

# Restricted Tangent Bundle of Space Curves

G. Hein, H. Kurke

April 4, 2003

## Contents

1	Introduction	1
2	The generalized Harder-Narasimhan polygon and the Shatz stratification	2
3	The semistability of the restricted tangent bundle	4
4	A result of R. Hernández	8

## 1 Introduction

The purpose of this paper is to investigate the restriction of the tangent bundle of  $\mathbb{P}^n$  to a curve  $X \subset \mathbb{P}^n$ . The corresponding question for rational curves was investigated by L. Ramella [7] and F. Ghione, A. Iarrobino and G. Sacchiero [2] in the case of rational curves. Let us also mention that D. Laksov [6] proved that the restricted tangent bundle of a projectively normal curve does not split unless the curve is rational. We will show the following theorem (See 3.1):

**Theorem** *In the variety of smooth connected space curves of genus  $g \geq 1$  and degree  $d \geq g + 3$  there exists a nonempty dense open subset where the restricted tangent bundle is semistable and moreover simple if  $g \geq 2$*

If the degree is high with respect to the genus ( $d > 3g$ ), we get a postulation formula for the strata with a given Harder-Narasimhan polygon, following results of R. Hernández [5].

In case of plane curves the situation is simpler due to

**Theorem** *If  $X$  is a smooth plane curve of degree  $d$ , the restricted tangent bundle is stable for  $d \geq 3$ , of splitting type  $(3, 3)$  for a conic, and of splitting type  $(2, 1)$  for a line.*

*Proof:* (following D. Huybrechts) We denote by  $E$  the tangent bundle of  $\mathbb{P}^2$  twisted by  $\mathcal{O}_{\mathbb{P}^2}(-1)$ . We first suppose that  $d > 2$ . We use the facts:

1.  $E$  is stable,  $c_1(E) = 1$  and  $c_2(E) = 1$ .
2. If  $E|_X$  is unstable, then we have a destabilizing quotient  $E|_X \rightarrow L$ . We define  $F = \ker(E \rightarrow E|_X \rightarrow L)$  and obtain a bundle  $F$  of rank 2 with  $\Delta(F) = c_1(F)^2 - 4c_2(F) \geq d^2 - 3 > 0$ .

3. By Bogomolov's inequality the bundle  $F$  is not semistable and if  $M \subset F$  is a subbundle of maximal degree and rank 1, then  $\deg(M) \geq 1$ , which contradicts the semistability of  $E$ .

Property 1 follows from the Euler sequence.

For property 2 we use

$$\begin{aligned} c_1(F) &= c_1(E) - [X] \\ c_2(F) &= c_2(E) + \deg(L) - c_1(E) \cdot [X] \quad \text{hence:} \\ \Delta(F) &= \Delta(E) + [X] \cdot [X] + 2c_1(E) \cdot [X] - 4\deg(L) \\ &= -3 + d^2 + 2(d - 2\deg(L)) \\ &\geq d^2 - 3 \end{aligned}$$

Property 3 follows because  $F/M \cong I \otimes M'$ , where  $I$  is the sheaf of ideals of a 0-dimensional subscheme. We set  $l = \text{length}(\mathcal{O}_{\mathbb{P}^2}/I)$  and obtain:

$$\begin{aligned} c_1(F) &= c_1(M) + c_1(M') \\ c_2(F) &= c_1(M)c_1(M') - l \quad \text{consequently:} \\ \Delta(F) &= [c_1(M) - c_1(M')]^2 + 4l \\ &= [2c_1(M) - c_1(F)]^2 + 4l, \end{aligned}$$

hence  $[2\deg(M) - 1 + d]^2 \geq \Delta(F) \geq d^2 - 3$ .

$M$  is destabilizing, so we must have  $\deg(M) \geq 1$ .

The same proof shows that in case of  $d = 2$  there can not exist a surjection  $E|_X \rightarrow L$  to a linebundle of degree less than one.

For  $d = 1$  the statement is obvious.  $\square$

The main idea we exploit in this papers is to consider degenerations of smooth curves into special reducible curves with ordinary double points and to extend the notion of the Harder-Narasimhan polygon to such curves. This idea was used by L. Ramella [7] in the case of rational curves.

## 2 The generalized Harder-Narasimhan polygon and the Shatz stratification

Let  $E$  be a vector bundle of rank  $r$  on a reduced curve  $X$ , with irreducible components  $X_i$  ( $i = 1, \dots, k$ ). We say a subsheaf  $F \subset E$  is of constant rank  $n$  if  $\text{rk}(F|_{X_i}) = n$  for all  $i = 1, \dots, k$ . In this case we write  $\text{rk}(F) = n$ . We define the function

$$\begin{aligned} f_E : \{0, \dots, r\} &\rightarrow \mathbb{Z} \\ n &\mapsto \sup\{\deg(F) \mid F \subset E \text{ and } \text{rk}(F) = n\} \end{aligned}$$

Then we define the generalized *Harder-Narasimhan polygon* ( $\text{HNP}(E)$ ) of  $E$  as the convex hull of this function.

*Remark* The degree of  $F$  is defined by  $\deg(F) = \chi(F) + n\chi(\mathcal{O}_X)$ . It is obvious that  $f(n) < \infty$  and  $\text{HNP}(E)$  coincides with the Harder-Narasimhan polygon in the case of a smooth curve  $X$ .

