ACTA ARITHMETICA
XCIX.1 (2001)

Restricted sums of subsets of $\mathbb{Z}$

by

ZHI-WEI SUN (Nanjing)

1. Introduction. Let

$$\{A_i\}_{i=1}^n \tag{1.1}$$

be a finite sequence of sets. If $a_1 \in A_1, \ldots, a_n \in A_n$, and $a_1, \ldots, a_n$ are pairwise different, then we call $\{a_i\}_{i=1}^n$ a *system of distinct representatives* (abbreviated to SDR) of (1.1). Apparently (1.1) has an SDR provided that

$$|A_i| \ge i \quad \text{for all } i = 1, \ldots, n. \tag{1.2}$$

If $A_1, \ldots, A_n$ are contained in a finite set $\{x_1, \ldots, x_k\}$ with cardinality k, then (1.1) has as many SDR's as $\{A_i^*\}_{i=1}^n$ does where $A_i^* = \{1 \le j \le k : x_j \in A_i\} \subseteq \{1, \ldots, k\}$.

Let $A_1, \ldots, A_n$ be finite subsets of an additive abelian group G. Their sumset is given by

$$A_1 + \ldots + A_n = \{a_1 + \ldots + a_n : a_1 \in A_1, \ldots, a_n \in A_n\}. \tag{1.3}$$

If we require the summands to be distinct, then we are led to the restricted sumset

$$S(\{A_i\}_{i=1}^n) = S(A_1, \ldots, A_n) = \Big\{ \sum_{i=1}^n a_i : \{a_i\}_{i=1}^n \text{ forms an SDR of } \{A_i\}_{i=1}^n \Big\}. \tag{1.4}$$

Of course there are many other kinds of restricted sumsets. An interesting problem is to provide a nontrivial lower bound for the cardinality of a restricted sumset of $A_1, \ldots, A_n$. In the light of the fundamental theorem on finitely generated abelian groups, it suffices to work within the ring $\mathbb{Z}$ of integers instead of a torsionfree abelian group G.

2000 *Mathematics Subject Classification*: Primary 11B75; Secondary 05A05, 11P70.

The research is supported by the Teaching and Research Award Fund for Outstanding Young Teachers in Higher Education Institutions of MOE, and the National Natural Science Foundation of P.R. China.

For a finite subset A of $\mathbb{Z}$, in 1995 M. B. Nathanson [N1] obtained the inequality

$$|n^\wedge A| \geq n|A| - n^2 + 1 \tag{1.5}$$

and determined when equality holds. (By $n^\wedge A$ we mean $S(\{A_i\}_{i=1}^n)$ with $A_1 = \ldots = A_n = A$.) Soon after this, Y. Bilu [B] gave the same result independently. Let p be a prime. In 1994 J. A. Dias da Silva and Y. O. Hamidoune [DH] proved the following generalization of a conjecture of P. Erdős and H. Heilbronn (cf. [EH] and [G]):

$$|n^\wedge A| \geq \min\{p, n|A| - n^2 + 1\} \quad \text{for any } A \subseteq \mathbb{Z}/p\mathbb{Z}. \tag{1.6}$$

By the so-called polynomial method, in 1996 N. Alon, M. B. Nathanson and I. Z. Ruzsa [ANR] got the following result: Let F be any field of characteristic p and $A_1, \ldots, A_n$ its subsets with $0 < |A_1| < \ldots < |A_n| < \infty$, then

$$|S(\{A_i\}_{i=1}^n)| \geq \min\left\{p, \sum_{i=1}^n |A_i| - \frac{n(n+1)}{2} + 1\right\}. \tag{1.7}$$

Their method does not allow one to determine when the bound can be attained. Provided that $A_1, \ldots, A_n$ are finite subsets of $\mathbb{Z}$ with $0 < |A_1| < \ldots < |A_n|$, we have

$$|S(\{A_i\}_{i=1}^n)| \geq 1 + \sum_{i=1}^n (|A_i| - i). \tag{1.8}$$

A purely combinatorial proof of this inequality was given by Hui-Qin Cao and Zhi-Wei Sun [CS], where the authors obtained some necessary conditions for the equality case.

Now we introduce our basic notations in this paper. For $A \subseteq \mathbb{Z}$ we put $-A = \{-x : x \in A\}$ and $a + A = A + a = \{a + x : x \in A\}$ for $a \in \mathbb{Z}$. An *arithmetic progression* A is a set of the form $\{a, a+d, \ldots, a+kd\}$ where a and $d, k > 0$ are integers; we use $d(A)$ to denote the (common) *difference* d of A. (A set having a single element is not considered as an arithmetic progression.) For the sake of convenience, AP will denote the class of all arithmetic progressions. For $a, b \in \mathbb{Z}$ we put

$$(a, b) = \{x \in \mathbb{Z} : a < x < b\}, \quad [a, b] = \{x \in \mathbb{Z} : a \leq x \leq b\},$$
$$[a, b) = \{x \in \mathbb{Z} : a \leq x < b\}, \quad (a, b] = \{x \in \mathbb{Z} : a < x \leq b\}.$$

In this paper we study lower bounds for cardinalities of various restricted sumsets of subsets of $\mathbb{Z}$. We use the powerful techniques developed in [CS].

In the next section we will prove the following general result on linearly restricted sums of subsets of $\mathbb{Z}$.

THEOREM 1.1. *Let* $A_1, \ldots, A_n$ *be finite subsets of* $\mathbb{Z}$, *and* V *a set of tuples* (s,t,μ,ν,w) *where* $1 \le s,t \le n$, $s \ne t$, $\mu,\nu \in \mathbb{Z}^* = \mathbb{Z}\setminus\{0\}$ *and* $w \in \mathbb{Z}$. *Set*

$$C = \{a_1 + \ldots + a_n : a_i \in A_i,\ \text{and}\ \mu a_i + \nu a_j \ne w\ \text{if}\ (i,j,\mu,\nu,w) \in V\}. \tag{1.9}$$

If each $V_i = \{(s,t,\mu,\nu,w) \in V : i \in \{s,t\}\}$ *has cardinality less than* $|A_i|$, *then*

$$|C| \ge \sum_{i=1}^{n} |A_i| - 2|V| - n + 1 = 1 + \sum_{i=1}^{n} (|A_i| - |V_i| - 1) > 0. \tag{1.10}$$

REMARK 1.1. If we replace $a_1 + \ldots + a_n$ by $\lambda_1 a_1 + \ldots + \lambda_n a_n$ in the definition (1.9) of C where $\lambda_1, \ldots, \lambda_n \in \mathbb{Z}^*$, then Theorem 1.1 remains valid. For, when $(i,j,\mu,\nu,w) \in V$, $a_i \in A_i$ and $a_j \in A_j$, we have

$$\mu a_i + \nu a_j = w \ \Leftrightarrow\ \lambda_i\lambda_j(\mu a_i + \nu a_j) = \lambda_i\lambda_j w \ \Leftrightarrow\ \mu'(\lambda_i a_i) + \nu'(\lambda_j a_j) = w'$$

where $\mu' = \lambda_j \mu$, $\nu' = \lambda_i \nu$ and $w' = \lambda_i\lambda_j w$.

Now we give several consequences of Theorem 1.1.

COROLLARY 1.1. *Let* $A_1, \ldots, A_n$ *be subsets of* $\mathbb{Z}$ *which are nonempty and finite. Then*

$$|A_1 + \ldots + A_n| \ge |A_1| + \ldots + |A_n| - n + 1. \tag{1.11}$$

Proof. Just apply Theorem 1.1 with $V = \emptyset$. ∎

REMARK 1.2. Corollary 1.1 is a known result. Equality in (1.11) holds if and only if all those A_i with $|A_i| \ge 2$ are arithmetic progressions with the same difference. See Theorems 1.4 and 1.5 of [N2].

