,a,r} \sum_{k=0}^r \left(\prod_{j=1}^k \alpha_{i,aq^{r-2j+1}} \right)^{-1}.$$

Theorem 4.10. *Let V be a quotient of $W_q(\lambda)$ for some $\lambda \in \mathcal{P}_q^+$. If $V_\mu \neq 0$, then $\mu \leq \lambda$.*

Proposition 4.11. *Let $V \in \widetilde{\mathcal{C}}_q$, $v \in V_\mu \setminus \{0\}$ for some $\mu \in \mathcal{P}_q$, and suppose $i \in I$ is such that $x_{i,r}^+ v = 0$ for all $r \in \mathbb{Z}$. Then, $\mu_i(u) \in \mathbb{C}(q)[u]$, and if $\mu_i(u) = \prod_{k=1}^m f_{i,a_k,r_k}(u)$ as in (1-14), then*

$$x_i^- v \in \sum_{k=1}^m \sum_{j=1}^{r_k} V_{\mu(\alpha_{i,a_k q_i^{r_k+1-2j}})^{-1}}.$$

Moreover, $\dim(V_{\mu(\alpha_{i,a_k q_i^{r_k-1}})^{-1}}) \geq \#\{1 \leq l \leq k : a_l = a_k\}$.

Given $V \in \widetilde{\mathcal{C}}_q$, let $\text{wt}_\ell(V) = \{\mu \in \mathcal{P}_q : V_\mu \neq 0\}$. A highest- ℓ -weight module V of highest ℓ -weight $\lambda \in \mathcal{P}_q^+$ is said to be special if $\text{wt}_\ell(V) \cap \mathcal{P}_q^+ = \{\lambda\}$.

Theorem 4.12. *If $\lambda \in \mathcal{P}_q^+$ is such that $V_q(\lambda)$ is special, then the output of the Frenkel–Mukhin algorithm with input λ is $\text{char}_\ell(V_q(\lambda))$.*

Theorem 4.13 [Hernandez 2007]. *If \mathfrak{g} is of type A , B , or G , then all minimal affinizations are special.*

Let \mathfrak{g} be of type A , B , or G , and let $\lambda \in P^+$. It follows from the above that if $V_q(\lambda)$ is a minimal affinization of $V_q(\lambda)$, then $\text{char}_\ell(V_q(\lambda))$ is given by the Frenkel–Mukhin algorithm. We will actually need only the following corollary of the algorithm. Let $V \in \widetilde{\mathcal{C}}_q$ and $v \in V_\mu \setminus \{0\}$ for some $\mu \in \mathcal{P}_q$, and suppose $i \in I$ is such that $x_{i,r}^+ v = 0$ for all $r \in \mathbb{Z}$. Using Proposition 4.11, we can write

$\mu_i(u) = \prod_{k=1}^m f_{i,a_k,r_k}(u)$ as in (1-14). Then, the algorithm implies that

$$(4-3) \quad \mu \alpha_{i,b}^{-1} \in \text{wt}_\ell(V_q(\lambda)) \quad \text{if and only if } b = a_k q_i^{r_k-1} \text{ for some } k = 1, \dots, m.$$

The next proposition will be crucial for the proof of Proposition 3.22.

Proposition 4.14. *Suppose \mathfrak{g} is of type A ; let $\lambda \in P^+$, $\lambda = \prod_{i \in I} \omega_{i,a_i,\lambda(h_i)}$, $\mu \in \text{wt}_\ell(V_q(\lambda))$, and $\lambda \mu^{-1} = \alpha_{j,b_j} \alpha_{j+1,b_{j+1}} \cdots \alpha_{k,b_k}$ for some $j \leq k$ and some $a_i, b_l \in \mathbb{C}(q)^\times$ and $i \in I, l = j, \dots, k$.*

- (a) *If $a_{i+1}/a_i = q^{\lambda(h_i)+\lambda(h_{i+1})+1}$ for all $i < n$, then $b_k = a_k q^{\lambda(h_k)-1}$.*
- (b) *If $a_{i+1}/a_i = q^{-(\lambda(h_i)+\lambda(h_{i+1})+1)}$ for all $i < n$, then $b_j = a_j q^{\lambda(h_j)-1}$.*

Proof. This is straightforward using induction on $k - j$ together with (4-3). □

4.15. Quantized relations.

Proof of Proposition 3.22. We now assume that \mathfrak{g} is orthogonal. To make the notation more uniform, we assume for the rest of the proof that \mathfrak{g} is of type B_n or D_{n+1} for $n \geq 2$. Before we begin, let us record the following corollary of Proposition 3.22 and Corollary 4.7.

Corollary 4.16. *If λ satisfies the conditions of Proposition 3.22 and Corollary 4.7, then Conjecture 3.20 holds. In particular, if $V_q(\lambda)$ is a minimal affinization of $V_q(\lambda)$, then $V_q(\lambda) \cong V_q(\lambda)$ as a $U_q(\mathfrak{g})$ -module.*

If λ is supported on an admissible subdiagram, Proposition 3.22 easily follows from Propositions 3.13 and 4.6. In particular, we can henceforth assume that the support of λ contains a spin node and that there exists $i < n$ such that $\lambda(h_i) \neq 0$. If \mathfrak{g} is of type D , we will prove Proposition 3.22 in the case $\lambda(h_{n+1}) = 0$; the other cases are proved similarly. Set

$$(4-4) \quad i_\lambda = \min\{i : \lambda(h_j) = 0 \text{ for all } i < j < n\}.$$

This definition of i_λ does not coincide with the one given in Section 4.5 for \mathfrak{g} of type B and $\lambda(h_n) > 1$.

From now on we assume that $\lambda \in \mathcal{P}_\mathbb{A}^{++}$ is such that $V = V_q(\lambda)$ is a minimal affinization of $V_q(\lambda)$ and $a \in \mathbb{C}$ is such that $\bar{\lambda} = \omega_{\lambda,a}$. We also fix a highest- ℓ -weight v vector of V and $a_i \in \mathbb{A}^\times$ for $i \in I$, such that $\lambda = \prod_{i \in I} \omega_{i,a_i,\lambda(h_i)}$.