**Theorem 2.1** *Let  $X \subset \mathbb{P}^n \times S$  be a flat family of reduced curves over  $S$ , and  $\mathcal{E}$  an  $X$  vector bundle of rank  $r$ . Then the map*

$$\begin{aligned} \text{HNP} : S &\longrightarrow \text{Polygons} \\ s &\mapsto \text{HNP}(\mathcal{E}_s) \end{aligned}$$

defines a finite and locally closed stratification on  $S$ , the so-called Shatz stratification.

The theorem follows from 2.2 and 2.3, since by 2.2 there are only finitely many polygons in the image of HNP above a given  $P$  and by 2.3 the set  $\{s \mid \text{HNP}(E_s) \geq P\}$  is therefore a closed set.

**Lemma 2.2** *There exists an integer  $M$ , such that  $f_{E_s}(n) < M$  for all  $s \in S$  and any  $n \in \{1, \dots, r\}$*

*Proof:* We assume  $S$  to be connected, then  $\chi(\mathcal{O}_{X_s})$  is constant. The function  $h^0 : S \rightarrow \mathbb{Z}$  assigning every  $s \in S$  the dimension of  $H^0(E_s)$  is upper semicontinuous.  $S$  is assumed to be a noetherian scheme, so there exists an upper bound  $M_1$  of  $h^0$ . Now, for any  $s \in S$  and  $F \subset E_s$  of rank  $n$  we have:

$$\begin{aligned} \deg(F) &= \chi(F) + n\chi(\mathcal{O}_{X_s}) \\ &\leq h^0(F) + n\chi(\mathcal{O}_{X_s}) \\ &\leq h^0(E_s) + n\chi(\mathcal{O}_{X_s}) \\ &\leq M_1 + r|\chi(\mathcal{O}_{X_s})|. \end{aligned}$$

We set  $M = M_1 + r|\chi(\mathcal{O}_{X_s})| + 1$ .  $\square$

**Lemma 2.3** *Under the assumptions made above for any  $\nu$   $0 \leq \nu \leq r$  the function  $f : S \rightarrow \mathbb{Z} \quad s \mapsto f_{E_s}(\nu)$  is upper semicontinuous.*

*Proof:* We suppose  $S$  to be irreducible. We have to show that the subset  $S_k = \{s \in S \mid f_{E_s}(\nu) \geq k\}$  is closed.

Let  $Q$  be the Quot scheme over  $S$  parametrizing quotients of  $E$  with Hilbert-polynomial  $\chi(n) = \chi(E) - k + \nu(\chi(\mathcal{O}_X))$ . The image of the natural morphism  $\psi : Q \rightarrow S$  is closed. This would be enough in case of a family  $X$  of integral schemes.

Assume now that  $s \in S$  is in  $\text{im}(\psi)$  but not in  $S_k$ . The problem occurring is that we might have different ranks over the irreducible components and we have to show that the quotient is not flatly smoothable to one of constant rank over all components. We will do this by the choice of a divisor which meets the quotient sheaf at every irreducible component in at least one point where this quotient is locally free.

Let  $X_s = X_1 \cup X_2 \cup \dots \cup X_m$  be an irreducible decomposition of  $X_s$  and  $E = \mathcal{E}_s$  the vector bundle on  $X_s$ .

We remark that any sheaf  $F$  on  $X_s$  which is a quotient of  $E$  has less than  $N := \chi(F) + h^1(E) + 1$  torsion points (i.e.  $\#\text{supp}(\text{tors}(F)) < N$ ). Otherwise  $F' = F/\text{tors}(F)$  would be a quotient of  $E$  with  $\chi(F') < -h^1(E)$ , which is impossible.

Now we choose a hypersurface  $H \subset \mathbb{P}^n$  which intersects  $X_s$  transversally and meets all irreducible components  $X_i$   $i = 1 \dots m$  at least  $N$ -times. We may assume (after a restriction to a smaller open subset, if necessary) that this property holds for all points of  $S$ . We now get a semicontinuous function  $R : Q \rightarrow \mathbb{Z}$ , assigning the minimum of the embedding dimensions of  $F_t$  at points of  $X_t \cap H$  to every quotient  $F_t$  of  $E_t$ . Thus the subset  $\{s' \in Q \mid \text{all } R(s') \geq r - \nu\}$  of  $Q$  is closed, hence its image in  $S$  is closed.

However,  $s$  can not be in the image because a quotient  $F$  of  $E_s$  with the Hilbert polynomial  $\chi$  must have a rank less than  $r - \nu$  at one component  $X_s$ . Therefore its embedding dimension in at least one point of  $X_s \cap H$  is less than  $r - \nu$ .  $\square$

Proceeding by the same method we obtain:

**Theorem 2.4** *Under the same assumptions as in 2.1 we have: The set of points  $s \in S$  where  $E_s$  is stable (respectively semistable) is open.*

### 3 The semistability of the restricted tangent bundle

We define  $\text{Hilb}(d, g)$  to be the Hilbert scheme of closed subschemes  $X \subset \mathbb{P}^3$  with Hilbert polynomial  $\chi(\mathcal{O}_X(n)) = dn + 1 - g$ . By  $\text{Hilb}_0(d, g)$  we define those quotients which are smooth irreducible curves. In [1] it is proved that  $\text{Hilb}_0(d, g)$  is irreducible if  $d > g + 2$ . Over the Hilbert scheme we have the universal curve  $C(d, g)$ , a closed subscheme of  $\text{Hilb}(d, g) \times \mathbb{P}^3$ . We consider the projection  $\pi_2 : C(d, g) \rightarrow \mathbb{P}^3$  and the tangent sheaf  $\Theta(-1)$  of the projective space twisted with  $\mathcal{O}_{\mathbb{P}^3}(-1)$ . This defines a bundle  $E = \pi_2^* \Theta(-1)$ . For any point  $s \in \text{Hilb}(d, g)$  the sheaf  $E_s$  is the restriction of  $\Theta(-1)$  to the curve parametrized by  $s$ .