COROLLARY 1.2. *Let* $A_1, \ldots, A_n$ *be finite subsets of* $\mathbb{Z}$ *such that* $|A_i| \ge |J_i|$ *for all* $i = 1, \ldots, n$ *where* $J_i = \{1 \le j \le n : A_i \cap A_j \ne \emptyset\}$. *Then*

$$|S(\{A_i\}_{i=1}^n)| \ge 1 + \sum_{i=1}^{n} (|A_i| - |J_i|). \tag{1.12}$$

Proof. Put $V = \{(i,j,1,-1,0) : 1 \le i < j \le n\ \&\ A_i \cap A_j \ne \emptyset\}$. Then

$$|V_i| = |\{1 \le j \le n : j \ne i\ \&\ A_i \cap A_j \ne \emptyset\}| = |J_i \setminus \{i\}| < |A_i| \quad \text{for } i \in [1,n].$$

Applying Theorem 1.1 we immediately get the desired inequality. ∎

COROLLARY 1.3. *Let* $\Lambda, A_1, \ldots, A_n$ *be finite subsets of* $\mathbb{Z}$ *such that*

$$|A_i| > \sum_{j \ne i} |(A_i + A_j) \cap \Lambda| \quad \text{for all } i = 1, \ldots, n.$$

Let $\lambda_1, \ldots, \lambda_n \in \mathbb{Z}^*$ *and*

$$L = \{\lambda_1 a_1 + \ldots + \lambda_n a_n : a_1 \in A_1, \ldots, a_n \in A_n,\ a_i + a_j \notin \Lambda\ \text{if}\ i \ne j\}.$$

Then

$$\sum_{i=1}^{n}|A_i|-|L|\le 2\sum_{1\le i<j\le n}|(A_i+A_j)\cap \Lambda|+n-1\le (n|\Lambda|+1)(n-1).$$

Proof. Set

$$V=\{(i,j,1,1,\lambda):1\le i<j\le n\ \&\ \lambda\in (A_i+A_j)\cap\Lambda\}.$$

Then

$$|V|=\sum_{1\le i<j\le n}|(A_i+A_j)\cap\Lambda|\le \binom{n}{2}|\Lambda|,$$

and $|V_i|=\sum_{j\ne i}|(A_i+A_j)\cap\Lambda|$ for $i=1,\ldots,n$. Thus the required result follows from Theorem 1.1 and Remark 1.1. ■

Corollary 1.4. *Let $A_1,\ldots,A_n$ be finite subsets of $\mathbb{Z}$, and*

$$S=\{a_1+\ldots+a_n:a_1\in A_1,\ldots,a_n\in A_n,\ a_i\ne\mu_{ij}a_j+\nu_{ij}\ \text{if } i\ne j\},$$

where $\mu_{ij}\in\mathbb{Z}^*$ and $\nu_{ij}\in\mathbb{Z}$. If $|A_i|\ge 2n-1$ for all $i=1,\ldots,n$, then

$$|S|\ge\sum_{i=1}^{n}|A_i|-2n^2+n+1.$$

Proof. Let $V=\{(i,j,1,-\mu_{ij},\nu_{ij}):1\le i,j\le n\ \&\ i\ne j\}$. If $1\le i\le n$ then $|V_i|=n-1+(n-1)=2n-2$. Clearly $2|V|+n-1=2(n^2-n)+n-1=2n^2-n-1$. So it suffices to apply Theorem 1.1. ■

Remark 1.3. For $1\le i<j\le n$ let $\mu_{ij}=1,\ \mu_{ji}=-1$ and $\nu_{ij}=\nu_{ji}=0$. Then the set S given in Corollary 1.4 becomes $\{\sum_{i=1}^{n}a_i:a_i\in A_i$ and all the a_i^2 are distinct$\}$.

Corollary 1.5. *For each $i=1,\ldots,n$ let $A_i\subseteq\mathbb{Z}$ and $3\le|A_i|<\infty$. Then the set*

$$\{a_1+\ldots+a_n:a_i\in A_i,\ a_i\ne a_{i+1}\ \text{if } i<n,\ \text{and } a_n\ne a_1\}$$

has cardinality at least $\sum_{i=1}^{n}|A_i|-3n+1$.

Proof. Let $V=\{(i,i+1,1,-1,0):i\in[1,n)\}\cup\{(n,1,1,-1,0)\}$. Then $|V|=n$, and $|V_i|=2<|A_i|$ for all $i\in[1,n]$. So the desired result follows immediately from Theorem 1.1. ■

Let F be a field of characteristic p where p is a prime, and $A_1,\ldots,A_n$ its finite subsets satisfying (1.2). Then Theorem 3.2 of [ANR] essentially asserts that

$$|S(\{A_i\}_{i=1}^{n})|\ge\min\Big\{p,1+\sum_{i=1}^{n}\min_{i\le j\le n}(|A_j|-j)\Big\}.$$

In the last section we will show the following general result by our combinatorial method.

THEOREM 1.2. *Let* $A_1, \ldots, A_n$ *be finite subsets of* $\mathbb{Z}$ *with* (1.2) *and* $|A_1| \le \ldots \le |A_n|$. *Then*

$$|S(\{A_i\}_{i=1}^n)| \ge 1 + \sum_{i=1}^{n} \min_{i\le j\le n}(|A_j| - j). \tag{1.13}$$

In the equality case, $\bigcup_{i=1}^m A_i = A_m$ *if* m *lies in*

$$M = \{1 \le i \le n : |A_i| - i < |A_j| - j \text{ for all } j \in (i,n]\}, \tag{1.14}$$

and providing $|A_i| > i$ *for all* $i \in [1,n]$ *the set* $\bigcup_{i=1}^n A_i = A_n$ *lies in* AP *with the only exceptions as follows*:

(i) $n = 1$ *or* $|A_n| = n + 1$;
(ii) $n = 2$, $|A_1| \in \{3,4\}$ *and* A_2 *has the form*

$$\{x_1, x_2, x_3, x_4\} \quad \text{with } x_1 < x_2 < x_3 < x_4 \text{ and } x_4 - x_3 = x_2 - x_1; \tag{1.15}$$

(iii) $n > 1$, $|A_{n-1}| = n$, A_{n-1} *and* $A_n \setminus A_{n-1}$ *belong to* AP, *and* $d(A_{n-1}) = d(A_n \setminus A_{n-1})$.

REMARK 1.4. Let $A_1, \ldots, A_n$ be finite subsets of $\mathbb{Z}$ with $k_i = |A_i| \ge i$ for all $i \in [1,n]$. Providing $k_s > k_{s+1}$ for some $s \in [1,n)$, we still have inequality (1.13). To see this, we exchange A_s and A_{s+1}, i.e. we arrange $A_1, \ldots, A_n$ in the order

$$\begin{gathered} A_1^* = A_1, \quad \ldots, \quad A_{s-1}^* = A_{s-1}, \quad A_s^* = A_{s+1}, \\ A_{s+1}^* = A_s, \quad A_{s+2}^* = A_{s+2}, \quad \ldots, \quad A_n^* = A_n. \end{gathered}$$

Clearly

$$|A_{s+1}^*| - (s+1) = k_s - s - 1 > k_{s+1} - (s+1)$$

and

$$\begin{aligned} \min\{|A_s^*| - s, |A_{s+1}^*| - (s+1)\} &= \min\{k_{s+1} - s, k_s - s - 1\} \\ &= k_{s+1} - s > k_{s+1} - (s+1) \\ &\ge \min\{k_s - s, k_{s+1} - (s+1)\}, \end{aligned}$$

thus

$$\min_{i\le j\le n}(|A_j^*| - j) \ge \min_{i\le j\le n}(k_j - j) \quad \text{ for all } i = 1, \ldots, n.$$

The following example shows that in Theorem 1.2 the lower bound (in terms of cardinalities $|A_1|, \ldots, |A_n|$) is best possible.

EXAMPLE 1.1. Let $k_1, \ldots, k_n$ be integers for which $k_1 \le \ldots \le k_n$ and $k_i \ge i$ for all $i = 1, \ldots, n$. Let $d_i = \min_{i\le j\le n}(k_j - j)$ for each $i = 1, \ldots, n$. Apparently $d_1 \le \ldots \le d_n$. Put $A_1 = [0, k_1 - 1], \ldots, A_n = [0, k_n - 1]$. Observe that $S(\{A_i\}_{i=1}^n)$ contains the following sets:

$$0+1+2+\ldots+(n-3)+(n-2)+[n-1,n-1+d_n],$$
$$0+1+2+\ldots+(n-3)+[n-2,n-2+d_{n-1}]+(n-1+d_n),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$0+[1,1+d_2]+(2+d_3)+\ldots+(n-2+d_{n-1})+(n-1+d_n),$$
$$[0,d_1]+(1+d_2)+(2+d_3)+\ldots+(n-2+d_{n-1})+(n-1+d_n).$$

Therefore

$$\begin{aligned}S(\{A_i\}_{i=1}^n) &\supseteq [0+1+\ldots+(n-1), d_1+(1+d_2)+\ldots+(n-1+d_n)]\\ &= \frac{n(n-1)}{2}+\Big[0,\sum_{i=1}^n d_i\Big].\end{aligned}$$

Suppose that $\max S(\{A_i\}_{i=1}^n)=\sum_{i=1}^n x_i$ where $x_1<\ldots<x_n$ and these n integers can be rearranged to form an SDR of $\{A_i\}_{i=1}^n$. Choose a permutation σ on $\{1,\ldots,n\}$ such that $x_{\sigma(i)}\in A_i$. When $1\le i\le n$, there exists a $j\in[i,n]$ such that $\sigma^{-1}(j)\notin(i,n]$ and hence $x_j\in A_{\sigma^{-1}(j)}\subseteq A_i$. So $x_i\in A_i$ for every $i=1,\ldots,n$. If $x_n<k_n-1$, then by substituting k_n-1 for x_n we would obtain an SDR of $\{A_i\}_{i=1}^n$ with the corresponding sum larger than $\sum_{i=1}^n x_i$. Thus $x_n=k_n-1=n-1+d_n$. Let $1\le i<n$ and assume that $x_j=j-1+d_j$ for all $j\in(i,n]$. When $i<j\le n$, we have $x_j=j-1+d_j\ge i+d_i$. If $x_i<i-1+d_i$ then by substituting $i-1+d_i\in A_i$ for x_i we would obtain a sum larger than $x_1+\ldots+x_n$, thus $x_i=i-1+d_i$. By the above,

$$\max S(\{A_i\}_{i=1}^n)=\sum_{i=1}^n x_i=\sum_{i=1}^n(i-1+d_i)=\frac{n(n-1)}{2}+\sum_{i=1}^n d_i.$$

Obviously

$$\min S(\{A_i\}_{i=1}^n)=0+1+\ldots+(n-1)=\frac{n(n-1)}{2}.$$

So we also have

$$S(\{A_i\}_{i=1}^n)\subseteq\frac{n(n-1)}{2}+\Big[0,\sum_{i=1}^n d_i\Big].$$

Therefore

$$S(\{A_i\}_{i=1}^n)=\Big[\frac{n(n-1)}{2},\frac{n(n-1)}{2}+\sum_{i=1}^n d_i\Big]$$

and hence $|S(\{A_i\}_{i=1}^n)|=1+\sum_{i=1}^n d_i$.