Let v' be the image of v in $L(\lambda)$. It again follows from Proposition 3.13 that $x_{a_i,j,1}^- v' = 0$ if $i = j$ or if $j < n$. If \mathfrak{g} is of type B , this implies $x_{a_i,n,1}^- v' = 0$ if $i > i_\lambda$. If \mathfrak{g} is of type D and $i_\lambda < i < n$, it follows that $x_{a_i,n,1}^- v' = x_{a_{i,n+1},1}^- v' = x_{\vartheta_i,1}^- v' = 0$. We claim that it remains to show that $x_{a_i,n,1}^- v' = 0$ for $i \leq i_\lambda$. In fact, if \mathfrak{g} is of type B this is clear from Proposition 4.6. If \mathfrak{g} is of type D , it follows that $x_{\vartheta_i,1}^- v' = [x_{a_{n+1}}^-, x_{a_i,n,1}^-] v' = 0$ provided $x_{a_i,n,1}^- v' = 0$.

Denote by \bar{v} the image of v in \bar{V} . It suffices to show that $x_{\alpha_{i,n},1}^- \bar{v} = ax_{\alpha_{i,n}}^- \bar{v}$ for all $i \leq i_\lambda$. Consider the elements

$$X_{\alpha_{i,j},r}^- = [x_{j,r}^-, [x_{j-1}^-, \dots [x_{i+1}^-, x_i^-] \dots]] \quad \text{and} \quad k_{i,j} = \prod_{l=i}^j k_l$$

for $i \leq j \leq n$ and $r \in \mathbb{Z}_{\geq 0}$. Set $X_{\alpha_{i,j}}^- := X_{\alpha_{i,j},0}^-$, and note that $X_{\alpha_{i,j},r}^- = [x_{j,r}^-, X_{\alpha_{i,j-1}}^-]$ if $i < j$. Clearly

$$X_{\alpha_{i,j},r}^- \in U_{\mathbb{A}}(\tilde{\mathfrak{n}}^-) \quad \text{and} \quad \overline{X_{\alpha_{i,j},r}^-} = x_{\alpha_{i,j},r}^-.$$

Lemma 4.17. *Suppose $V \in \mathcal{C}_q$, $\mu \in P$, $w \in V_\mu \setminus \{0\}$, and $i \in I$ are such that $\mu(h_l) = 0$ and $x_l^+ w = 0$ for $l > i$. Then $X_{\alpha_{i,j}}^- w = x_j^- x_{j-1}^- \dots x_i^- w$.*

Proof. This is a straightforward computation using the commutation relations $[x_{l,s}^-, x_{l',s'}^-] = 0$ for l, l' such that $c_{l,l'} = 0$ and $x_l^- w = 0$ if $l > i$. □

Lemma 4.18. *Suppose w is a highest- ℓ -weight vector of $V_q(\omega_{i,\alpha_i,m})$ for some $i \in I$, and some $m \in \mathbb{Z}_{\geq 0}$. Then $x_{i,1}^- w = a_i q_i^m x_i^- w$.*

Proof. Use the relation $[h_{i,1}, x_i^-] = -[2]_{q_i} x_{i,1}^-$ and [Proposition 4.9](#). □

Let λ' be such that $\lambda = \lambda' \omega_{n,\alpha_n,\lambda(h_n)}$. Let also v_1 and v_2 be highest- ℓ -weight vectors of $V_q(\lambda')$ and $V_q(\omega_{n,\alpha_n,\lambda(h_n)})$. By [Proposition 4.3](#) and [Corollary 4.4](#), either

$$\begin{aligned} V_q(\lambda) &\cong U_q(\tilde{\mathfrak{g}})(v_1 \otimes v_2) \subseteq V_q(\lambda') \otimes V_q(\omega_{n,\alpha_n,\lambda(h_n)}) \quad \text{or} \\ V_q(\lambda) &\cong U_q(\tilde{\mathfrak{g}})(v_2 \otimes v_1) \subseteq V_q(\omega_{n,\alpha_n,\lambda(h_n)}) \otimes V_q(\lambda'). \end{aligned}$$

We assume we are in the first case (the latter is proved similarly using part (b) of [Proposition 4.14](#) instead of part (a)). In particular, by [Theorem 3.2](#), we must have

$$(4-5) \quad a_{i+1}/a_i = q^{d_i \lambda(h_i) + d_{i+1} \lambda(h_{i+1}) + r^\vee} \quad \text{for all } i < n.$$

By [Lemmas 1.5](#) and [1.6](#), modulo elements of the form $x(v_1 \otimes v_2)$ with $x \in U_{\mathbb{A}}(\tilde{\mathfrak{g}}) \otimes U_{\mathbb{A}}(\tilde{\mathfrak{g}})$ such that $\bar{x} = 0$, we have

$$\begin{aligned} X_{\alpha_{i,n}}^- (v_1 \otimes v_2) &= x_n^- X_{\alpha_{i,n-1}}^- (v_1 \otimes v_2) - X_{\alpha_{i,n-1}}^- x_n^- (v_1 \otimes v_2) \\ &= x_n^- ((X_{\alpha_{i,n}}^- v_1) \otimes v_2) - X_{\alpha_{i,n-1}}^- (v_1 \otimes (x_n^- v_2)) \\ &= (x_n^- X_{\alpha_{i,n-1}}^- v_1) \otimes (q^{-\lambda(h_n)} v_2) + (X_{\alpha_{i,n-1}}^- v_1) \otimes (x_n^- v_2) \\ &\quad - (X_{\alpha_{i,n-1}}^- v_1) \otimes (k_{i,n-1}^{-1} x_n^- v_2) - v_1 \otimes (X_{\alpha_{i,n-1}}^- x_n^- v_2) \\ &= q^{-\lambda(h_n)} (x_n^- X_{\alpha_{i,n-1}}^- v_1) \otimes v_2 + (1 - q^{-2}) (X_{\alpha_{i,n-1}}^- v_1) \otimes (x_n^- v_2) \\ &\quad - v_1 \otimes (X_{\alpha_{i,n-1}}^- x_n^- v_2). \end{aligned}$$

On the other hand,

$$\begin{aligned}
 X_{\alpha_i, n, 1}^-(v_1 \otimes v_2) &= x_{n, 1}^- X_{\alpha_i, n-1}^-(v_1 \otimes v_2) - X_{\alpha_i, n-1}^- x_{n, 1}^-(v_1 \otimes v_2) \\
 &= x_{n, 1}^- ((X_{\alpha_i, n}^- v_1) \otimes v_2) - X_{\alpha_i, n-1}^-(v_1 \otimes (x_{n, 1}^- v_2)) \\
 &= (x_{n, 1}^- X_{\alpha_i, n-1}^- v_1) \otimes (q^{\lambda(h_n)} v_2) + (X_{\alpha_i, n-1}^- v_1) \otimes (x_{n, 1}^- v_2) \\
 &\quad - (X_{\alpha_i, n-1}^- v_1) \otimes (k_{i, n-1}^{-1} x_{n, 1}^- v_2) - v_1 \otimes (X_{\alpha_i, n-1}^- x_{n, 1}^- v_2) \\
 &= q^{\lambda(h_n)} (x_{n, 1}^- X_{\alpha_i, n-1}^- v_1) \otimes v_2 + (1 - q^{-2}) (X_{\alpha_i, n-1}^- v_1) \otimes (x_{n, 1}^- v_2) \\
 &\quad - v_1 \otimes (X_{\alpha_i, n-1}^- x_{n, 1}^- v_2).
 \end{aligned}$$