**Theorem 3.1** *If  $g \geq 1$  and  $d > g + 2$ , then for a general  $s \in \text{Hilb}_0(d, g)$  the vector bundle  $E_s$  is semistable.*

The proof of the theorem divides into three steps:

*Step 1:* We show that the statement is true for  $g = 1$  and  $d \geq 4$ . (See 4.5)

*Step 2:* Under the assumption that in  $\text{Hilb}_0(d, g)$  there exists a point  $s$  parametrizing a smooth curve  $Y$  with:

- $E_s$  is semistable
  - $H^1(\mathcal{O}_Y(1)) = 0$
  - $E_s$  is not isomorphic to a direct sum of two vector bundles,
- we show that there exists a point  $t$  in  $\text{Hilb}(d + 1, g + 1)$  parametrizing a curve  $X$  satisfying:
- $E_t$  is semistable
  - $H^1(\mathcal{O}_X(1)) = 0$
  - $\dim(\text{End}(E_t)) = 1$ .

*Step 3:* The curve  $X$  obtained in the previous step corresponds to a smooth point in  $\text{Hilb}(d + 1, g + 1)$  and is in the closure of  $\text{Hilb}_0(d + 1, g + 1)$ , because of  $H^1(\mathcal{O}_X(1)) = 0$  (see [4] 1.2). However semistability is an open condition (2.4) and, hence holds for an open subset of  $\text{Hilb}_0(d + 1, g + 1)$  too. The same applies to  $\dim(\text{End}(E_t)) = 1$  and  $H^1(\mathcal{O}_X(1)) = 0$  which implies that on a nonempty open subset of  $\text{Hilb}_0(d + 1, g + 1)$  the necessary conditions of step 2 are fulfilled.

The rest of this section is devoted to the proof of the second step.

Let  $X$  be a connected curve with two ordinary double points and two irreducible components  $Y$  and  $Z$  of genus  $g_Y$  and 0 (i.e.  $Z \cong \mathbb{P}^1$ ), which intersect in two points  $P$  and  $Q$ . Then we have an exact sequence:

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_Y \oplus \mathcal{O}_Z \rightarrow k(P) \oplus k(Q) \rightarrow 0$$

Hence  $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Y) - 1$  or  $g_X = g_Y + 1$ .

**Lemma 3.2** *Let  $E$  be a vector bundle of rank  $n$  on  $X$  such that  $E_Y = E \otimes \mathcal{O}_Y$  is semistable of degree  $d$  and  $E_Z = E \otimes \mathcal{O}_Z$  is globally generated and of degree 1. Let  $F$  be a subsheaf of constant rank  $r$  with maximal degree among subsheaves of constant rank  $r$ . If  $F$  is destabilizing then  $F$  is a subbundle of  $E$  and  $F_Z$  is of degree 1.*

*Proof:* If  $F_Y$  and  $F_Z$  are the subbundles of  $E_Y$  and  $E_Z$  generated by the images of  $F$  in  $E_Y$  resp.  $E_Z$ , then

$F \subset \tilde{F} = E \cap (F_Y \times F_Z) \subset E_Y \times E_Z$  and  $\tilde{F}/F$  has finite support. Since  $F$  has maximal degree it follows that  $F = \tilde{F}$  and we obtain an exact sequence

$$0 \rightarrow F \rightarrow F_Y \times F_Z \rightarrow F \otimes \mathcal{O}_D \rightarrow 0$$

(with  $D = P + Q$ , as a subscheme of  $X$ ) If  $F$  is destabilizing then

$$r(\deg(E_Y) + 1) < n(\deg(F_Y) + \deg(F_Z) - [l(F \otimes \mathcal{O}_D) - r \cdot l(\mathcal{O}_D)])$$

$$r(\deg(E_Y) + 1) < r \cdot \deg(E_Y) + n \cdot \deg(F_Z) - n[l(F \otimes \mathcal{O}_D) - r \cdot l(\mathcal{O}_D)]$$

$$\text{hence } r + n[l(F \otimes \mathcal{O}_D) - r \cdot l(\mathcal{O}_D)] < n \cdot \deg(F_Z)$$

Since  $\deg(F_Z) \leq 1$  this implies  $\deg(F_Z) = 1$  and  $l(F \otimes \mathcal{O}_D) = r \cdot l(\mathcal{O}_D)$ , i.e.  $F$  is a subbundle.  $\square$

Let  $V$  be a vector space of dimension 4. From now on we consider a fixed smooth curve  $Y$  of genus  $g_Y \geq 1$  and a quotient  $V \otimes \mathcal{O}_Y \rightarrow E_Y$  for which we suppose:

(i)  $E_Y$  is a semistable vector bundle of rank 3 and degree  $d$ .

(ii)  $E_Y$  is not decomposable.

(iii) The morphism  $Y \rightarrow \mathbb{P}(V^\vee)$  defined by the surjection  $V \otimes \mathcal{O}_Y \rightarrow E_Y$  is an embedding. Therefore we will identify  $Y$  with its image in  $\mathbb{P}(V^\vee)$ .