Remark 1.5. Example 1.1 was realized by Alon, Nathanson and Ruzsa [ANR], but they did not go into details. Let $k_1,\ldots,k_n$ and $A_1,\ldots,A_n$ be as in Example 1.1. For $i=1,\ldots,n$ put $A_i^*=\{a+jd: j\in[0,k_i)\}$ where

$a \in \mathbb{Z}$ and $d \in \mathbb{Z}^*$. By Example 1.1,

$$|S(\{A_i^*\}_{i=1}^n)| = |S(\{A_i\}_{i=1}^n)| = 1 + \sum_{i=1}^n \min_{i \le j \le n} (|A_j^*| - j).$$

As for the exceptions (i) and (ii), here we give

EXAMPLE 1.2. Let A be a finite subset of $\mathbb{Z}$ with $|A| \ge n \ge 1$, and $A_1, \ldots, A_n$ subsets of $\mathbb{Z}$ with $\bigcup_{i=1}^n A_i = A_n = A$. Suppose that $|A_i| - i \ge |A_n| - n$ for all $i = 1, \ldots, n$ (i.e. the set M defined by (1.14) only contains n). If $\{a_i\}_{i=1}^n$ is an SDR of $\{A_i\}_{i=1}^n$, then $\{a_1, \ldots, a_n\}$ is a subset of A with cardinality n. If $S \subseteq A$ and $|S| = n$, then for each $i \in [1, n]$ we have

$$|S \cap A_i| \ge |S| - |A \setminus A_i| = n - (|A_n| - |A_i|) \ge i,$$

therefore $\{S \cap A_i\}_{i=1}^n$ has an SDR $\{a_i\}_{i=1}^n$ and hence $S = \{a_1, \ldots, a_n\}$. Thus $S(\{A_i\}_{i=1}^n) = n^\wedge A$, (1.13) is equivalent to (1.5), and the equality case of (1.13) is the same as that of (1.5). A result of Nathanson says that $|n^\wedge A| = n|A| - n^2 + 1$ if and only if $n \in \{1, |A| - 1, |A|\}$, or $A \in$ AP, or $n = 2$ and A can be written in the form (1.15). (See Section 3 of [N1] and Section 1.3 of [N2].) Thus, if $n = 1$ or $|A| = n + 1$, whether $A \in$ AP or not, the two sides of (1.13) are always equal; this corresponds to the exception (i). In the case $n = 2$, if $A_2 = A$ is of the form (1.15), then $|A_1| \in \{|A_2| - 1, |A_2|\} = \{3, 4\}$ and

$$\begin{aligned}|S(\{A_i\}_{i=1}^2)| &= |2^\wedge A| = 2|A| - 2^2 + 1 = 5 \\ &= 1 + \min\{|A_1| - 1, |A_2| - 2\} + |A_2| - 2\end{aligned}$$

though we may not have $A_2 = A \in$ AP.

For the equality case of (1.13), Example 1.2 shows that the necessary conditions given by Theorem 1.2 are also sufficient in the case $M = \{n\}$.

From Theorem 1.2 we have

COROLLARY 1.6. *Let $A_1, \ldots, A_n$ be finite subsets of $\mathbb{Z}$ with $|A_1| \le \ldots \le |A_n|$ and $\min_{1 \le i \le n}(|A_i| - i) = 0$. Put $m = \max\{1 \le i \le n : |A_i| = i\}$. Suppose that the two sides of* (1.13) *are equal. Then $A_n \setminus A_m \in$ AP unless we have one of the following*:

(i′) *$m \in \{n - 1, n\}$ or $|A_n| = n + 1$;*

(ii′) *$m = n - 2$, $|A_{n-1}| \in \{n + 1, n + 2\}$ and $A_n \setminus A_{n-2}$ is of the form* (1.15);

(iii′) *$m < n - 1$, $|A_{n-1}| = n$, $A_{n-1} \setminus A_m$ and $A_n \setminus A_{n-1}$ lie in* AP, *and $d(A_{n-1} \setminus A_m) = d(A_n \setminus A_{n-1})$.*

Proof. Write $M = \{m_1, \ldots, m_l\}$ where $m_0 = 0 < m_1 < \ldots < m_l = n$. Clearly $m_1 = m$. For any $j \in [1, l]$ set $A_i^* = A_{m_j}$ for all $i \in (m_{j-1}, m_j]$. By Theorem 1.2, $A_i \subseteq A_{m_j}$ for all $i = 1, \ldots, m_j$. In the light of Example 1.2, any $m_j - m_{j-1}$ distinct elements of A_{m_j} can be arranged to form an SDR of $\{A_i\}_{m_{j-1}<i\le m_j}$. So

$$\begin{aligned}S(\{A_i\}_{i=1}^n) &= S(\{A_i^*\}_{i=1}^n)\\ &= \Big\{\sum_{x\in A_m} x + \sum_{m<i\le n} a_i : a_i \in A_i^* \setminus A_m,\ \text{all the } a_i \text{ are distinct}\Big\}\\ &= \sum_{x\in A_m} x + S(\{A_i^*\setminus A_m\}_{i\in(m,n]})\end{aligned}$$

where we regard $S(\emptyset)$ as $\{0\}$. Observe that

$$\begin{aligned}\sum_{i=1}^n \min_{i\le j\le n}(|A_j|-j) &= \sum_{j=1}^l \sum_{m_{j-1}<i\le m_j}(|A_{m_j}|-m_j)\\ &= \sum_{m<i\le n}\min_{i\le j\le n}(|A_j^*|-j)\\ &= \sum_{m<i\le n}\min_{i\le j\le n}(|A_j^*\setminus A_m|-(j-m)).\end{aligned}$$

Thus

$$\begin{aligned}|S(\{A_i^*\setminus A_m\}_{i\in(m,n]})| &= |S(\{A_i\}_{i=1}^n)|\\ &= 1+\sum_{m<i\le n}\min_{i\le j\le n}(|A_j^*\setminus A_m|-(j-m)).\end{aligned}$$

If $i\in(m,n]$, then $|A_i|-i > |A_m|-m = 0$ and hence $|A_i^*\setminus A_m| = |A_i^*|-m > i-m$.

Below we assume that $m\ne n$. Let us apply Theorem 1.2 to the sets $A_{m+1}^*\setminus A_m,\dots,A_n^*\setminus A_m$. If $A_n^*\setminus A_m = A_n\setminus A_m \notin \mathrm{AP}$, then we are led to the exceptions corresponding to (i)–(iii) in Theorem 1.2. Obviously

$$|(m,n]| = 1 \ \Leftrightarrow\ m=n-1 \quad \text{and} \quad |A_n^*\setminus A_m| = (n-m)+1 \ \Leftrightarrow\ |A_n| = n+1.$$

In the case $n-m=2$, $A_n^*\setminus A_m = A_n\setminus A_{n-2}$ and

$$|A_{n-1}^*\setminus A_m| \in |A_n^*\setminus A_m|+\{0,-1\} \ \Leftrightarrow\ |A_{n-1}^*|\in|A_n|+\{0,-1\} \ \Leftrightarrow\ n-1\notin M,$$

if $|A_n^*\setminus A_m| = |A_n\setminus A_{n-2}| = 4$ then $|A_n| = |A_{n-2}|+4 = n+2$ and

$$|A_{n-1}^*\setminus A_m|\in\{3,4\} \ \Leftrightarrow\ |A_{n-1}|\in|A_n|+\{0,-1\} = \{n+1,n+2\}.$$

When $n-m>1$, we have

$$\begin{aligned}|A_{n-1}^*\setminus A_m| = n-m \ \&\ (A_n^*\setminus A_m)\setminus(A_{n-1}^*\setminus A_m)\in\mathrm{AP}\\ \Leftrightarrow\ |A_{n-1}^*| = n,\ A_{n-1}^*\ne A_n^* = A_n \ \&\ A_n^*\setminus A_{n-1}^*\in\mathrm{AP}\\ \Leftrightarrow\ n-1\in M,\ |A_{n-1}| = n \ \&\ A_n\setminus A_{n-1}\in\mathrm{AP}\\ \Leftrightarrow\ |A_{n-1}| = n \ \&\ A_n\setminus A_{n-1}\in\mathrm{AP}.\end{aligned}$$

In view of this, we have (i′) or (ii′) or (iii′) if $A_n \setminus A_m \notin \mathrm{AP}$. ∎

REMARK 1.6. Clearly (i), (ii) and (iii) correspond to (i′), (ii′) and (iii′) with $m = 0$ and $A_0 = \emptyset$. The proof of Corollary 1.6 shows that in the equality case of (1.13) those A_m with $m \in M$ are vital.