Using Lemma 4.18 we get

$$\begin{aligned}
 X_{\alpha_i, n, 1}^-(v_1 \otimes v_2) &= q^{\lambda(h_n)} (x_{n, 1}^- X_{\alpha_i, n-1}^- v_1) \otimes v_2 \\
 &\quad + a_n q^{\lambda(h_n)} ((1 - q^{-2}) (X_{\alpha_i, n-1}^- v_1) \otimes (x_n^- v_2) - v_1 \otimes (X_{\alpha_i, n-1}^- x_n^- v_2)) \\
 &= a_n q^{\lambda(h_n)} X_{\alpha_i, n}^- (v_1 \otimes v_2) + q^{\lambda(h_n)} (x_{n, 1}^- X_{\alpha_i, n-1}^- v_1) \otimes v_2 - a_n (x_n^- X_{\alpha_i, n-1}^- v_1) \otimes v_2.
 \end{aligned}$$

Hence, it suffices to show that

$$(4-6) \quad q^{\lambda(h_n)} (x_{n, 1}^- X_{\alpha_i, n-1}^- v_1) \otimes v_2 = a_n (x_n^- X_{\alpha_i, n-1}^- v_1) \otimes v_2.$$

If $i > i_\lambda$, both sides of the above equality vanish. If $i \leq i_\lambda$, we proceed as follows. Notice that $x_{n, r}^+ X_{\alpha_i, n-1}^- v_1 = 0$ for all $r \in \mathbb{Z}$, and let W be the $U_q(\tilde{\mathfrak{g}}_n)$ -submodule of $V_q(\lambda')$ generated by $X_{\alpha_i, n-1}^- v_1$. Then, by Proposition 4.14(a), the highest- ℓ -weight of W is $\omega_{n, a_{n-1} q^{r^\vee \lambda(h_{n-1})}, r^\vee}$. Moreover, by (4-3), W is a minimal affinization. Hence, by Lemma 4.18,

$$x_{n, 1}^- X_{\alpha_i, n-1}^- v_1 = a_{n-1} q^{r^\vee (\lambda(h_{n-1})+1)} x_n^- X_{\alpha_i, n-1}^- v_1.$$

This and (4-5) imply (4-6). □

5. Graded characters of restricted limits of minimal affinizations

5.1. Preliminaries. Although Theorem 3.2 tells which objects of $\tilde{\mathcal{C}}_q$ correspond to minimal affinizations, it does not say anything about their $U_q(\mathfrak{g})$ -structure, unless \mathfrak{g} is of type A . In some few cases this is known; see [Chari 1995; CM 2006; 2007]. In principle, the $U_q(\mathfrak{g})$ -structure can be read off the ℓ -character. In practice, this is not so easy to do, even in the situations that the Frenkel–Mukhin algorithm does produced the ℓ -character. We will now apply the techniques of [CM 2006; 2007] to prove Proposition 3.24 and, hence, Conjecture 3.20 in those cases. As a byproduct, we obtain closed formulas for the character of the minimal affinizations if λ is as

in Proposition 3.24. We shall also prove an analogue of Conjecture 3.20 in the case of multiple minimal affinizations for \mathfrak{g} of type D_4 .

Lemma 5.2 [CM 2007, Section 1.5]. *Consider the three-dimensional Heisenberg algebra \mathfrak{H} spanned by elements x, y and z , where z is central and $[x, y] = z$. Suppose that V is a representation of \mathfrak{H} , and let $0 \neq v \in V$ be such that $x^r v = 0$. Then for all $k, s \in \mathbb{Z}_{\geq 0}$, the element $y^k z^s v$ is in the span of elements of the form $x^a y^b z^c v$ with $0 \leq c < r, a + c = s$, and $b + c = k + s$.*

Let V be a finite-dimensional \mathfrak{g} -module, and let $\lambda \in P^+$. Denote by $m_\lambda(V)$ the multiplicity of the irreducible module $V(\lambda)$ as an irreducible constituent of V . Set $d'_i = d_i/r^\vee$. Hence, if \mathfrak{g} is simply laced, $d'_i = d_i = 1$. If \mathfrak{g} is of type B , $d'_i = 1$ if $i < n$ and $d'_n = 1/2$. The symbol $[m]$ means the largest integer not greater than m .

5.3. Type B. Given $i \in I$ and $m \in \mathbb{Z}_{\geq 0}$, let $v_{i,m}$ be the image of 1 in $M(m\omega_i)$. The following was proved in.

Lemma 5.4 [Chari 2001; CM 2006]. (a) $M(m\omega_1) \cong V(m\omega_1)$.

$$(b) \ M(m\omega_2)[l] = \begin{cases} 0 & \text{if } l > [d'_2 m], \\ U(\mathfrak{n}^-)(x_{\theta_{1,1}}^-)^l v_{2,m} \cong V(m\omega_2 - l\theta_{1,1}) & \text{if } 0 \leq l \leq [d'_2 m]. \end{cases}$$

$$(c) \ M(m\omega_3)[l] = \begin{cases} 0 & \text{if } l > [d'_3 m], \\ U(\mathfrak{n}^-)(x_{\theta_{2,1}}^-)^l v_{3,m} \cong V(m\omega_3 - l\theta_{2,2}) & \text{if } 0 \leq l \leq [d'_3 m]. \end{cases}$$

Also, $(x_{\theta_{1,2,1}}^-)^{r_{1,2}}(x_{\theta_{2,2,1}}^-)^{r_{2,2}}v_{3,m}$ is a multiple of $(x_{\alpha_1}^-)^{r_{1,2}}(x_{\theta_{2,2,1}}^-)^{r_{1,2}+r_{2,2}}v_{3,m}$.

The “also” part of this lemma can also be proved using Lemma 5.2.