Given two different points  $P$  and  $Q$  of  $Y$  we denote by  $Z(P, Q)$  the line in  $\mathbb{P}(V^\vee)$  through  $P$  and  $Q$  and by  $X(P, Q)$  the union of  $Y$  and  $Z(P, Q)$ . Again we define  $E_{X(P, Q)}$  to be the restriction of the tangent bundle of  $\mathbb{P}(V^\vee)$  twisted by  $\mathcal{O}_{\mathbb{P}(V^\vee)}(-1)$  to  $X(P, Q)$ . Restricting the Euler sequence to  $X$  gives:

$$\begin{aligned} & \text{a surjection } V \otimes \mathcal{O}_{X(P, Q)} \rightarrow E_{X(P, Q)} \\ & \text{with } E_Y = E_{X(P, Q)} \otimes \mathcal{O}_Y \\ & \text{and } E_{Z(P, Q)} = E_{X(P, Q)} \otimes \mathcal{O}_{Z(P, Q)} \cong \mathcal{O}_Z(1) \oplus \mathcal{O}_Z^{\oplus 2}. \end{aligned}$$

We will show that for two general points  $P$  and  $Q$  of  $Y$  the bundle  $E_{X(P, Q)}$  is semistable. Of course we have to choose  $P$  and  $Q$  such that  $Z(P, Q)$  meets  $Y$  in exactly these two points and, moreover, quasi transversally, i.e. not tangentially. But this is always possible because  $Y$  is not a strange curve. We will call the corresponding line  $Z(P, Q)$  the *bisecant* to  $Y$ , determined by  $P$  and  $Q$ .

We have  $E_{Z(P, Q)} \cong \mathcal{O}_Z(1) \oplus \mathcal{O}_Z^{\oplus 2}$  and therefore a canonic subbundle of rank and degree 1 in  $E_{Z(P, Q)}$ , namely the tangent bundle of  $Z(P, Q)$  twisted with  $\mathcal{O}(-1)$ . This defines a one-dimensional subspace of  $E \otimes k(P)$ , which we denote by  $T(P, Q)$ . We will frequently use the following obvious lemma:

**Lemma 3.3** *Let  $P_0, P_1, P_2, \dots, P_m$  be different points of  $Y$ . Then we have one dimensional subspaces  $T(P_0, P_i)$  of  $E \otimes k(P_0)$  and*

$$\dim\left[\sum_{i=1}^m T(P_0, P_i)\right] = \dim W, \quad \text{where } W \subset \mathbb{P}(V^\vee) \text{ is the linear subspace spanned by the points } P_i.$$

where  $W \subset \mathbb{P}(V^\vee)$  is the linear subspace spanned by the points  $P_i$ .

**Lemma 3.4** *Let  $Y$  and  $E_Y$  be as before and suppose that, for two given points  $P$  and  $Q$ , the bundle  $E_{X(P, Q)}$  is not semistable. Then there exists a subbundle  $F_Y \subset E_Y$  such that:*

$$\begin{aligned} (i) \quad & \mu(F_Y) + \frac{1}{\text{rk}(F_Y)} > \mu(E_Y) + \frac{1}{\text{rk}(E_Y)} \\ (ii) \quad & T(P, Q) \subset F_Y \otimes k(P) \\ (iii) \quad & T(Q, P) \subset F_Y \otimes k(Q). \end{aligned}$$

*Proof:* Let  $F \subset E_{X(P,Q)}$  be a subsheaf with constant rank and  $\mu(F) > \mu(E_{X(P,Q)})$ . Because of 3.2,  $F$  is a subbundle.

We set  $F_Y = F \otimes \mathcal{O}_Y$  and  $F_Z = F \otimes \mathcal{O}_{Z(P,Q)}$ . We have:  
 $\mu(F) = \mu(F_Y) + \mu(F_Z) \geq \mu(E_Y) + \mu(E_Z) = \mu(E)$  hence  $\deg(F_Z) = 1$  and  
therefore:  $\mu(F_Y) + \frac{1}{\text{rk}(F)} \geq \mu(E_Y) + \frac{1}{n}$ ,  $T(P, Q) \subset F \otimes k(P) = F_Y \otimes k(P)$  and  
 $T(Q, P) \subset F \otimes k(Q) = F_Y \otimes k(Q)$ .  $\square$

Analogously we obtain:

**Lemma 3.5** *Let  $Y$  and  $E_Y$  be as above and suppose that, for two given points  $P$  and  $Q$ , the bundle  $E_{X(P,Q)}$  contains a destabilizing subbundle of constant rank 2. Then there exists a subbundle  $F_Y \subset E_Y$  of rank 2 and a plane  $H \subset \mathbb{P}(V^\vee)$  such that:*

$$\begin{aligned} (i) \quad & \mu(F_Y) + \frac{1}{2} > \mu(E_Y) + \frac{1}{\text{rk}(E_Y)} \\ (ii) \quad & P \in H \text{ and } Q \in H \\ (iii) \quad & \Theta_H(-1) \otimes k(P) = F_Y \otimes k(P) \\ (iv) \quad & \Theta_H(-1) \otimes k(Q) = F_Y \otimes k(Q). \quad \square \end{aligned}$$

We denote by  $\Theta_Y(-1)$  the tangent bundle of  $Y$  twisted with  $\mathcal{O}_{\mathbb{P}(V^\vee)}(-1)$ .  $\Theta_Y(-1)$  is a sublinebundle of  $E_Y$  of degree  $2 - 2g_Y - d$ .