Let $A_1, \ldots, A_n$ be finite subsets of $\mathbb{Z}$ satisfying (1.2). Theorem 1.2, together with Example 1.1, Remark 1.5 and Corollary 1.6, shows that we have completely determined the set $\bigcup_{i=1}^{n} A_i = A_n$ in the equality case of (1.13).

COROLLARY 1.7. *Let* $A_1, \ldots, A_n$ *be finite subsets of* $\mathbb{Z}$ *with* (1.2) *and* $|A_1| \le \ldots \le |A_n|$. *Then*

$$|S(\{A_i\}_{i=1}^{n})| \ge 1 + \sum_{i=1}^{n} (|A_i| + h_i - n) \tag{1.16}$$

where

$$h_i = |\{|A_j| : 1 \le j \le n \ \&\ |A_j| > |A_i|\}|. \tag{1.17}$$

Furthermore, when the lower bound in (1.16) *is reached*, $A_i \subseteq A_m$ *for all* $i = 1, \ldots, m$ *if* $|A_m| < |A_{m+1}| - 1$ *or* $m = n$; *also* $|A_l| < \ldots < |A_n|$ *where* l *is the least index with* $|A_l| < |A_{l+1}| - 1$ *or* $l = n$; *and providing* $\min\{n, |A_1| - 1, \ldots, |A_n| - n\} \ge 2$ *we have* $A_n \in \mathrm{AP}$ *unless* A_n *is of the form* (1.15).

Proof. Let $k_i = |A_i|$ for $i \in [1, n]$. When $i \in [1, n)$, if $k_i = k_{i+1}$ then $h_i = h_{i+1}$, if $k_i \le k_{i+1} - 1$ then $h_i = h_{i+1} + 1$; thus $k_i + h_i \le k_{i+1} + h_{i+1}$, and $k_i + h_i < k_{i+1} + h_{i+1}$ if and only if $k_i < k_{i+1} - 1$. For $i \in [1, n]$, if $j \in [i, n]$ then $k_i + h_i - n \le k_j + h_j - n \le k_j - j$, so $k_i + h_i - n \le d_i = \min_{i \le j \le n}(k_j - j)$. Thus (1.16) holds by Theorem 1.2.

Clearly $k_1 + h_1 = \ldots = k_l + h_l$ by the above, and $d_1 = \ldots = d_l$ since $k_1 - 1 \ge \ldots \ge k_l - l$. When $k_i + h_i - n = d_i$ for all $i = 1, \ldots, n$, for each $m \in [1, n)$ we have

$$m \in M \ \Leftrightarrow\ d_m < d_{m+1} \ \Leftrightarrow\ k_m + h_m < k_{m+1} + h_{m+1} \ \Leftrightarrow\ k_m < k_{m+1} - 1,$$

so $l \in M$ and $k_l + h_l - n = d_l = k_l - l$, therefore $h_l = n - l$ and $|A_l| < \ldots < |A_n|$. Conversely, if $|A_l| < \ldots < |A_n|$, then $k_l - l \le \ldots \le k_n - n$ and hence $d_i = k_i - i = k_i + h_i - n$ for all $i \in [l, n]$. So $k_i + h_i - n = d_i$ for all $i \in [1, n]$ if and only if $k_l < \ldots < k_n$.

Suppose that the two sides of (1.16) are equal. Then the two sides of (1.13) are equal, and $k_l < \ldots < k_n$ by the above. In view of Theorem 1.2, $\bigcup_{i=1}^{m} A_i = A_m$ provided that $k_m < k_{m+1} - 1$ or $m = n$. If $n \ge 2$ and $d_1 = \min_{1 \le i \le n}(k_i - i) \ge 2$, then either $A_n \in \mathrm{AP}$, or $n = 2$ and A_2 can be written in the form (1.15). ∎

REMARK 1.7. In the case $A_1 = \ldots = A_n = A$, we have $h_1 = \ldots = h_n = 0$ and Corollary 1.7 reduces to Theorem 2 of Nathanson [N1]. When

$|A_1| < \ldots < |A_n|$, Corollary 1.7 is a slight improvement on the main theorem of Cao and Sun [CS].

COROLLARY 1.8. *Let $A_1, \ldots, A_n$ be finite subsets of $\mathbb{Z}$ with* (1.2). *Then*

$$|S(\{A_i\}_{i=1}^n)| \geq \sum_{i=1}^n |A_i| - n^2 + 1. \tag{1.18}$$

Providing $2 \leq n \leq |A_n| - 2$ and $|A_n| \neq 4$, the two sides are equal if and only if $A_1 = \ldots = A_n \in \mathrm{AP}$.

Proof. If we rearrange the order of $A_1, \ldots, A_n$, both sides of (1.16) keep unchanged. Suppose that $|A_{\sigma(1)}| \leq \ldots \leq |A_{\sigma(n)}|$ where σ is a permutation on $\{1, \ldots, n\}$. If $|A_{\sigma(i)}| < i$, then

$$[i, n] \subseteq \{1 \leq j \leq n : |A_j| \geq i\} \subseteq \{\sigma(j) : j \in (i, n]\},$$

which is impossible. So $|A_{\sigma(i)}| \geq i$ for all $i \in [1, n]$. By Corollary 1.7, (1.16) holds and hence (1.18) follows. If both sides of (1.18) are equal, then $h_i = 0$ for all $i = 1, \ldots, n$ and hence $|A_1| = \ldots = |A_n|$, as $\bigcup_{i=1}^n A_i = A_n$ by Corollary 1.7 we must have $A_1 = \ldots = A_n$. Now it suffices to apply the Nathanson result. ∎

For the equality case of (1.13), let us look at one more example.

EXAMPLE 1.3. Let k and n be integers with $k > n > 1$. Let $A_1, \ldots, A_{n-1}$ be subsets of $A_n = [0, k-1]$ with $A_1 = [0, k-n] \setminus \{k-n-1\}$ and $|A_{i+1}| - |A_i| \in \{0, 1\}$ for all $i \in (1, n)$. We assert that

$$S = S(\{A_i\}_{i=1}^n) = \left[\frac{n(n-1)}{2}, kn - \frac{n(n+1)}{2}\right] \Big\backslash \left\{kn - \frac{n(n+1)}{2} - 1\right\}$$

and hence

$$|S| = kn - n^2 = 1 + (|A_1| - 1) + (n-1)(k-n) = 1 + \sum_{i=1}^n \min_{i \leq j \leq n} (|A_j| - j).$$

Since $M = \{1, n\}$, by the arguments in the proof of Corollary 1.6, we may assume $A_2 = \ldots = A_n$ without any loss of generality.

In the case $k = n + 1$, clearly $A_1 = \{1\}$ and $A_i = [0, n]$ for $i \in (1, n]$; setting $A = [0, n] \setminus \{1\}$ we then have

$$\begin{aligned} S &= 1 + (n-1)^\wedge A = 1 + \Big\{\sum_{x \in A} x - a : a \in A\Big\} = \sum_{i=1}^n i - A \\ &= \frac{n(n+1)}{2} - ([0, n] \setminus \{1\}) = \left[\frac{n(n-1)}{2}, \frac{n(n+1)}{2}\right] \Big\backslash \left\{\frac{n(n+1)}{2} - 1\right\} \\ &= \left[\frac{n(n-1)}{2}, kn - \frac{n(n+1)}{2}\right] \Big\backslash \left\{kn - \frac{n(n+1)}{2} - 1\right\}. \end{aligned}$$

Below we verify the assertion on the condition $k > n+1$. By Example 1.1,

$$S \subseteq S([0, k-n], A_2, \ldots, A_n) = \frac{n(n-1)}{2} + \Big[0, \sum_{i=1}^{n}(k-n)\Big]$$
$$= \Big[\frac{n(n-1)}{2}, kn - \frac{n(n+1)}{2}\Big]$$

and S contains

$$S([0, k-n-2], A_2, \ldots, A_n) = \frac{n(n-1)}{2} + [0, k-n-2+(n-1)(k-n)]$$
$$= \Big[\frac{n(n-1)}{2}, kn - \frac{n(n+1)}{2} - 2\Big].$$

Observe that

$$\max S = k-n+(k-n-1)+\ldots+(k-1) = kn - \frac{n(n+1)}{2}.$$

Now it suffices to show that $kn - n(n+1)/2 - 1 \notin S$. On the contrary, we can write

$$kn - \frac{n(n+1)}{2} - 1 = k-n+(k-i_1)+\ldots+(k-i_{n-1})$$

where $1 \le i_1 < \ldots < i_{n-1} \le k$ and $n \notin \{i_1, \ldots, i_{n-1}\}$. Apparently

$$i_1+\ldots+i_{n-1} = \frac{n(n+1)}{2} + 1 - n, \quad \text{i.e.} \quad \sum_{j=1}^{n-1}(i_j - j) = 1.$$

So $i_t - t = 1$ for some $t \in [1, n)$, and $i_j = j$ for all $j \in [1, n) \setminus \{t\}$. As $i_{n-1} \ne n$, we have $t < n-1$ and hence $i_t = t+1 = i_{t+1}$. This contradicts $i_t < i_{t+1}$.