Proposition 5.5. *Suppose $\lambda \in P^+$ is such that $\lambda(h_i) = 0$ for $i > 2$. Then,*

$$M(\lambda)[k] \cong \begin{cases} V(\lambda - k\theta_{1,1}) & \text{if } 0 \leq k \leq [d'_2 \lambda(h_2)], \\ 0 & \text{otherwise.} \end{cases}$$

Also, $M(\lambda) \cong T(\lambda)$.

Proof. Let v be the image of 1 in $M(\lambda)$. Equation (4-1) gives $R^+(\lambda, 1) \supseteq R^+ \setminus \{\theta_{1,1}\}$. Together with the PBW Theorem, this implies that

$$M(\lambda) = \bigoplus_{k \geq 0} U(\mathfrak{n}^-)(x_{\theta_{1,1}}^-)^k v.$$

It follows from Lemma 2.3 that $m_\mu(M(\lambda)) \leq 1$ for every μ and that $m_\mu(M(\lambda))$ may be nonzero only when $\mu = \lambda - k\theta_{1,1}$ for some $k \in \mathbb{Z}_{\geq 0}$. Since $\theta_{1,1} = (d'_2)^{-1}\omega_2$, $\mu - k\theta_{1,1} \in P^+$ if and only if $k \leq [d'_2 \lambda(h_2)]$.

Let v_i be a nonzero element in the top weight space of $M(\lambda(h_i)\omega_i)$ for $i = 1, 2$. Then, by [CM 2006], $(x_{\theta_{1,1}}^-)^k v_2$ is the highest-weight vector of the irreducible \mathfrak{g} -submodule of $M(\lambda(h_2)\omega_2)[k]$ for $k = 0, \dots, [d'_2 \lambda(h_2)]$, while $M(\lambda(h_1)\omega_1)$ is an

irreducible \mathfrak{g} -module itself. Therefore, $(x_{\theta_{1,1},1}^-)^k(v_1 \otimes v_2) = v_1 \otimes (x_{\theta_{1,1},1}^-)^k v_2 \neq 0$, proving that $T(\lambda)[k] \neq 0$. Hence, $T(\lambda)[k] \cong M(\lambda)[k]$ and we are done. \square

Remark. This proposition reproves one of the main results of [Chari 1995] using a different method.

Now assume $n \geq 3$ and suppose $\lambda \in P^+$ is such that $\lambda(h_i) = 0$ for $i > 3$. In this case, Equation (4-1) implies

$$(5-1) \quad R^+(\lambda, 1) \supseteq R^+ \setminus \{\theta_{2,2}, \theta_{1,2}, \theta_{1,1}\}.$$

Observe that $\theta_{1,1} = \omega_2$, $\theta_{1,2} = \omega_1 - \omega_2 + (d'_3)^{-1}\omega_3$ and $\theta_{2,2} = (d'_3)^{-1}\omega_3 - \omega_1$. In particular, $\{\theta_{2,2}, \theta_{1,2}, \theta_{1,1}\}$ is a linearly independent subset of \mathfrak{h}^* . Let \mathbf{e}_j for $j \in \mathbb{Z}_{\geq 0}^3$ be the standard basis of $\mathbb{Z}_{\geq 0}^3$, set

$$(5-2) \quad \mathcal{A}_3(\lambda) = \{\mathbf{r} = (r_1, r_2, r_3) \in \mathbb{Z}_{\geq 0}^3 : r_3 \leq \lambda(h_2), r_2 \leq \lambda(h_1), r_1 + r_2 \leq [d'_3 \lambda(h_3)]\}$$

and, given $\mathbf{r} \in \mathbb{Z}_{\geq 0}^3$, define

$$(5-3) \quad \mathbf{y}_{\mathbf{r}} = (x_{\theta_{2,2},1}^-)^{r_1} (x_{\theta_{1,2},1}^-)^{r_2} (x_{\theta_{1,1},1}^-)^{r_3}.$$

Notice that the elements $x_{\theta_{2,2},1}^-$, $x_{\theta_{1,2},1}^-$, $x_{\theta_{1,1},1}^-$ commute among themselves.

Lemma 5.6. *Let v be the image of 1 in $M(\lambda)$. For every $\mathbf{s} \in \mathbb{Z}_{\geq 0}^3$,*

$$\mathbf{y}_{\mathbf{s}}v \in \sum_{\mathbf{r}} U(\mathfrak{n}^-)\mathbf{y}_{\mathbf{r}}v$$

where the sum is over the elements $\mathbf{r} \in \mathbb{Z}_{\geq 0}^3$ such that $r_3 \leq \lambda(h_2)$ and $r_2 \leq \lambda(h_1)$.

Proof. By Lemma 5.2 with $x = x_{\alpha_2}^-$, $y = x_{\theta_{1,2},1}^-$, and $z = x_{\theta_{1,1},1}^-$, we have that $\mathbf{y}_{\mathbf{s}}v$ is in the span of elements of the form $(x_{\alpha_2}^-)^a \mathbf{y}_{\mathbf{s}' }v$ with $a > 0$ and \mathbf{s}' with $s'_3 \leq \lambda(h_2)$. Using Lemma 5.2 once more, this time with $x = x_{\alpha_1}^-$, $y = x_{\theta_{2,2},1}^-$, and $z = x_{\theta_{1,2},1}^-$, it follows that an element $\mathbf{y}_{\mathbf{s}' }v$ with \mathbf{s}' as above belongs to the span of elements of the form $(x_{\alpha_1}^-)^a \mathbf{y}_{\mathbf{r}}v$ with $a > 0$ and \mathbf{r} as claimed. \square

Given $\mathbf{r} \in \mathbb{Z}_{\geq 0}^3$, define

$$(5-4) \quad \text{wt}(\mathbf{r}) = r_1\theta_{2,2} + r_2\theta_{1,2} + r_3\theta_{1,1} \quad \text{and} \quad \text{gr}(\mathbf{r}) = r_1 + r_2 + r_3.$$

Since $\{\theta_{2,2}, \theta_{1,2}, \theta_{1,1}\}$ is linearly independent, wt is an injective function.

