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Restricted Three-Body Problem in Different Coordinate Systems

—II-In Sidereal Spherical Coordinates System

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ABSTRACT

In this paper of the series, the equations of motion for the spatial circular restricted three-body problem in sidereal spherical coordinates system were established. Initial value procedure that can be used to compute both the spherical and Cartesian sidereal coordinates and velocities was also developed. The application of the procedure was illustrated by numerical example and graphical representations of the variations of the two sidereal coordinate systems.

Keywords: Spatial Restricted Circular Three Body Problem; Regularization; Coordinate Transformations

1. Introduction

In a previous communication to this journal [1], hereafter referred to as Paper I we started our studies towards establishing new differential equations for the different forms of the three-body problem using some important coordinate systems. By this, we aims at obtaining differential equations (see [1] for details) which are: 1) Regular; 2) Suitable for the geometry to which they referred; 3) Producing slow variations in the coordinates during the orbital motion, a property which produces more stable numerical integration procedures. In Paper I, the equations of motion for spatial restricted circular three body problem in cylindrical coordinates system was established together with a computational algorithm that can be used to compute both the cylindrical and Cartesian coordinates and velocities. In the present paper, the equations of motion for the spatial circular restricted three-body problem in sidereal spherical coordinates system were established. Initial value procedure that can be used to compute both the spherical and Cartesian sidereal coordinates and velocities was also developed. The application of the procedure was illustrated by numerical example and graphical representations of the variations of the two sidereal coordinate systems.

2. Circular Restricted Three-Body Problem in Sidereal System

If two of the bodies, say m_1 and m_2 in the three-body problem move in circular, coplanar orbits about their common center of mass and the mass say m_3 of the third body is too small to affect the motion of the other bodies, the problem of the motion of the third body is called the circular, restricted, three body problem. The two revolving bodies are called the primaries; their masses are arbitrary but have such internal mass distributions that they may be considered point masses.

The equations of motion of the third body in a dimensionless sidereal (inertial) coordinate (x, y, z) system with the mean motion $n = 1$, are [2]

$$\ddot{x} = \frac{\partial V}{\partial x}, \tag{1}$$

$$\ddot{y} = \frac{\partial V}{\partial y}, \tag{2}$$

$$\ddot{z} = \frac{\partial V}{\partial z}, \tag{3}$$

where $V = V(x, y, z)$ is given as

$$V = \frac{1-\mu}{r_1} + \frac{\mu}{r_2} \tag{4}$$

μ denotes the mass of the smaller primary when the total mass of the primaries has been normalized to unity.

$$r_i^2 = (x - x_i)^2 + (y - y_i)^2 + z^2; i = 1, 2 \tag{5}$$

and $r_i; i = 1, 2$ are the distances of the third body from the primaries which are located at $(x_i, y_i, 0)$; $i = 1, 2$, these coordinates are functions of the time t and are

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given as

$$x_1 = \mu\cos t;\ x_2 = -(1-\mu)\cos t;$$
$$y_1 = \mu\sin t;\ y_2 = -(1-\mu)\sin t \tag{6}$$

3. The Equations of Motion in Sidereal Spherical Coordinate System

Corresponding to the Cartesian sidereal coordinate system (x,y,z), the coordinate system related to the system (x,y,z) by certain transformation, is also called sidereal coordinate system. In this respect the system (u_1,u_2,u_3) of Equation (7) is called sidereal spherical coordinate system.

In what follows we shall establish, the differential equations for the spatial circular restricted three body-problem in sidereal spherical coordinate system.

3.1. Coordinate and Velocity Transformations

$$x = u_1\sin u_2\cos u_3;\ y = u_1\sin u_2\sin u_3;\ z = u_1\cos u_2 \tag{7}$$

$$\dot{x} = \dot{u}_1\cos u_3\sin u_2 + u_1\dot{u}_2\cos u_2\cos u_3 - u_1\dot{u}_3\sin u_2\sin u_3 \tag{8.1}$$

$$\dot{y} = \dot{u}_1\sin u_2\sin u_3 + \dot{u}_2 u_1\cos u_2\sin u_3 + u_1\dot{u}_3\cos u_3\sin u_2 \tag{8.2}$$

$$\dot{z} = \dot{u}_1\cos u_2 - u_1\dot{u}_2\sin u_2 \tag{8.3}$$

where

$$0 \le u_1 < \infty,\ 0 < u_2 \le \pi,\ -\pi < u_3 < \pi \tag{9}$$

3.2. Inverse Transformations

From Equation (7) we have

$$u_1 = \left(x^2+y^2+z^2\right)^{1/2};$$
$$u_2 = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right); \tag{10}$$
$$u_3 = \tan^{-1}\left(\frac{y}{x}\right)$$