Let $A_1, \ldots, A_{n-1}$ be subsets of $A_n = [0, k_n - 1]$ with the two sides of (1.13) equal. Set $A'_i = \{k_n - 1 - x : x \in A_i\}$ for $i = 1, \ldots, n$. Then

$$|S(\{A'_i\}_{i=1}^n)| = |S(\{A_i\}_{i=1}^n)| = 1 + \min_{i \le j \le n}(|A'_j| - j).$$

If $\min A_1 + \max A_1 \ge k_n$, then $\min A'_1 + \max A'_1 = 2(k_n - 1) - \min A_1 - \max A_1 < k_n$. So, to discuss the equality case of (1.13) with $A_n \in$ AP, we may simply take $A_n = [0, k_n - 1]$ and assume that $\min A_1 + \max A_1 < k_n$.

Now we pose a conjecture which essentially determines the equality case of (1.13).

Conjecture 1.1. *Let $A_1, \ldots, A_n$ be finite subsets of $\mathbb{Z}$ with $|A_1| \le \ldots \le |A_n|$, $k_i = |A_i| > i$ for $i \in [1, n]$, and $\bigcup_{i=1}^m A_i = A_m$ for all $m \in M$. Suppose that $A_n = [0, k_n - 1]$ and $\min A_1 + \max A_1 < k_n$. If the two sides of (1.13) are equal, then $A_m = [0, k_m - 1]$ for all $m \in M$, unless*

$$\text{(1.19)} \qquad M = \{1, n\}, \quad k_n - k_1 = n \quad \text{and} \quad A_1 = [0, k_1] \setminus \{k_1 - 1\}.$$

Though we are unable to solve this conjecture, we have found evidence to support it through computer calculations.

2. Proof of Theorem 1.1. We use induction on n. In the case $n = 1$, the inequality is obvious since $C = A_1$ and $V_1 = V = \emptyset$. So we proceed to the induction step.

Let $n > 1$ and assume the assertion holds for smaller values of n. Set $a = \min A_n$ and

$$V' = \{(s, t, \mu, \nu, w) \in V : 1 \le s, t \le n - 1\}.$$

For each $i = 1, \ldots, n-1$ let A'_i consist of those $a_i \in A_i$ for which $\mu a_i + \nu a \ne w$ if $(i, n, \mu, \nu, w) \in V$, and $\mu a + \nu a_i \ne w$ if $(n, i, \mu, \nu, w) \in V$. Apparently

$$|A'_i| \ge |A_i| - |\{(s, t, \mu, \nu, w) \in V : \{s, t\} = \{i, n\}\}|,$$

and thus

$$\begin{aligned} V'_i &= \{(s, t, \mu, \nu, w) \in V' : i \in \{s, t\}\} \\ &= V_i \setminus \{(s, t, \mu, \nu, w) \in V : \{s, t\} = \{i, n\}\} \end{aligned}$$

has cardinality not greater than $|V_i| + |A'_i| - |A_i| < |A'_i|$. Let

$$C' = \{a_1 + \ldots + a_{n-1} : a_i \in A'_i, \text{ and } \mu a_i + \nu a_j \ne w \text{ if } (i, j, \mu, \nu, w) \in V'\}.$$

By the induction hypothesis,

$$|C'| \ge 1 + \sum_{i=1}^{n-1} (|A'_i| - |V'_i| - 1) \ge 1 + \sum_{i=1}^{n-1} (|A_i| - |V_i| - 1) > 0.$$

Write $\max C' = \sum_{i=1}^{n-1} a'_i$ where $a'_1 \in A'_1, \ldots, a'_{n-1} \in A'_{n-1}$, and $\mu a'_i + \nu a'_j \ne w$ if $(i, j, \mu, \nu, w) \in V'$. Let A'_n consist of those $a_n \in A_n$ for which $\mu a'_i + \nu a_n \ne w$ if $(i, n, \mu, \nu, w) \in V$, and $\mu a_n + \nu a'_i \ne w$ if $(n, i, \mu, \nu, w) \in V$. Note that $a \in A'_n$ and $|A'_n| \ge |A_n| - |V_n| > 0$. Clearly

$$(C' + a) \cup (a'_1 + \ldots + a'_{n-1} + A'_n) \subseteq C$$

and

$$\max(C' + a) = a'_1 + \ldots + a'_{n-1} + a = \min(a'_1 + \ldots + a'_{n-1} + A'_n).$$

Therefore

$$\begin{aligned} |C| &\ge |C' + a| + |a'_1 + \ldots + a'_{n-1} + A'_n| - 1 = |C'| + |A'_n| - 1 \\ &\ge 1 + \sum_{i=1}^{n-1} (|A_i| - |V_i| - 1) + |A_n| - |V_n| - 1 = 1 + \sum_{i=1}^{n} (|A_i| - |V_i| - 1). \end{aligned}$$

Since $\sum_{i=1}^{n} |V_i| = 2|V|$, we are done.

3. Several lemmas. We first check the exception (iii) given in Theorem 1.2.

LEMMA 3.1. *Let* $A_1, \ldots, A_n$ $(n > 1)$ *be finite subsets of* $\mathbb{Z}$ *such that* $|A_i| > i$ *for all* $i \in [1, n]$, $|A_{n-1}| = n < |A_n| - 1$ *and* $\bigcup_{i=1}^{n-1} A_i = A_{n-1} \subseteq A_n$. *Then the two sides of* (1.13) *are equal if and only if* $A_{n-1}, A_n \setminus A_{n-1} \in \mathrm{AP}$ *and* $d(A_{n-1}) = d(A_n \setminus A_{n-1})$.

Proof. Let $S = S(\{A_i\}_{i=1}^n)$ and $k_i = |A_i|$ for all $i = 1, \ldots, n$. Write $A_{n-1} = \{x_1, \ldots, x_n\}$ and $A_n \setminus A_{n-1} = \{y_1, \ldots, y_{k_n - k_{n-1}}\}$ where $x_1 < \ldots < x_n$ and $y_1 < \ldots < y_{k_n - k_{n-1}}$. Since $k_i - i \geq 1 = k_{n-1} - (n-1)$ for all $i \in [1, n-1]$, $S(\{A_i\}_{i=1}^{n-1}) = (n-1)^\wedge A_{n-1}$ as pointed out in Example 1.2. Thus

$$S = \bigcup_{i=1}^{n} \{x_1 + \ldots + x_n - x_i + y : y \in \{x_i, y_1, \ldots, y_{k_n - k_{n-1}}\}\}$$
$$= x_1 + \ldots + x_n + (\{0\} \cup \{y_j - x_i : i \in [1, n],\ j \in [1, k_n - k_{n-1}]\})$$

and hence $|S| = 1 + |(A_n \setminus A_{n-1}) - A_{n-1}|$ where we let $A - B = A + (-B) = \{a - b : a \in A,\ b \in B\}$ for $A, B \subseteq \mathbb{Z}$. By a known result (cf. Lemma 1.3 and Theorem 1.5 of [N2]), for any finite subsets A and B of $\mathbb{Z}$ with $|A| \geq 2$ and $|B| \geq 2$, $|A+B| = |A| + |B| - 1$ if and only if $A, B \in \mathrm{AP}$ and $d(A) = d(B)$. So

$$|S| = 1 + \sum_{i=1}^{n} \min_{i \leq j \leq n} (k_j - j) = 1 + (n-1)(k_{n-1} - (n-1)) + k_n - n = k_n$$
$$\Leftrightarrow \ |(A_n \setminus A_{n-1}) - A_{n-1}| = k_n - 1 = |A_n \setminus A_{n-1}| + |-A_{n-1}| - 1$$
$$\Leftrightarrow \ x_{i+1} - x_i = y_{j+1} - y_j \quad \text{for all } i \in [1, n) \text{ and } j \in [1, k_n - k_{n-1}). \ \blacksquare$$

The following lemma is an improvement on Lemma 2 of [CS].

LEMMA 3.2. *Let* A_1 *and* A_2 *be finite subsets of* $\mathbb{Z}$ *with* $|A_1| \geq 3$, $A_1 \subset A_2$, $\min A_1 = \min A_2$, $\max A_1 \neq \max A_2$ *and* $|S(A_1, A_2)| = |A_1| + |A_2| - 2$. *Then* $A_2 \in \mathrm{AP}$ *unless* $|A_1| = 3$ *and* A_2 *can be written in the form* (1.15).