Proposition 5.7. *For every $\lambda \in P^+$ as above, we have $M(\lambda) \cong T(\lambda)$ and*

$$M(\lambda)[l] \cong \bigoplus_{\mathbf{r} \in \mathcal{A}_3(\lambda): \text{gr}(\mathbf{r})=l} V(\lambda - \text{wt}(\mathbf{r})).$$

Proof. Let v be the image of 1 in $M(\lambda)$. Equation (5-1), together with the PBW theorem, implies that

$$M(\lambda) = \sum_{\mathbf{r} \in \mathbb{Z}_{\geq 0}^3} U(\mathfrak{n}^-) \mathbf{y}_{\mathbf{r}} v.$$

Lemma 5.6 implies that this sum can be restricted to \mathbf{r} such that $r_3 \leq \lambda(h_2)$ and $r_2 \leq \lambda(h_1)$. This, together with Lemma 2.3, implies that $m_{\mu}(M(\lambda)) \leq 1$ and equality may occur only if $\mu = \lambda - \text{wt}(\mathbf{r})$ for some \mathbf{r} as above. Also, $\text{wt}(\mathbf{r}) \in P^+$ only if $r_1 + r_2 \leq [d'_3 \lambda(h_3)]$, and hence \mathbf{r} must be in $\mathcal{A}_3(\lambda)$. It follows that $M(\lambda)[l]$ is a quotient of $\bigoplus_{\mathbf{r} \in \mathcal{A}_3(\lambda); \text{gr}(\mathbf{r})=l} V(\lambda - \text{wt}(\mathbf{r}))$. To complete the proof, it suffices to show that $T(\lambda)[l]$ contains a submodule isomorphic to $V(\lambda - \text{wt}(\mathbf{r}))$ for every $\mathbf{r} \in \mathcal{A}_3(\lambda)$ such that $\text{gr}(\mathbf{r}) = l$.

Thus, let $v_i = v_{i, \lambda(h_i)}$, $i = 1, 2, 3$, and let v_i^j be the image of v_i in $M(\lambda(h_i)\omega_i)(j)$ for $j \geq 0$. Then, if $\mathbf{r} \in \mathcal{A}_3(\lambda)$, Lemma 5.4 implies

$$\begin{aligned} \mathbf{y}_{\mathbf{r}}(v_1 \otimes v_2 \otimes v_3^{r_1+r_2}) &= v_1 \otimes (x_{\theta_{1,1,1}}^-)^{r_3} v_2 \otimes (x_{\theta_{2,2,1}}^-)^{r_1} (x_{\theta_{1,2,1}}^-)^{r_2} v_3^{r_1+r_2} \\ (5-5) \qquad \qquad \qquad &= v_1 \otimes (x_{\theta_{1,1,1}}^-)^{r_3} v_2 \otimes (x_{\alpha_1}^-)^{r_2} (x_{\theta_{2,2,1}}^-)^{r_1+r_2} v_3^{r_1+r_2} \neq 0. \end{aligned}$$

Given $r \leq \lambda(h_2)$ and $s \leq [d'_3 \lambda(h_3)]$, let $T_{r,s}(\lambda)$ be the $\mathfrak{g}[t]$ -submodule of

$$M(\lambda(h_1)\omega_1) \otimes M(\lambda(h_2)\omega_2)(r) \otimes M(\lambda(h_3)\omega_3)(s)$$

generated by $v_{r,s} := v_1 \otimes v_2^r \otimes v_3^s$. Clearly $T_{r,s}(\lambda)$ is a quotient of $T(\lambda)(r+s)$. Set $\mathbf{r}_0 = (s, 0, r)$ and $\mathbf{r}_j = \mathbf{r}_{j-1} + (\mathbf{e}_2 - \mathbf{e}_1)$ for $1 \leq j \leq s' := \min(\lambda(h_1), s)$. Notice that

$$T_{r,s}(\lambda)[r+s] = \sum_{j=0}^{s'} U(\mathfrak{n}^-) \mathbf{y}_{\mathbf{r}_j} v_{r,s} \quad \text{and} \quad (\lambda - \text{wt}(\mathbf{r}_j))(h_1) = \lambda(h_1) + s.$$

In particular, $\lambda - \text{wt}(\mathbf{r}_0)$ is the unique maximal weight of $T_{r,s}(\lambda)[r+s]$. We prove inductively on $k = 0, 1, \dots, s'$ that

$$\sum_{j=0}^k U(\mathfrak{n}^-) \mathbf{y}_{\mathbf{r}_j} v_{r,s} \cong \bigoplus_{j=0}^k V(\lambda - \text{wt}(\mathbf{r}_j))$$

as a \mathfrak{g} -module. Since every $\mathbf{r} \in \mathcal{A}_3(\lambda)$ is of the form \mathbf{r}_j for some r, s, j as above, this completes the proof.

It is clear from Lemma 5.4 and (5-5) that $\mathfrak{n}^+ \mathbf{y}_{\mathbf{r}_0} v_{r,s} = 0$ and hence generates a \mathfrak{g} -submodule isomorphic to $V(\lambda - \text{wt}(\mathbf{r}_0))$. In particular, we can assume $s' > 0$. Notice that the weight space of $V(\lambda - \text{wt}(\mathbf{r}_j))$ of weight $\lambda - \text{wt}(\mathbf{r}_j) - (k-j)\alpha_1$ is one-dimensional for $0 \leq j \leq k$. Using the induction hypothesis on k , we know that the weight space of $\sum_{j=0}^k U(\mathfrak{n}^-) \mathbf{y}_{\mathbf{r}_j} v_{r,s}$ of weight $\lambda - \text{wt}(\mathbf{r}_0) - (k+1)\alpha_1$ has dimension $k+1$. Since the elements $(x_{\alpha_1}^-)^j \mathbf{y}_{\mathbf{r}_{k+1-j}} v_{r,s}$ for $0 \leq j \leq k+1$ are

clearly linearly independent, it follows that $V(\lambda - \text{wt}(\mathbf{r}_{k+1}))$ is a submodule of $\sum_{j=0}^{k+1} U(\mathfrak{n}^-) \mathbf{y}_{r_j} v_{r,s}$. □

Remark. Suppose $n > 3$ and that $\lambda \in P^+$ is such that $\text{supp}(\lambda) \subseteq \{1, 2, 3, n\}$ with $\lambda(h_n) = 1$. Since $R^+(n, 1, 1) = R^+$, it follows that all of the above can be carried out and [Proposition 5.7](#) remains valid (notice $d'_3 = 1$ in this case).