Differentiating the first and the third of Equation (10) and the third of Equation (7) with respect to the time t we get:

$$\dot{u}_1 = (x\dot{x}+y\dot{y}+z\dot{z})/u_1;$$
$$\dot{u}_2 = \left(z\dot{u}_1 - \dot{z}u_1\right)\Big/u_1\left(x^2+y^2\right)^{1/2}; \tag{11}$$
$$\dot{u}_3 = \left(x\dot{y}-y\dot{x}\right)\Big/\left(x^2+y^2\right)$$

where u_1 and $\dot{u}_1$ are given in terms of (x,y,z) and $(\dot{x},\dot{y},\dot{z})$ from the previous equations.

3.3. The Equations of Motion

The kinetic energy of a particle of unit mass in the spherical coordinate system is

$$T = \frac{1}{2}\left(\dot{u}_1^2 + u_1^2\dot{u}_2^2 + u_1^2\dot{u}_3^2\sin^2 u_2\right) \tag{12}$$

By using the transformation equations (Equations (7)), the gravitational potential V could be expressed in term of (u_1,u_2,u_3).

$$\mathrm{d}/\mathrm{d}t\left(\partial T/\partial\dot{u}_j\right) - \partial T/\partial u_j = \partial V/\partial u_j;\ j=1,2,3$$

Consequently, we deduce for the equations of motion in sidereal spherical coordinate system, the forms

$$\ddot{u}_1 = u_1\left(\dot{u}_2^2 + \dot{u}_3^2\sin^2 u_2\right) + \partial V/\partial u_1 \tag{13.1}$$

$$\ddot{u}_2 = 1/2\,\dot{u}_3^2\sin 2u_2 + 1/u_1^2\left(\partial V/\partial u_2 - 2u_1\dot{u}_1\dot{u}_2\right) \tag{13.2}$$

$$\ddot{u}_3 = 1/u_1^2\left(\partial V/\partial u_3\csc^2 u_2 - 2u_1\dot{u}_3\left[\dot{u}_1 + u_1\dot{u}_2\cot u_2\right]\right), \tag{13.3}$$

where $\dfrac{\partial V}{\partial u_j}$ are given as

$$\frac{\partial V}{\partial u_j} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial u_j} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial u_j} + \frac{\partial V}{\partial z}\frac{\partial z}{\partial u_j};\ j=1,2,3, \tag{14}$$

$\partial x/\partial u_j$, $\partial y/\partial u_j$ and $\partial z/\partial u_j$; $j=1,2,3$ can be computed from Equation (7), while $\partial V/\partial x$, $\partial V/\partial y$ and $\partial V/\partial z$ can be computed from Equations (1)-(3), so we get

$$\frac{\partial V}{\partial u_1} = \Big(\cos u_2\left((\mu-1)Q_2 - \mu Q_1\right) + \sin u_2\left((Q_1-Q_2)(Q_3\cos u_3 + Q_4\sin u_3)\right)\Big)\Big/Q_1Q_2,$$

$$\frac{\partial V}{\partial u_2} = u_1\Big(\sin u_2\left(\mu Q_1 - Q_2(\mu-1)\right) + \cos u_2\left(\cos u_3\left(Q_3Q_1 - Q_5Q_2\right) + \sin u_3\left(Q_6Q_1 - Q_4Q_2\right)\right)\Big)\Big/Q_1Q_2,$$

$$\frac{\partial V}{\partial u_3} = Q_7u_1\left(Q_1-Q_2\right)/Q_1Q_2,$$

where

$$Q_1 = \left(\mu^2 - 2\mu u_1\cos(t-u_3)\sin u_2 + u_1^2\right)^{3/2},$$
$$Q_2 = \left((\mu-1)^2 - 2u_1(\mu-1)\cos(t-u_3)\sin u_2 + u_1^2\right)^{3/2},$$
$$Q_3 = \mu\left((\mu-1)\cos t - u_1\cos u_3\sin u_2\right),$$
$$Q_4 = (\mu-1)\left(\mu\sin t - u_1\sin u_2\sin u_3\right),$$
$$Q_5 = (\mu-1)\left(\mu\cos t - u_1\sin u_2\cos u_3\right),$$
$$Q_6 = \mu\left((\mu-1)\sin t - u_1\sin u_3\sin u_2\right),$$

$Q_7 = (\mu - 1)\mu u_1 \sin u_2 \sin(t - u_3)$.