**Lemma 3.6** *For any 1-dimensional subspace  $W \subset V$ , corresponding to a point  $P \in \mathbb{P}(V^\vee)$ , we denote by  $L^P \subset E_Y$  the subbundle of  $E_Y$  generated by the image of  $W \otimes \mathcal{O}_Y \rightarrow V \otimes \mathcal{O}_Y \rightarrow E_Y$ . The sublinebundle  $L^P \subset E_Y$  satisfies:*

$$\begin{aligned} (i) \quad & L^P \otimes k(Q) = T(Q, P) \text{ for all points } Q \in Y \text{ with } Q \neq P, \\ (ii) \quad & L^P \otimes k(P) = \Theta_Y(-1) \otimes k(P) \text{ if } P \in Y, \\ (iii) \quad & \deg(L^P) = 0 \text{ if } P \notin Y, \\ & = 1 \text{ if } P \in Y. \end{aligned}$$

*Proof:* obvious

We will now prove that for  $\text{rk}(E_Y) = 3$  and two general points  $P$  and  $Q$  of  $Y$  the bundle  $E_{X(P,Q)}$  has no destabilizing subbundle of rank one or two. Assuming the contrary we will derive the existence of certain subsheaves of  $E_Y$  by using 3.3 and 3.4 which leads to contradictions. The proof splits into three cases, depending on  $\deg(E_Y)$  modulo 3.

**Lemma 3.7** *Let  $E_Y$  be a semistable bundle on  $Y$  of rank 3 and degree  $d = 3k$ . If  $E_Y$  is indecomposable, then  $E_{X(P,Q)}$  has no destabilizing subsheaf of constant rank 1 for two general points  $P$  and  $Q$  of  $Y$ .*

*Proof:* We take four points  $P, Q_1, Q_2, Q_3$  of  $Y$  which span  $\mathbb{P}(V^\vee)$  and define bisecants  $Z(P, Q_i)$  to  $Y$ . If  $E_{X(P,Q_i)}$  were not semistable for  $i = 1, 2, 3$ , we would have 3 linebundles  $L_i \subset E_Y$  with  $\deg(L_i) = k$  and  $L_i \otimes k(P) = T(P, Q_i)$  by 3.4. We define  $E' = L_1 + L_2 + L_3$ . By 3.3  $E'$  is a subsheaf of  $E_Y$  of rank 3 and degree at least  $3k$ . However this would imply that  $E_Y = L_1 \oplus L_2 \oplus L_3$   $\square$

**Lemma 3.8** *Let  $E_Y$  be a semistable and indecomposable vector bundle on  $Y$  of rank 3 and degree  $d = 3k(k > 1)$ . Then  $E_{X(P,Q)}$  has no destabilizing subsheaf of constant rank 2 for two general points  $P$  and  $Q$  of the curve  $Y$ .*

*Proof:* As before we assume that for all pairs  $P, Q$  of points of  $Y$  there exists a subsheaf  $F_Y \subset E_Y$  of rank 2 and degree  $2k$ , see 3.5. We distinguish three cases.

*Case 1:*  $E_Y$  has no sublinebundle of degree  $k$ .

We choose points  $P, Q_1, Q_2$  of  $Y$  defining bisecants  $Z(P, Q_i)$  and rank 2 subbundles  $F_1, F_2$  of rank 2 and degree  $2k$  according to 3.5, such that  $T(P, Q_1) \subset F_1 \otimes k(P)$

$T(P, Q_2) \not\subset F_1 \otimes k(P)$  and  $T(P, Q_2) \subset F_2 \otimes k(P)$

We define  $F = F_1 + F_2$  and  $G$  to be the kernel of the surjection  $F_1 \oplus F_2 \rightarrow F$ . Then  $G$  must have rank 1 and we have an injection from  $G$  to  $F_2$ , hence an injection from  $G$  to  $E_Y$ , therefore  $\deg(G) < k$ . But this gives  $\deg(F) > 3k$ , which is impossible.

*Case 2:*  $E_Y$  has two (or more) sublinebundles  $L_1$  and  $L_2$  of degree  $k$

We take a point  $P$  of  $Y$  such that  $L_1 \otimes k(P) \neq L_2 \otimes k(P)$  in  $E_Y \otimes k(P)$ . We define  $W = L_1 \otimes k(P) + L_2 \otimes k(P)$ . Now we choose a point  $Q$  in  $Y$  such that  $T(P, Q) \not\subset W$  and both points define a bisecant. Again we suppose, there were an  $F \subset E_Y$  of rank 2 and  $\deg F = 2k$  and  $T(P, Q) \subset F \otimes k(P)$ . This implies that at most one of the linebundles  $L_i$  can be contained in  $F$ . We suppose  $L_1 \not\subset F$  and find  $E_Y = F \oplus L_1$  as before.

*Case 3:*  $E_Y$  has exactly one sublinebundle  $L$  of degree  $k$ .

If there were a bundle  $F \subset E_Y$  of rank 2 and degree  $2k$  not containing  $L$ , then we would have  $E_Y = L \oplus F$ . So we can assume that all subbundles of  $E_Y$  with degree  $2k$  and rank 2 contain  $L$ .

For any  $P \in Y$  we denote the line through  $P$  with direction  $L \otimes k(P)$  in  $P$  by  $Z(L, P)$ . We choose two points  $P, Q \in Y$  such that  $Z(L, P)$  and  $Z(L, Q)$  differ from the line through  $P$  and  $Q$ . However, because of 3.5 these three lines are in a plane  $H \subset \mathbb{P}(V^\vee)$ . We denote the intersection point  $Z(L, P) \cap Z(L, Q)$  by  $Q_0$ .