5.8. Type D. Define the set $\mathcal{A}_3(\lambda)$ as in (5-2) and the maps wt and gr as in (5-4).

Proposition 5.9. *If $\lambda \in P^+$ is such that $\lambda(h_i) = 0$ if $3 < i < n-1$, then $M(\lambda) \cong T(\lambda)$.*

(a) *If $n = 4$, then*

$$M(\lambda)[l] \cong \begin{cases} V(\lambda - l\theta_{1,1}) & \text{if } 0 \leq l \leq \lambda(h_2), \\ 0 & \text{otherwise.} \end{cases}$$

(b) *If $n > 4$, then*

$$M(\lambda)[l] \cong \bigoplus_{\mathbf{r} \in \mathcal{A}_3(\lambda); \text{gr}(\mathbf{r})=l} V(\lambda - \text{wt}(\mathbf{r})).$$

Proof. The proof is essentially the same as that of [Proposition 5.7](#) using that $R^+(m\omega_i, 1) = R^+$ if i labels a spin node. □

In particular, [Proposition 5.9](#) gives the description of the graded characters of $M(\lambda)$ in types D_4 and D_5 for any $\lambda \in P^+$. If $\text{supp}(\lambda)$ contains at most one of the spin nodes, it follows that [Proposition 5.9](#) describes the character of the minimal affinizations of $V_q(\lambda)$ as well. Otherwise, it is just a lower bound.

Now, let $m \in \mathbb{Z}_{\geq 0}$ and set

$$\mathcal{A}(m) = \{\mathbf{r} = (r_1, r_2, \dots, r_{[(n-2)/2]}) \in \mathbb{Z}_{\geq 0}^{[(n-2)/2]} : m \geq r_1 \geq \dots \geq r_{[(n-2)/2]}\}.$$

Define

$$\text{wt}(\mathbf{r}) = \sum_{j=1}^{[(n-2)/2]} r_j \theta_{n-2j, n-2j} \quad \text{and} \quad \text{gr}(\mathbf{r}) = \sum_{j=1}^{[(n-2)/2]} r_j.$$

It was proved in [[Chari 2001](#); [CM 2006](#)] that

$$M(m\omega_{n-2})[l] = \sum_{\mathbf{r} \in \mathcal{A}(m); \text{gr}(\mathbf{r})=l} \mathbf{y}_{\mathbf{r}} v_{n-2,m} \cong \bigoplus_{\mathbf{r} \in \mathcal{A}(m); \text{gr}(\mathbf{r})=l} V(m\omega_{n-2} - \text{wt}(\mathbf{r})),$$

where

$$\mathbf{y}_{\mathbf{r}} = \prod_{j=1}^{[(n-2)/2]} (x_{\theta_{n-2j, n-2j, 1}}^-)^{r_j}.$$

Proceeding similarly to the proof of [Proposition 5.7](#), one also proves the following.

Proposition 5.10. *Let $\lambda \in P^+$ such that $\lambda(h_i) = 0$ if $i < n - 2$. Then $M(\lambda) \cong T(\lambda)$ and*

$$M(\lambda)[l] \cong \bigoplus_{\mathbf{r} \in \mathcal{A}(\lambda(h_{n-2})): \text{gr}(\mathbf{r})=l} V(\lambda - \text{wt}(\mathbf{r})).$$

5.11. Multiple minimal affinizations: The regular case. Let \mathfrak{g} be of types D or E and $i_0 \in I$ be the unique node triply connected. Let also $J_1, J_2, J_3 \subseteq I$ be an enumeration of the three maximal subdiagrams of type A of the Dynkin diagram of \mathfrak{g} (they are not admissible). Let also $J'_k = J_l \cap J_m$ for $\{k, l, m\} = \{1, 2, 3\}$. It follows from [CP 1996b, Theorem 6.1] that, if $\lambda(h_{i_0}) \neq 0$ and λ is supported on the three connected components of $I \setminus \{i_0\}$, then $V_q(\lambda)$ has exactly three equivalence classes of minimal affinizations. Moreover:

Theorem 5.12. *Let $\lambda \in \mathcal{P}_q^+$ be such that $\text{wt}(\lambda) = \lambda$, where λ is as above. Then $V_q(\lambda)$ is a minimal affinization of $V_q(\lambda)$ if and only if there exists $k \in \{1, 2, 3\}$ such that $V_q(\lambda_{J_l})$ is a minimal affinization of $V_q(\lambda_{J_l})$ for $l \neq k$.*

Definition 5.13. Given $\lambda \in P^+$ and $k \in \{1, 2, 3\}$, let $M_k(\lambda)$ be the quotient of $A(\lambda)$ by the submodule generated by the vectors $x_{\alpha, 1}^- v_\lambda$ for all $\alpha \in R_{J_l}^+$ with $l \neq k$. Suppose $\lambda \in \mathcal{P}_{\mathbb{A}}^{++}$ is such that $V_q(\lambda_{J_l})$ is a minimal affinization of $V_q(\lambda_{J_l})$ for $l \neq k$. Set $T_k(\lambda)$ to be the $\mathfrak{g}[t]$ -submodule of $M(\lambda^{J'_k}) \otimes L(\lambda^{I \setminus J'_k})$ generated by the top weight space.

It is easy to see that $M_k(\lambda)$ is a restricted $\mathfrak{g}[t]$ -module and that $M(\lambda)$ is a quotient of $M_k(\lambda)$ for all k . Also, proceeding similarly to the proofs of Propositions 3.21 and 3.22 we get the following analogue (we omit the details).

Proposition 5.14. *Let $\lambda \in \mathcal{P}_{\mathbb{A}}^{++}$ and $k \in \{1, 2, 3\}$ be such that $V_q(\lambda_{J_l})$ is a minimal affinization of $V_q(\lambda_{J_l})$ for $l \neq k$. Then there exist surjective $\mathfrak{g}[t]$ -module maps $M_k(\lambda) \twoheadrightarrow L(\lambda) \twoheadrightarrow T_k(\lambda)$.*

Conjecture 5.15. *Suppose $\lambda \in P^+$ is supported on the three connected components of $I \setminus \{i_0\}$. Then, $T_k(\lambda)$ and $M_k(\lambda)$ are isomorphic for every $k \in \{1, 2, 3\}$.*

Corollary 5.16. *Suppose $\lambda \in \mathcal{P}_{\mathbb{A}}^{++}$ and $k \in \{1, 2, 3\}$ are such that $V_q(\lambda_{J_l})$ is a minimal affinization of $V_q(\lambda_{J_l})$ for $l \neq k$ and $\text{wt}(\lambda)$ is supported on the three connected components of $I \setminus \{i_0\}$. Then, $T_k(\lambda) \cong L(\lambda) \cong M_k(\lambda)$.*