Proof. Let $A_1 = \{a_1, \ldots, a_k\}$ and $A_2 = \{b_1, \ldots, b_l\}$ where $a_1 < \ldots < a_k$ and $b_1 < \ldots < b_l$. By the proof of Lemma 2 of [CS], $a_i \in \{b_i, b_{i+1}\}$ for all $i \in [1, k]$,

$$S(A_1, A_2) = \{a_1 + b_2, \ldots, a_1 + b_{l-1}, a_1 + b_l, \ldots, a_k + b_l\},$$

and $A_2 \in \mathrm{AP}$ if $a_3 < b_{l-1}$.

Suppose that $a_3 = b_{l-1}$. Then $k = 3$ since $a_3 \leq a_k < b_l$. As $a_1 + b_{l-1} < a_2 + b_{l-1} < a_2 + b_l$, we must have $a_2 + b_{l-1} = a_1 + b_l$, i.e. $b_l - b_{l-1} = a_2 - a_1$. If $a_3 = b_3$, then $l = 4$, $a_2 = b_2$ and hence $b_4 - b_3 = b_2 - b_1$, so A_2 is of the form (1.15). Below we let $a_3 = b_4$. Then $l = 5$ and $b_5 - b_4 = a_2 - a_1$. As $a_1 + b_4 < a_3 + b_2 = b_4 + b_2 \leq a_2 + b_4 = a_1 + b_5$, we must have $a_2 = b_2 < b_3$. Observe that

$$a_1 + b_3 < a_2 + b_3 < a_2 + b_4 = a_1 + b_5 < a_3 + b_3 < a_3 + b_5.$$

So $a_2 + b_3 = a_1 + b_4$ and $a_3 + b_3 = a_2 + b_5$, therefore $A_2 \in \mathrm{AP}$. ■

We now present a lemma reflecting some symmetry.

LEMMA 3.3. *Let $A_1, \ldots, A_n$ be finite subsets of $\mathbb{Z}$ with $A_1 = \ldots = A_m \subseteq A_{m+1} = \ldots = A_n$ and $0 < |A_m| - m \le |A_n| - n$ where $m \in [1, n]$. Define the dual sequence $\{B_j\}_{j=1}^{|A_n|-n}$ of $\{A_i\}_{i=1}^n$ as follows:*

$$B_i = A_n \setminus A_m \quad \text{for each } i \in [1, |A_n| - n - (|A_m| - m)]$$

and

$$B_j = A_n \quad \text{for all } j \in (|A_n| - n - (|A_m| - m), |A_n| - n].$$

Then $|S(\{A_i\}_{i=1}^n)| = |S(\{B_i\}_{i=1}^{|A_n|-n})|$ and

$$\sum_{i=1}^{n} \min_{i \le j \le n} (|A_j| - j) = \sum_{i=1}^{|A_n|-n} \min_{i \le j \le n} (|B_j| - j).$$

Proof. Let $k_m = |A_m|$ and $k_n = |A_n|$. Suppose that $A_m = \{x_1, \ldots, x_{k_m}\}$ and $A_n \setminus A_m = \{y_1, \ldots, y_{k_n - k_m}\}$. Then $S(\{A_i\}_{i=1}^n)$ consists of integers of the form $\sum_{i\in I} x_i + \sum_{j\in J} y_j$ where $I \subseteq [1, k_m]$, $J \subseteq [1, k_n - k_m]$, $|I| + |J| = n$ and $|I| \ge m$, in other words the elements of $S(\{A_i\}_{i=1}^n)$ are integers of the form

$$\sum_{i=1}^{k_m} x_i - \sum_{i\in \bar{I}} x_i + \sum_{j=1}^{k_n-k_m} y_j - \sum_{j\in \bar{J}} y_j$$

where $\bar{I} \subseteq [1, k_m]$, $\bar{J} \subseteq [1, k_n - k_m]$, $|\bar{I}| + |\bar{J}| = k_m + (k_n - k_m) - n = k_n - n$ and $|\bar{J}| \ge k_n - k_m - (n - m) = k_n - n - (k_m - m)$. Thus

$$S(\{A_i\}_{i=1}^n) = \sum_{x\in A_n} x - S(\{B_i\}_{i=1}^{k_n-n})$$

and so

$$|S(\{A_i\}_{i=1}^n)| = |S(\{B_i\}_{i=1}^{k_n-n})|.$$

Clearly

$$\sum_{i=1}^{n} \min_{i\le j\le n} (|A_j| - j) = m(k_m - m) + (n - m)(k_n - n).$$

Also,

$$\begin{aligned}
&\sum_{i=1}^{k_n-n} \min_{i\le j\le n} (|B_j| - j) - (k_m - m)(|A_n| - (k_n - n)) \\
&\qquad = (k_n - n - (k_m - m))(|A_n \setminus A_m| - (k_n - n - (k_m - m))) \\
&\qquad = (n - m)(k_n - n) + (m - n)(k_m - m).
\end{aligned}$$

This concludes the proof. ■

Let $A_1 \subseteq A_2 \subseteq \mathbb{Z}$, $|A_1| = 3$ and $|A_2| = 4$. Then the dual sequence of $\{A_i\}_{i=1}^2$ is the sequence A_2, A_2. Thus the example (given by Nathanson) with $|2^\wedge A_2| = 2|A_2| - 2^2 + 1$ and $A_2 \notin \mathrm{AP}$, induces the exception (ii) in Theorem 1.2.

4. Reduction of Theorem 1.2. Let $A_1, \ldots, A_n$ be finite subsets of $\mathbb{Z}$ with (1.2) and $|A_1| \le \ldots \le |A_n|$. Put $d_i = \min_{i \le j \le n}(|A_j| - j)$ and $k'_i = d_i + i$ for $i = 1, \ldots, n$. Clearly $k'_n = |A_n|$ and $k'_i < k'_{i+1}$ for all $i \in [1, n)$. As $k'_i \le |A_i|$, we can choose a subset A'_i of A_i with $|A'_i| = k'_i$. Obviously $A'_n = A_n$ and $\sum_{i=1}^n |A'_i| \le \sum_{i=1}^n |A_i|$. By the Theorem of Cao and Sun [CS], we have

$$|S(\{A_i\}_{i=1}^n)| \ge |S(\{A'_i\}_{i=1}^n)| \ge 1 + \sum_{i=1}^n (k'_i - i) = 1 + \sum_{i=1}^n d_i.$$

So (1.13) holds. If equality is valid in (1.13), then

$$|S(\{A'_i\}_{i=1}^n)| = 1 + \sum_{i=1}^n (k'_i - i),$$

hence by the Theorem of [CS] we have $\bigcup_{i=1}^m A'_i = A'_m \subseteq A_m$ for any m in the set

$$\begin{aligned} M &= \{1 \le i < n : k'_i < k'_{i+1} - 1\} \cup \{n\} = \{1 \le i \le n : d_i < d_{i+1}\} \cup \{n\} \\ &= \{1 \le i \le n : |A_i| - i < |A_j| - j \text{ for all } j \in (i, n]\}. \end{aligned}$$

For any $i = 1, \ldots, n$, if $a_i \in A_i$ then we can select $A'_i \subseteq A_i$ so that $a_i \in A'_i$ and $|A'_i| = k'_i$. Thus, in the equality case of (1.13) we have $\bigcup_{i=1}^m A_i \subseteq A_m$ for all $m \in M$.

Let $1 \le i \le n$. Then

$$k'_i > i \ \Leftrightarrow\ d_i > 0 \ \Leftrightarrow\ |A_j| > j \text{ for all } j \in [i, n].$$

Thus

$$|A_i| > i \text{ for all } i \in [1, n] \ \Leftrightarrow\ |A'_i| > i \text{ for all } i \in [1, n].$$

Recall that $A'_n = A_n$. When $n = 2$ and $A'_2 = A_2$ is of the form (1.15), clearly

$$|A_1| \in \{3, 4\} \ \Leftrightarrow\ |A_1| - 1 \ge |A_2| - 2 \ \Leftrightarrow\ d_1 = 2 \ \Leftrightarrow\ k'_1 = 3.$$

In the case $n > 1$ and $|A_n| > n$, we have

$$|A_{n-1}| = n \ \Leftrightarrow\ d_{n-1} = 1 \ \Leftrightarrow\ k'_{n-1} = n,$$

thus $A_{n-1} = A'_{n-1}$ providing $|A_{n-1}| = n$ or $k'_{n-1} = n$.

In view of the above and Lemma 3.1, Theorem 1.2 can be reduced to the following

Theorem 4.1. *Let $A_1, \dots, A_n$ be subsets of $\mathbb{Z}$ with $|A_1| < \dots < |A_n| < \infty$ and $|A_i| > i$ for all $i = 1, \dots, n$. If*

$$|S(\{A_i\}_{i=1}^n)| = 1 + \sum_{i=1}^n (|A_i| - i), \tag{4.1}$$

then $A_n \in \mathrm{AP}$ unless we have (i) *or* (iii), *or* (ii) *with $|A_1| = 3$.*

Remark 4.1. Let k be a positive integer. By the previous reasoning, if Theorem 4.1 holds for those subsets $A_1, \dots, A_n$ of $\mathbb{Z}$ with $|A_1| + \dots + |A_n| \le k$, then so does Theorem 1.2.