We now see that for a general point  $P'$  of  $Y$  not contained in  $H$  the line  $Z(L, P')$  must intersect with  $Z(L, P)$  and  $Z(L, Q)$ . This is possible only if  $Q_0 \in Z(L, P')$  for all  $P' \in Y$ . But this immediately yields:  $L = L^{Q_0}$   $\square$

Now we come to the case of  $\deg(E_Y) = 3k + 1$ . For numerical reasons  $E_{X(P, Q)}$  can not have a destabilizing subsheaf of constant rank 2 (see 3.4). So only the subsheaves of constant rank 1 have to be considered:

**Lemma 3.9** *Let  $E_Y$  be a stable bundle on  $Y$  of rank 3 and degree  $d = 3k + 1$ . If, moreover,  $d \geq 5$ , then  $E_{X(P, Q)}$  has no destabilizing subsheaf of constant rank 1, for two general points  $P$  and  $Q$  of  $Y$ .*

*Proof:* We take a general hyperplane  $H$  of  $\mathbb{P}(V^\vee)$ , such that  $Y \cap H = \{P, Q_1, \dots, Q_{d-1}\}$  consists of  $d$  different points in general position. Moreover, we take a point  $Q$  of  $Y$  which is not contained in  $H$ . Now we suppose that for all  $i = 1, \dots, d-1$  there is a sublinebundle  $L_i$  of  $E_Y$  with  $\deg(L_i) = k$  and  $T(P, Q_i) = L_i \otimes k(P)$  in  $E \otimes k(P)$ , see 3.4. We see that  $L_1 + L_2 \cong L_1 \oplus L_2$ , therefore  $F = L_1 + L_2 + L_3$  is of rank 2 or 3.

*Case 1:*  $\text{rk}(F) = 2$

Here we have a non-zero morphism  $L_3 \rightarrow L_1 + L_2 \cong L_1 \oplus L_2$ . However, from the choice of the points  $Q_i$  it follows that neither  $L_3 \subset L_1$  nor  $L_3 \subset L_2$  holds, hence  $L_1 \cong L_2$ .

Let now  $L$  be a sublinebundle of  $E_Y$  such that  $\deg(L) = k$  and  $T(P, Q) = L \otimes k(P)$ . We see that  $G = L + L_1 + L_2 \cong L_1 \oplus L_1 \oplus L$ . We obtain a short exact sequence  $0 \rightarrow G \rightarrow E_Y \rightarrow T \rightarrow 0$ , where  $T$  is a torsion sheaf of length one. So we have  $\dim(\text{Ext}^1(T, E_Y)) = 3$  and  $\dim(\text{Hom}(G, E_Y)) \geq 5$ , which implies  $\dim(\text{End}(E_Y)) \geq 2$ . This is impossible because a stable sheaf is simple.

*Case 2:*  $\text{rk}(F) = 3$

This is only possible when  $F \cong L_1 \oplus L_2 \oplus L_3$ . Now we consider the linebundle  $L_4 \subset E_Y$  of degree  $k$  with  $T(P, Q_4) = L_4 \otimes k(P)$ . (Here we really need the assumption  $d \geq 5$ .) It follows from the construction that  $L_4$  is even a sublinebundle of  $F$  and as before we get  $L_1 \cong L_2$ . The rest we conclude like in the first case.  $\square$

We need the statement of the previous lemma also for the case of  $g = 1$  and  $d = 4$ .

**Lemma 3.10** *Let  $Y$  be an elliptic curve embedded in  $\mathbb{P}(V^\vee)$  as a curve of degree 4, and  $E_Y$  be stable. Then, for two general point  $P$  and  $Q$  of  $Y$ , the bundle  $E_{X(P,Q)}$  has no destabilizing subsheaf of constant rank one.*

*Proof:* Let  $Q(E_Y, 2, 3)$  be the Quot scheme of Quotients  $E_Y \rightarrow F$  with  $\deg(F) = 3$  and  $\text{rk}(F) = 2$ . Considering the kernel of these surjections we obtain a morphism  $\phi : Q(E_Y, 2, 3) \rightarrow \text{Pic}^1(Y)$ . For a linebundle  $L$  of degree one we have:

$$\begin{aligned} \text{Hom}(E_Y, L) &\cong \text{Ext}^1(L, E_Y)^\vee && \text{(Serre duality)} \\ \text{Hom}(E_Y, L) &= 0 && E_Y \text{ is stable, hence:} \\ \dim(\text{Hom}(L, E_Y)) &= 1. \end{aligned}$$

The same argument shows that  $Q(E_Y, 2, 3)$  is smooth of dimension 1 and so  $\phi : Q(E_Y, 2, 3) \rightarrow \text{Pic}^1(Y)$  is an isomorphism. On the other hand, we have a family of sublinebundles of  $E_Y$  parametrized by  $Y$  (3.6). So it follows that all linebundles  $L \subset E_Y$  of degree 1 are the linebundles  $L^P$  for a  $P \in Y$ .  $\square$

Now we come to the easy case where  $\deg(E_Y) = 3k+2$ . Here we see immediately that for all points  $P$  and  $Q$  the bundle  $E_{X(P,Q)}$  is semistable. But we have to show a little bit more:

**Lemma 3.11** *Let  $E_Y$  be stable. Then, for any two points  $P$  and  $Q$  of  $Y$ , we have  $\dim(\text{End}(E_{X(P,Q)})) = 1$*