We now prove Conjecture 5.15 for \mathfrak{g} of type D_4 . Thus, let $\lambda \in P^+$ be such that $\lambda(h_i) \neq 0$ for all $i \neq 2$, and let λ be such that $V_q(\lambda)$ is a minimal affinization of $V_q(\lambda)$. Set also $J_1 = \{1, 2, 3\}$, $J_2 = \{1, 2, 4\}$, and $J_3 = \{2, 3, 4\}$. Without loss of generality we can assume that $V_q(\lambda_{J_1})$ and $V_q(\lambda_{J_2})$ are minimal affinizations. We want to show that $T_3(\lambda) \cong M_3(\lambda)$ in this case. We also assume that

$$\lambda = \omega_{1,a,\lambda(h_1)} \omega_{2,aq^{\lambda(h_1)+\lambda(h_2)+1},\lambda(h_2)} \omega_{3,aq^{\lambda(h_2)+\lambda(h_3)+1},\lambda(h_3)} \omega_{4,aq^{\lambda(h_2)+\lambda(h_4)+1},\lambda(h_4)}$$

for some $a \in \mathbb{C}^\times$. The case

$$\lambda = \omega_{1,a,\lambda(h_1)} \omega_{2,aq^{-(\lambda(h_1)+\lambda(h_2)+1)},\lambda(h_2)} \omega_{3,aq^{-(\lambda(h_2)+\lambda(h_3)+1)},\lambda(h_3)} \omega_{4,aq^{-(\lambda(h_2)+\lambda(h_4)+1)},\lambda(h_4)}$$

is proved similarly. If $\lambda(h_2) \neq 0$, these two cases cover all minimal affinizations such that $V_q(\lambda_{J_1})$ and $V_q(\lambda_{J_2})$ are also minimal affinizations. Otherwise, there are two more possibilities for λ (see the closing remark of Section 5.18).

Let v be the image of 1 in $M_3(\lambda)$. By the definition of $M_3(\lambda)$, we have the relations

$$(5-6) \quad x_{\alpha_i,1}^- v = x_{\alpha_2+\alpha_j,1}^- v = x_{\alpha_1+\alpha_2+\alpha_3,1}^- v = x_{\alpha_1+\alpha_2+\alpha_4,1}^- v = 0$$

for all $i, j \in I$ and $j \neq 2$. Using the commutation relations $[x_\alpha^-, x_\beta^-] = x_{\alpha+\beta}^-$ (up to multiple) we also get

$$(5-7) \quad x_{\alpha,2}^- v = 0 \quad \text{for all } \alpha \in R^+.$$

Let $\vartheta_1 = \sum_{i=1}^4 \alpha_i = \omega_1 + \omega_3 + \omega_4 - \omega_2$, $\vartheta_2 = \vartheta_1 - \alpha_1 = \omega_3 + \omega_4 - \omega_1$, and $\theta = \theta_{1,1} = \vartheta_1 + \alpha_2 = \omega_2$. It follows that

$$M_3(\lambda) = \sum_{\mathbf{r} \in \mathbb{Z}_{\geq 0}^3} U(\mathfrak{n}^-) \mathbf{y}_{\mathbf{r}} v, \quad \text{where } \mathbf{y}_{\mathbf{r}} = (x_{\theta,1}^-)^{r_3} (x_{\vartheta_2,1}^-)^{r_2} (x_{\vartheta_1,1}^-)^{r_1}.$$

Since $\{\vartheta_1, \vartheta_2, \theta\}$ is a linearly independent subset of \mathfrak{h}^* , it follows as before that $m_\mu(M_3(\lambda)) \leq 1$ for every $\mu \in P^+$, with equality only if $\mu = \lambda - r_1\vartheta_1 - r_2\vartheta_2 - r_3\theta$ for some $r_j \in \mathbb{Z}_{\geq 0}$. But such elements are dominant if and only if

$$r_1 \leq \lambda(h_1) + r_2, \quad r_3 \leq \lambda(h_2) + r_1, \quad r_1 + r_2 \leq \min\{\lambda(h_3), \lambda(h_4)\}.$$

Set

$$(5-8) \quad \mathfrak{D}_3(\lambda) = \{\mathbf{r} \in \mathbb{Z}_{\geq 0}^3 : r_1 \leq \lambda(h_1), r_3 \leq \lambda(h_2), r_1 + r_2 \leq \min\{\lambda(h_3), \lambda(h_4)\}\}.$$

Proceeding similarly to the proof of Lemma 5.6, one proves that

$$M_3(\lambda) = \sum_{\mathbf{r} \in \mathfrak{D}_3(\lambda)} U(\mathfrak{n}^-) \mathbf{y}_{\mathbf{r}} v.$$

Given $\mathbf{r} \in \mathbb{Z}_{\geq 0}^3$, define

$$(5-9) \quad \text{wt}(\mathbf{r}) = r_1\vartheta_1 + r_2\vartheta_2 + r_3\theta \quad \text{and} \quad \text{gr}(\mathbf{r}) = r_1 + r_2 + r_3.$$

Since $\{\vartheta_1, \vartheta_2, \theta\}$ is linearly independent, it follows that wt is an injective function. To complete the proof of Conjecture 5.15 in this case, it suffices to prove that $m_\mu(T_3(\lambda)) \geq 1$ if $\mu = \lambda - \text{wt}(\mathbf{r})$ for some $\mathbf{r} \in \mathfrak{D}_3(\lambda)$. In particular, it will follow

that

$$(5-10) \quad M_3(\lambda)[l] = \bigoplus_{r \in \mathcal{D}_3(\lambda): \text{gr}(r)=l} V(\lambda - \text{wt}(r)).$$

Proposition 5.17. *Let*

$$b \in \mathbb{A}^\times, \quad \mu = m_3\omega_3 + m_4\omega_4 \in P^+, \quad \boldsymbol{\mu} = \omega_{3,b,m_3}\omega_{4,bq^{m_4-m_3},m_4}.$$

Then, $L(\boldsymbol{\mu})[l] \cong V(\mu - l\vartheta_2)$ for $0 \leq l \leq \min\{m_3, m_4\}$ and $L(\boldsymbol{\mu})[l] = 0$ otherwise.