5. Proof of Theorem 4.1. We proceed by induction on $k = \sum_{i=1}^n |A_i|$. Apparently $k \ge |A_1| > 1$.

If $k = 2$, then $n = 1$ and $|A_1| = 2$. In the case $n = 1$, both (4.1) and (i) hold.

Below we let $k > 2$ and $n \ge 2$, and assume that the result holds if $|A_1| + \dots + |A_n| < k$. Now let $|A_1| + \dots + |A_n| = k$. For all $i \in [1, n]$ we set

$$k_i = |A_i| \quad \text{and} \quad d_i = \min_{i \le j \le n} (k_j - j) = k_i - i. \tag{5.1}$$

Obviously $1 \le d_1 \le \dots \le d_n$. Put

$$a = \min \bigcup_{i=1}^n A_i, \quad I = \{1 \le i \le n : a \in A_i\}, \quad r = \min I, \quad t = \max I. \tag{5.2}$$

For $i \in I$ let

$$A_i' = \begin{cases} A_i \setminus \{a\} & \text{if } i \ne r, \\ \{a\} & \text{if } i = r; \end{cases} \tag{5.3}$$

and for $i \in \bar{I} = [1, n] \setminus I$ set

$$A_i' = \begin{cases} A_i \setminus \{a_i\} & \text{if } r < i < t \text{ and } i \notin M, \\ A_i & \text{otherwise,} \end{cases} \tag{5.4}$$

where a_i is an arbitrary element of A_i. Write $k_i' = |A_i'|$ for $i \in [1, n] \setminus \{r\}$. Then $1 < k_1' < \dots < k_{r-1}' < k_r \le k_{r+1}' < \dots < k_n'$ and $\sum_{i \ne r} k_i' < \sum_{i=1}^n k_i = k$. For $i \in [1, n] \setminus \{r\}$ we set

$$d_i' = \begin{cases} k_i' - i & \text{if } i < r, \\ k_i' - (i - 1) & \text{if } i > r. \end{cases} \tag{5.5}$$

Let $S = S(\{A_i\}_{i=1}^n)$, and assume that (4.1) holds. By the Theorem of [CS] and its proof, $\bigcup_{i=1}^m A_i = A_m$ for all $m \in M$, and

$$|S(\{A_i'\}_{i \ne r})| = \sum_{i \ne r} k_i' - \frac{n(n-1)}{2} + 1 = 1 + \sum_{i \ne r} d_i'.$$

Also $t = n$ and $(r, t) \cap \bar{I} \cap M = \emptyset$ (see (12) and (14) of [CS]), therefore $k'_i = k_i - 1$ for $i \in (r, n]$ and $d'_i = d_i$ for all $i \in [1, n] \setminus \{r\}$.

Clearly $b = \max \bigcup_{i=1}^n A_i \neq a$ (otherwise $|A_n| = |\{a\}| < n$), $-b = \min \bigcup_{i=1}^n (-A_i)$ and

$$|S(\{-A_i\}_{i=1}^n)| = |S| = 1 + \sum_{i=1}^n \min_{i \le j \le n} (|-A_j| - j).$$

Like the fact that $a \in A_t = A_n$ we should also have $-b \in -A_n$. Thus $b \in A_n \setminus \{a\}$.

Let s denote the least index such that $b \in A_s$. By p. 166 of [CS], there exists an $l \in [r, n]$ such that $k_l - l = k_r - r$ (i.e. $d_r = \ldots = d_l$), and $l = s = r < n$ is impossible.

From now on we assume that none of (i)–(iii) (in Theorem 1.2) holds. Then $k_n > n + 1$. If $k_{n-1} = n$, then $n - 1 \in M$ and $\bigcup_{i=1}^{n-1} A_i = A_{n-1} \subseteq A_n$, thus (iii) holds by Lemma 3.1. Now that (iii) fails, we must have $k_{n-1} > n$.

We claim that $A_n^* = A_n \setminus \{a\} \in \mathrm{AP}$. For this conclusion, it suffices to work under the condition $A_n^* \notin \mathrm{AP}$.

Case 1. $r < n-1$. Apparently $n > 2$, $k'_n = k_n - 1 > n = (n-1)+1$ and $k'_{n-1} = k_{n-1} - 1 > n - 1 = (n-2)+1$. As $A'_n = A_n^* \notin \mathrm{AP}$, by the induction hypothesis, $n - 1 = 2$, $r = 1$, $k'_2 = 3$ and $A'_3 = A_3 \setminus \{a\}$ is of the form (1.15). Note that $k_2 = k'_2 + 1 = 4$ and $k_3 = k'_3 + 1 = 5$. If $k_1 > 2$, then $k_1 = 3$ and $M = \{3\}$, hence $S = 3^\wedge A_3$ and $A_3 \in \mathrm{AP}$ by Example 1.2. Thus $k_1 = 2$, $k_2 = 4$ and $k_3 = 5$. Observe that $|S| = 1 + (2-1) + (4-2) + (5-3) = 6$. If $1 \le i < j \le 4$, then x_i or x_j lies in A_2 (since $A_2 \subseteq A_3$ and $k_3 - k_2 = 1$), therefore $a + x_i + x_j \in S$. Thus S contains the following 5 integers:

$$a+x_1+x_2,\ a+x_1+x_3,\ a+x_1+x_4 = a+x_2+x_3,\ a+x_2+x_4,\ a+x_3+x_4.$$

Suppose that $A_1 = \{a, x_i\}$ where $1 \le i \le 4$. If $i \in \{3, 4\}$, then both $x_4 + x_3 + x_1$ and $x_4 + x_3 + x_2$ belong to S, this contradicts the fact that $|S| = 6 < 5+2$. So $i \in \{1, 2\}$, and S consists of the above 5 integers and the number $x_i + x_3 + x_4$. Apparently S also contains $x_1 + x_2 + x_3$ and $x_1 + x_2 + x_4$. Since $a + x_2 + x_3 < x_1 + x_2 + x_3 < x_1 + x_2 + x_4 < x_i + x_3 + x_4$, we must have $x_1 + x_2 + x_3 = a + x_2 + x_4$ and $x_1 + x_2 + x_4 = a + x_3 + x_4$. Thus $x_4 - x_3 = x_1 - a = x_3 - x_2$ and hence $A_n = A_3 \in \mathrm{AP}$.

Case 2. $A_{n-1} \subset A_n^*$. As $n - 1 \in M$, $a \notin A_{n-1} = \bigcup_{i=1}^{n-1} A_i$ and so $r = n$. Clearly $k_1 < \ldots < k_{n-1} < k_n^* = |A_n^*| = k_n - 1$. Let S^* denote the set $S(A_1, \ldots, A_{n-1}, A_n^*)$. Then $a + \min S(\{A_i\}_{i=1}^{n-1}) = \min S < \min S^*$. So $|S^*| \le |S| - 1 = \sum_{i=1}^n (k_i - i)$ and hence $|S^*| = |S| - 1 = 1 + \sum_{i=1}^{n-1} (k_i - i) + (k_n^* - n)$.

Recall that $k_n^* = k_n - 1 > k_{n-1} \ge n + 1$. By the induction hypothesis, $n = 2$, $k_1 = 3$, A_2^* has the form (1.15) and hence $k_2 = 5$. For any two distinct elements x and y of A_2^* we have $x+y \in S^*$ since one of them belongs to A_1. All the $1+(3-1)+(4-2)=5$ elements of S^* are as follows:

$$x_1+x_2,\ x_1+x_3,\ x_1+x_4=x_2+x_3,\ x_2+x_4,\ x_3+x_4.$$

As $|a+A_1|=3$, $\max(a+A_1)<x_1+x_4$ and $|S|=1+(3-1)+(5-2)=6$, we must have

$$S=(a+A_1)\cup\{x_i+x_4 : i=1,2,3\}.$$

Evidently $x_4 \in A_1$ and $x_1+x_3=a+x_4$ since $x_1+x_3 \in a+A_1$, also $x_3 \in A_1$ and $x_1+x_2=a+x_3$ since $x_1+x_2 \in a+A_1$. So $x_4-x_3=x_1-a=x_3-x_2$ and hence $A_n=A_2 \in \mathrm{AP}$.

CASE 3. $r=n-1$, or $r=n$ and $A_{n-1}=A_n^*$. Let $\bar{r}=n$ if $r=n-1$, and $\bar{r}=n-1$ if $r=n$. Clearly $A'_{\bar{r}}=A_n^*$ and $k'_{\bar{r}}=|A_n^*|=k_n-1>n=(n-1)+1$.

Let us handle the case $n=2$. Note that $k_1=k_{n-1}>n=2$. If $A_1=A_2^*$, then $\min(-A_1)=\min(-A_2)$ and $\max(-A_1)<\max(-A_2)=-a$, hence $-A_2 \in \mathrm{AP}$ (i.e. $A_2 \in \mathrm{AP}$) by Lemma 3.2 since (ii) fails. When $r=1$, we have $\min A_1=\min A_2$, if $s=2$ (i.e. $\max A_1 \neq \max A_2$) then $A_2 \in \mathrm{AP}$ by Lemma 3.2. In the case $r=s=1$, we have $l>1$ because $l=r=s<n$ is impossible, hence $k_1=k_2-1$ since $k_r-r=k_l-l$, thus $S=2^\wedge A_2$ and $A_2 \in \mathrm{AP}$ by Example 1.2. (Recall that (ii) fails.)