*Proof:* On the one hand this follows from the fact that the only endomorphisms of the stable bundle  $E_Y$  are the multiplications and on the other hand that an endomorphism of  $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ , which is the multiplication at two different points  $P$  and  $Q$  of  $\mathbb{P}^1$ , is itself a multiplication.  $\square$

## 4 A result of R. Hernández

Here we give a short review on a result of R. Hernández [5], which was the starting point for this work. For a fixed smooth curve  $X$  of genus  $g$  Hernández considered the Quot scheme  $Q(m, n, d)$  of quotients from  $\mathcal{O}_X^{\oplus m}$  of rank  $n$  ( $n < m$ ) and degree  $d$ . We denote by  $Q_0(m, n, d) \subset Q(m, n, d)$  those quotients  $E$  which are vector bundles and satisfy  $H^1(X, E) = 0$ . It is obvious that  $Q_0$  is an open subset whose existence is given by the following lemma.



**Lemma 4.1**  $Q_0(m, n, d)$  is nonempty if and only if  $d > ng$ .

*Proof:* By the Riemann Roch theorem it follows that  $d > ng$  is necessary. (The only exception occurs when  $g = 0$ ). It remains to show that the condition is sufficient.

We proceed by induction on  $n$ . For  $n = 1$  we have to show that a general line bundle  $L$  on  $X$  with  $\deg(L) > g$  is globally generated. For a general line bundle  $L$  of degree  $d > g$  it is well known that  $H^1(L) = 0$ .

Let  $Z = \{\alpha \in \text{Pic}^{d-1} \mid h^0(\alpha) \geq d - g + 1\}$ . We show that  $\text{codim}(Z, \text{Pic}^{d-1}) \geq 2$ . To do so, we regard the surjection  $pr : S^{d-1}X \rightarrow \text{Pic}^{d-1}$ . Now the fibre of this morphism is at least of dimension  $d - g$  over  $Z$ . At the other hand we have  $\text{codim}(pr^{-1}(Z), S^{d-1}X) \geq 1$ . Combining these two facts we get the stated codimension.

Now for any line bundle  $L \in \text{Pic}^d$  we define the following map:

$$\begin{aligned} \phi_L : X &\rightarrow \text{Pic}^{d-1} \\ p &\mapsto L(-p) \end{aligned}$$

We can choose  $L$  such that  $\text{im}(\phi_L) \cap Z = \emptyset$ , which means that  $L$  is base-point-free and, therefore, generated by its global sections.

Now we assume that the assertion is true for  $n - 1$ , so that we have a quotient  $E \in Q_0(m - 1, n - 1, d - g)$ . Dualizing we get a short exact sequence:

$$0 \rightarrow E^\vee \rightarrow \mathcal{O}_X^{\oplus m-1} \rightarrow L \rightarrow 0.$$

On the other hand, we take an effective divisor  $D$  of degree  $g$ , such that  $h^0(\mathcal{O}_X(D)) = 1$  and a section  $s \in H^0(L(D))$  not vanishing at the points of  $D$ . This yields the following diagram with exact rows:

$$\begin{array}{ccccccc} 0 \rightarrow & \mathcal{O}_X^{\oplus m-1} & \rightarrow & \mathcal{O}_X^{\oplus m} & \rightarrow & \mathcal{O}_X & \rightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \rightarrow & L & \rightarrow & L(D) & \rightarrow & \mathcal{O}_D & \rightarrow 0 \end{array}$$

Denoting the kernel of the vertical morphism in the middle by  $G$  we obtain the kernel-cokernel sequence:

$$0 \rightarrow E^\vee \rightarrow G \rightarrow \mathcal{O}_X(-D) \rightarrow 0$$

and we obviously conclude  $G^\vee \in Q_0(m, n, d)$   $\square$

Let  $V$  be a vector space of dimension  $n$  and  $Q(V, r, d)$  be the Quot scheme of quotients of  $V \otimes \mathcal{O}_X$  of degree  $d$  and rank  $r$ . As before we denote by  $Q_0(V, r, d)$  the open subset where the quotients have no first cohomology and are locally free. Now we fix a convex polygon  $P = \{(0, 0); (r_1, d_1); (d_2, r_2) \dots (r_l, d_l)\}$  with  $r_l = r$  and  $d_l = d$  and consider the subset  $Q_0(V, P)$  in  $Q_0(V, r, d)$  which parametrizes quotients  $E$  with  $\text{HNP}(E) = P$ . By 2.1  $Q_0(V, P)$  is locally closed. We set  $r_0 = d_0 = 0$  and  $r_{-1} = r_l - n$ . Under the assumption that  $Q_0(V, P) \neq \emptyset$  we have (See [5]):

**Theorem 4.2**  $Q_0(V, P)$  is smooth irreducible and

$$\dim(Q_0(V, P)) = \sum_{i=0}^{l-1} [(r_{i+1} - r_{i-1})(d_l - d_i) + (r_i - r_{i-1})(r_l - r_i)(1 - g)]$$

Let  $X$  be a smooth curve of genus  $g$  and  $V$  a vector space of dimension  $m$ . As before we denote by  $Q_0(V, n, d)$  the quotients  $V \otimes \mathcal{O}_X \rightarrow E$  that are bundles and satisfy  $h^1(E) = 0$ . (So we obviously require  $m > n$  and  $d > ng$  by 4.1.) We define now two convex polygons by:

$$P_{\min} = \{(0, 0); (n, d)\} \text{ and } P_{\max} = \{(0, 0); (1, d - 1 - g(n - 1)); (n, d)\}.$$