Proof. Let v be a highest-weight vector of $L(\boldsymbol{\mu})$. Clearly v satisfies relations (5-6) and (5-7). Moreover, proceeding as above, we get

$$L(\boldsymbol{\mu}) = \sum_{r \in \mathcal{D}_3(\boldsymbol{\mu})} U(\mathfrak{g})\mathbf{y}_r v = \bigoplus_{r=0}^{\min\{m_3, m_4\}} U(\mathfrak{n}^-)(x_{\vartheta_2,1}^-)^r v$$

and, by Lemma 5.2 once more,

$$(5-11) \quad (x_{\vartheta_1,1}^-)^{r_1} (x_{\vartheta_2,1}^-)^{r_2} v = (x_{\alpha_1}^-)^{r_1} (x_{\vartheta_2,1}^-)^{r_1+r_2} v.$$

Without loss of generality, assume $m_4 \geq m_3 \geq 1$ and observe that

$$\boldsymbol{\mu} = \left(\prod_{j=0}^{m_3-1} \omega_{\omega_3+\omega_4, bq^{1-m_3+2j}} \right) \left(\prod_{j=0}^{m_4-m_3-1} \omega_{4, bq^{m_3+1+2j}} \right).$$

Then by Proposition 4.3 and its corollary, $V_q(\boldsymbol{\mu})$ is the $U_q(\tilde{\mathfrak{g}})$ -submodule of

$$\begin{aligned} (V_q(\omega_{\omega_3+\omega_4, bq^{1-m_3}}) \otimes \cdots \otimes V_q(\omega_{\omega_3+\omega_4, bq^{m_3-1}})) \\ \otimes (V_q(\omega_{4, bq^{m_3+1}}) \otimes \cdots \otimes V_q(\omega_{4, bq^{2m_4-m_3-1}})) \end{aligned}$$

that is generated by the top weight space. Let $M'(\omega_3 + \omega_4)$ be the pullback of $\overline{V_q(\omega_{\omega_3+\omega_4, bq^m})}$ by τ_b , where $m \in \mathbb{Z}$, and let $T'(\mu)$ be the $\mathfrak{g}[t]$ -submodule of $M'(\omega_3 + \omega_4)^{\otimes m_3} \otimes M(\omega_4)^{\otimes m_4-m_3-1}$. As before, it follows from Lemma 2.20 that $T'(\mu)$ is a quotient of $L(\boldsymbol{\mu})$. Hence, we are left to show that $T'(\mu)$ has $V(\mu - l\vartheta_2)$ as an irreducible \mathfrak{g} -submodule for every $0 \leq l \leq m_3$. Moreover, it suffices to consider the case $m_4 = m_3 = m \in \mathbb{Z}_{>0}$. Observe that $V_q(\omega_{\omega_3+\omega_4, bq^m})$ is not a minimal affinization and that $V(\omega_3) \otimes V(\omega_4) \cong V(\omega_3 + \omega_4) \oplus V(\omega_1)$. In other words, the proposition is proved for $m_3 = m_4 = 1$. Finally, let v_j for $j = 1, \dots, m$ be a highest-weight vector of the j -th copy of $M'(\omega_3 + \omega_4)$ in $M'(\omega_3 + \omega_4)^{\otimes m}$, and let v_j^0 be its image in $M'(\omega_3 + \omega_4)(0)$. Then

$$\begin{aligned} (x_{\vartheta_2,1}^-)^l (v_1 \otimes v_2 \otimes \cdots \otimes v_l \otimes v_{l+1}^0 \otimes \cdots \otimes v_m^0) \\ = (x_{\vartheta_2,1}^- v_1) \otimes (x_{\vartheta_2,1}^- v_2) \otimes \cdots \otimes (x_{\vartheta_2,1}^- v_l) \otimes v_{l+1}^0 \otimes \cdots \otimes v_m^0 \end{aligned}$$

and we are done using a simple induction on l . □

Let v_1 be a highest-weight vector of $M(\lambda^{\{1,2\}})$, and let v_2 be a highest-weight vector of $L(\lambda^{\{3,4\}})$. It follows from Proposition 5.9 and (5-11) that if $r \in \mathfrak{D}_3(\lambda)$, then

$$(5-12) \quad \mathbf{y}_r(v_1 \otimes v_2) = ((x_{\theta,1}^-)^{r_3} v_1) \otimes ((x_{a_1}^-)^{r_1} (x_{\theta_2,1}^-)^{r_1+r_2} v_2).$$

The proof of (5-10) is completed similarly to the end of the proof of Proposition 5.7.

5.18. Multiple minimal affinizations: The irregular case. Keep the notation of Section 5.11. If λ is supported on the three connected components of $I \setminus \{i_0\}$ and $\lambda(h_{i_0}) = 0$, it follows from [CP 1996a] that the number of equivalence classes of minimal affinizations of $V_q(\lambda)$ is not uniformly bounded (it grows as λ “grows”). If \mathfrak{g} is of type E , write I as the disjoint union of two connected subdiagrams of type A , say I_1 and I_2 , and the subdiagram of type D_4 , say J . For \mathfrak{g} of type D we write I as the disjoint union of a subdiagram I_1 of type A and the subdiagram J of type D_4 (for convenience we set $I_2 = \emptyset$ and $\lambda^{\emptyset} = 0$). Similarly to the proof of Proposition 3.21, one proves this:

Proposition 5.19. *Suppose $\lambda \in P^+$ and $\lambda \in \mathfrak{P}_q^+$ are such that $V_q(\lambda)$ is a minimal affinization of $V_q(\lambda)$. Then $L(\lambda)$ projects onto the $\mathfrak{g}[t]$ -submodule of $L(\lambda^{I_1}) \otimes L(\lambda^J) \otimes L(\lambda^{I_2})$ generated by the top weight space.*

We then have a natural conjecture, the discussion of which we postpone to a forthcoming publication.

Conjecture 5.20. *Suppose $\lambda \in P^+$ and $\lambda \in \mathfrak{P}_q^+$ are such that $V_q(\lambda)$ is a minimal affinization of $V_q(\lambda)$. Then $L(\lambda)$ is isomorphic to the $\mathfrak{g}[t]$ -submodule of $L(\lambda^{I_1}) \otimes L(\lambda^J) \otimes L(\lambda^{I_2})$ generated by the top weight space.*

Remark. Let λ be as in [CP 1996a, Theorem 2.2 (a)_{3,4} or (b)_{3,4}]. If conditions (a)_{3,4} are satisfied, the results of Section 5.11 apply, and hence the graded character of $L(\lambda)$ is given by the right side of (5-10). To prove the conjecture of remark (1) following that theorem, it suffices to show that if conditions (b)_{3,4} are satisfied, then the graded character of $L(\lambda)$ is also given by the right side of (5-10). The proof is essentially the same as for the former case replacing Proposition 5.17 by its appropriate obvious modification. We omit the details.

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