Let $n-1=2$, $k_1=k'_1=3$ and $A'_{\bar{r}}$ have the form (1.15). Observe that $n=3<k_{n-1}=k_2\le k_3-1=|A_3^*|=|A'_{\bar{r}}|=4$. So $M=\{3\}$ and hence $A_3 \in \mathrm{AP}$ by Example 1.2.

Now we assume that $n>2$, and $n\neq 3$ or $k'_1 \neq 3$ or $A'_{\bar{r}}$ is not of the form (1.15). As $A'_{\bar{r}}=A_n^* \notin \mathrm{AP}$, by the induction hypothesis, $k_{n-2}=k'_{n-2}=n-1$, also $A_{n-2}=A'_{n-2}$ and $A_n^*\setminus A_{n-2}=A'_{\bar{r}}\setminus A'_{n-2}$ form arithmetic progressions with the same difference d. Since $k_{n-2}=n-1<n<k_{n-1}$, we have $n-2\in M$ and hence $\bigcup_{i=1}^{n-2}A_i=A_{n-2}\subseteq A_n^*$. Let $A_{n-1}^*=A_{n-1}\setminus\{a\}$, $k_{n-1}^*=|A_{n-1}^*|$ and $S^*=S(A_1,\ldots,A_{n-2},A_{n-1}^*,A_n^*)$. Then

$$\begin{aligned}
&1<k_1<\ldots<k_{n-2}=n-1<k_{n-1}^*\le k_n^*<k_n,\\
&d_n^*=k_n^*-n=k_n-1-n=d_n-1>0,\\
&d_{n-1}^*=\min\{k_{n-1}^*-(n-1),k_n^*-n\}=k_{n-1}-n=d_{n-1}-1>0,\\
&d_i^*=\min\{k_i-i,\ldots,k_{n-2}-(n-2),d_{n-1}^*\}=1=d_i \text{ for } i\in[1,n-2].
\end{aligned}$$

Write $A_{n-2}=\{x_1,\ldots,x_{n-1}\}$ and $A_n^*\setminus A_{n-2}=\{y_1,\ldots,y_{k_n-1-(n-1)}\}$ where $x_1<\ldots<x_{n-1}$ and $y_1<\ldots<y_{k_n-n}$. In view of Example 1.2, $S(\{A_i\}_{i=1}^{n-2})$ $=(n-2)^\wedge A_{n-2}=\{x-x_i : 1\le i\le n-1\}$ where $x=\sum_{i=1}^{n-1}x_i$. As $A_{n-1}^*\subseteq A_n^*$ all elements of S^* have the form $x-x_i+y_j+z$ where $1\le i\le n-1$, $1\le j\le k_n-n$ and $z\in\{x_i,y_1,\ldots,y_{k_n-n}\}\setminus\{y_j\}$, they are all greater than $x-x_{n-1}+y_1+a$. If $x-x_{n-1}+y_2+a=x-x_i+y_j+z$ where i,j,z are as above, then $j=1$ and $z=x_i$ since $a+y_2<\min\{x_i+y_2,y_1+y_2\}$, hence $-x_{n-1}+y_2+a=-x_i+y_1+x_i=y_1$ and $x_{n-1}-a=y_2-y_1=d=x_{n-1}-x_{n-2}$;

this is impossible. So $x - x_{n-1} + y_1 + a, x - x_{n-1} + y_2 + a \notin S^*$. However, both $x - x_{n-1} + y_1 + a$ and $x - x_{n-1} + y_2 + a$ lie in S, for, $a \in A_{n-1}$ if $r = n-1$, and $y_1, y_2 \in A_{n-1}$ if $A_{n-1} = A_n^*$. Therefore

$$|S^*| \le |S| - 2 = 1 + \sum_{i=1}^{n} d_i - 2 = 1 + \sum_{i=1}^{n} d_i^*.$$

If $A_{n-1} = A_n^*$, then $k_{n-1}^* = k_{n-1} > n$. Since $A_n^* \notin \mathrm{AP}$, by Remark 4.1 and the induction hypothesis we have either

(i*) $k_n - 1 = k_n^* = n+1$ and hence $k_{n-1} = n+1$, or

(iii*) $|A_{n-1}^*| = n$ (whence $r = n-1$), and A_{n-1}^* and $A_n \setminus A_{n-1} = A_n^* \setminus A_{n-1}^*$ form arithmetic progressions with the same difference.

Assume (i*). Let $B_1 = \ldots = B_{n-2} = A_{n-2}$ and $B_{n-1} = B_n = A_n$. As $M = \{n-2, n\}$, by the idea in Example 1.2 or the proof of Corollary 1.6, $S = S(\{B_i\}_{i=1}^n)$ and $|S(\{B_i\}_{i=1}^n)| = 1 + \sum_{i=1}^n \min_{i \le j \le n}(|B_j| - j)$. The dual sequence of $\{B_i\}_{i=1}^n$ is the sequence $A_n \setminus A_{n-2}$, A_n with $|A_n \setminus A_{n-2}| = n+2-(n-1) = 3$, $|A_n| = n+2 > 4$ and $|A_n \setminus A_{n-2}| + |A_n| < (n+1) + k_n \le k = k_1 + \ldots + k_n$. In view of Lemma 3.3 and the induction hypothesis, we have $A_n \in \mathrm{AP}$.

Now we consider the case (iii*). Clearly $k_{n-1} = n+1$ and $k_n - k_{n-1} \ge 2$, so $n-1 \in M$ and $A_{n-2} \subset A_{n-1} \subset A_n$. Write $A_{n-1} = \{a, x_1, \ldots, x_{n-1}, y_j\}$ where $1 \le j \le k_n - n$. Then $A_n \setminus A_{n-1} = \{y_1, \ldots, y_{k_n - n}\} \setminus \{y_j\}$. Since $d(A_{n-1}^*) = d(A_n \setminus A_{n-1}) \ge d$, we must have $y_j \in \{x_1 - d, x_{n-1} + d\}$. Now that $d(A_n \setminus A_{n-1}) = d(A_{n-1}^*) = d$, j must be 1 or $k_n - n$. If $y_1 \in A_{n-1}$ (i.e. $j = 1$), then $y_1 + d = y_2 \ne x_1$ and hence $y_1 = x_{n-1} + d$, thus $A_n^* = \{x_1, \ldots, x_{n-1}, y_1, \ldots, y_{k_n-n}\} \in \mathrm{AP}$. If $y_{k_n-n} \in A_{n-1}$ (i.e. $j = k_n - n$), then $y_{k_n-n} - d = y_{k_n-n-1} \ne x_{n-1}$ and hence $y_{k_n-n} = x_1 - d$, thus $A_n^* = \{y_1, \ldots, y_{k_n-n}, x_1, \ldots, x_{n-1}\} \in \mathrm{AP}$.

By the above, we do have $A_n \setminus \{a\} \in \mathrm{AP}$ in either case. As $-b = \min \bigcup_{i=1}^n (-A_i)$, by analogy $-A_n \setminus \{-b\} \in \mathrm{AP}$. Because $k_n > n+1 \ge 3$, and $A_n \setminus \{\min A_n\}$ and $A_n \setminus \{\max A_n\}$ are both in AP, the set A_n must form an arithmetic progression.

The induction step is now complete and the proof of Theorem 4.1 is finished.

References

[ANR] N. Alon, M. B. Nathanson and I. Z. Ruzsa, *The polynomial method and restricted sums of congruence classes*, J. Number Theory 56 (1996), 404–417.

[B] Y. Bilu, *Addition of sets of integers of positive density*, ibid. 64 (1997), 233–275.

[CS] H. Q. Cao and Z. W. Sun, *On sums of distinct representatives*, Acta Arith. 87 (1998), 159–169.

[DH] J. A. Dias da Silva and Y. O. Hamidoune, *Cyclic space for Grassmann derivatives and additive theory*, Bull. London Math. Soc. 26 (1994), 140–146.

[EH] P. Erdős and H. Heilbronn, *On the addition of residue classes* mod p, Acta Arith. 9 (1964), 149–159.

[G] R. K. Guy, *Unsolved Problems in Number Theory*, 2nd ed., Springer, New York, 1994, 129–131.

[N1] M. B. Nathanson, *Inverse theorems for subset sums*, Trans. Amer. Math. Soc. 347 (1995), 1409–1418.

[N2] —, *Additive Number Theory*: *Inverse Problems and the Geometry of Sumsets*, Grad. Texts in Math. 165, Springer, New York, 1996.

Department of Mathematics
Nanjing University
Nanjing 210093
The People's Republic of China
E-mail: zwsun@nju.edu.cn

Received on 5.4.2000
and in revised form on 17.10.2000 (3800)