**Theorem 4.3** *A convex polygon  $P$  from  $(0, 0)$  to  $(n, d)$  arises in the image of  $\text{HNP} : Q_0(V, n, d) \rightarrow \text{Polygons}$   $E \mapsto \text{HNP}(E)$  if and only if  $P_{\min} \leq P \leq P_{\max}$ .*

*Proof:* Let  $V \otimes \mathcal{O}_X \rightarrow E$  be a point in the Quot scheme, then  $P_{\min} \leq P$  holds by definition of  $\text{HNP}(E)$ . Let  $F \subset E$  be a subsheaf of  $E$ . We can assume  $E' = E/F$  to be a vector bundle. (Otherwise  $E$  would have a subsheaf of same rank as  $F$  but with a higher degree.)  $E'$  is generated by its global sections and satisfies  $h^1(E') = 0$ , hence by 4.1:

$$\begin{aligned} \deg(E') &\geq \text{rk}(E')g + 1 && \text{which implies} \\ \deg(F) &\leq d - 1 - gn + \text{rk}(F)g && \text{and therefore} \\ P &\leq P_{\max}. \end{aligned}$$

Conversely given any convex polygon  $P = \{(0, 0); (r_1, d_1); \dots; (n, d)\}$  of length  $l$  with  $P_{\min} \leq P \leq P_{\max}$ . Then, again by 4.1, there exist semistable bundles  $E_i$   $i = 1, \dots, l$  with  $\text{rk}(E_i) = r_i - r_{i-1}$  and  $\deg(E_i) = d_i - d_{i-1}$  which are globally generated and satisfy  $h^1(E_i) = 0$ .

Their direct sum is therefore a quotient with the given Harder-Narasimhan polygon.  $\square$

*Example:* Let  $X$  be an elliptic curve and let  $V$  be a 4-dimensional vector space. We consider the open subset  $Q_0$  of  $\text{Quot}(V \otimes \mathcal{O}_X, 3, 9)$   $\dim(Q_0) = 36$ , and the possible Harder-Narasimhan polygons (beside  $P_{\min}$ ) are:

$$\begin{array}{ll} P_1 = \{(0, 0); (1, 4); (3, 9)\} & \text{codim}(\text{HNP}^{-1}(P_1), Q_0) = 3 \\ P_2 = \{(0, 0); (1, 5); (3, 9)\} & \text{codim}(\text{HNP}^{-1}(P_2), Q_0) = 6 \\ P_3 = \{(0, 0); (1, 6); (3, 9)\} = P_{\max} & \text{codim}(\text{HNP}^{-1}(P_3), Q_0) = 9 \\ P_4 = \{(0, 0); (2, 7); (3, 9)\} & \text{codim}(\text{HNP}^{-1}(P_4), Q_0) = 3 \\ P_5 = \{(0, 0); (1, 4); (2, 7); (3, 9)\} & \text{codim}(\text{HNP}^{-1}(P_5), Q_0) = 4. \end{array}$$

Let now  $X \subset \mathbb{P}^m \times S$  be a flat family of smooth curves over  $S$ . Then we have the Quot scheme  $Q = \text{Quot}(V \otimes \mathcal{O}_X, n, d)$  together with a morphism  $\pi : Q \rightarrow S$ . Again we define  $Q_0$  to be the open subset of  $Q$  that parametrizes bundles with vanishing first cohomology. Let  $P$  be a convex polygon given by  $\{(0, 0), (r_1, d_1), \dots, (r_l, d_l)\}$  with  $r_l = n$  and  $d_l = d$ , then we can define the subset  $Q_0(V, P)$  of  $Q_0$  as before. By 4.2 and 4.3 we know the fibres of  $\pi|_{Q_0(V, P)} : Q_0(V, P) \rightarrow S$ . Neither their existence nor their dimension depend on  $s \in S$ , which gives:

**Theorem 4.4**  $Q_0(V, P) \neq \emptyset$  if and only if  $P_{\min} \leq P \leq P_{\max}$ .

$$\dim(Q_0(V, P)) = \sum_{i=0}^{l-1} [(r_{i+1} - r_{i-1})(d_l - d_i) + (r_i - r_{i-1})(r_l - r_i)(1 - g)] + \dim S$$

If  $S$  is irreducible (resp. smooth), then  $Q_0(V, P)$  is so.

**Corollary 4.5** *If  $d > 3g$ , then the general curve in  $\text{Hilb}_0(d, g)$  has a semistable restricted tangent bundle.*

## References

- [1] L. Ein , Hilbert schemes of smooth space curves, Ann. Sc. Ec. Norm. Sup., 4. sér., 19, (1986), p. 469-478
- [2] F. Ghione, A. Iarrobino and G. Sacchiero, Restricted tangent bundles of rational curves in  $\mathbb{P}^r$ , preprint, (1988)
- [3] A. Grothendieck, Techniques de construction et théorèmes d'existence en géométrie algébrique, IV. Les schémas de Hilbert, Seminaire Bourbaki 221 (1960/61)
- [4] R. Hartshorne and A. Hirschowitz, Smoothing algebraic space curves, Lect. Notes Math 1124 (1978), 98-131
- [5] R. Hernandez, On Harder-Narasimhan stratification over Quotschemes, J. reine ang. Math 371 (1986), 114-24
- [6] D. Laksov, Indecomposability of restricted tangent bundles, Astérisque 87/88, (1981), p. 207-219
- [7] L. Ramella, Sur le schéma de Hilbert de courbes rationales, These de doctorat, Nice, (1988)