4. Computational Development

4.1. Initial Value Procedure

In what follows, we shall establish a procedure that can be used to compute $\forall t_0 \le t \le t_f$ (say) both:

1) The spherical sidereal coordinates and velocities $(u_1, u_2, u_3, \dot{u}_1, \dot{u}_2, \dot{u}_3)$, and

2) The Cartesian sidereal coordinates and velocities $(x, y, z, \dot{x}, \dot{y}, \dot{z})$.

So, such procedure is a double usefulness computational algorithm, for which a differential solver can be used for the spherical sidereal six order system to obtain $(u_1, u_2, u_3, \dot{u}_1, \dot{u}_2, \dot{u}_3)$. While the Cartesian sidereal coordinates and velocities $(x, y, z, \dot{x}, \dot{y}, \dot{z})$ are obtained by the substitutions in the direct transformation formulae (Equations (7) and (8)), rather than solving the six order system of Equations (1), (2) and (3). By this way, great time can be saved.

This initial value procedure using sidereal spherical coordinate system will be described through its basic points, input, output and computational steps.

Input: 1) $x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0$ at $t = t_0$,

2) the final time $t = t_f$

3) $\frac{\partial V}{\partial x} = F_1(x, y, z);$

$\frac{\partial V}{\partial y} = F_2(x, y, z); \frac{\partial V}{\partial z} = F_3(x, y, z);$

Output: 1) $u_j; \dot{u}_j; j = 1, 2, 3 \quad \forall t_0 \le t \le t_f$

2) $x, y, z; \dot{x}, \dot{y}, \dot{z} \quad \forall t_0 \le t \le t_f$

Computational steps

1) Using the given values $x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0$ at $t = t_0$ and the inverse transformations to compute the initial values $u_{0j}; j = 1, 2, 3, 4, 5, 6$.

2) Using the partial derivatives $\frac{\partial V}{\partial u_1}; \frac{\partial V}{\partial u_2}; \frac{\partial V}{\partial u_3}$ (functions of $u_j; j = 1, 2, 3$) to construct the analytical forms of equations of motion as first order system.

Table 1. The values of sidereal spherical coordinates and velocities.

t	$u_1(t)$	$u_2(t)$	$u_3(t)$	$u_4(t)$	$u_5(t)$	$u_6(t)$
0	0.97841	1.1671	1.7427	-2.46793×10^{-10}	1.484571×10^{-10}	2.37483×10^{-11}
0.4	0.871726	1.22588	1.74187	−0.551644	0.353516	−0.00503351
0.8	0.527597	1.72526	1.73568	−0.95018	3.59417	−0.041217
1.2	1.30109	3.01557	−1.33442	3.79127	−1.05605	−0.618301
1.6	2.88497	2.82617	−1.37945	4.03252	−0.207042	−0.0206263
2	4.50548	2.77375	−1.384	4.06276	−0.0826788	−0.00629347
2.4	6.13255	2.74966	−1.38574	4.07109	−0.0437919	−0.00301006
2.8	7.7617	2.73592	−1.38666	4.07424	−0.0269405	−0.00176013
3.2	9.39171	2.72707	−1.38723	4.07564	−0.0181821	−0.00115414
3.6	11.0221	2.7209	−1.38762	4.07632	−0.0130683	−0.000814998
4	12.6527	2.71636	−1.3879	4.07667	−0.00983058	−0.00060616
4.4	14.2834	2.71289	−1.38811	4.07685	−0.00765465	−0.000468466
4.8	15.9142	2.71016	−1.38828	4.07693	−0.00612371	−0.000372901
5.2	17.545	2.70794	−1.38842	4.07696	−0.00500672	−0.000303871
5.6	19.1758	2.70611	−1.38853	4.07696	−0.00416736	−0.000252387
6	20.8065	2.70458	−1.38862	4.07695	−0.00352102	−0.000212967
6.4	22.4373	2.70328	−1.3887	4.07692	−0.00301297	−0.000182117
6.8	24.0681	2.70216	−1.38877	4.07689	−0.00260654	−0.00015752
7.2	25.6988	2.70118	−1.38882	4.07686	−0.00227643	−0.000137593
7.6	27.3296	2.70033	−1.38888	4.07682	−0.00200475	−0.000121223
8.	28.9603	2.69957	−1.38892	4.07678	−0.00177854	−0.000107611

Table 2. The values of sidereal Cartesian coordinates and velocities.

t	$x(t)$	$y(t)$	$z(t)$	$\dot{x}(t)$	$\dot{y}(t)$	$\dot{z}(t)$
0	−0.15391	0.886499	0.83434	-1.72682×10^{-10}	-2.54539×10^{-10}	-1.10303×10^{-10}
0.4	−0.139664	0.808409	0.294745	0.0747123	−0.408197	−0.476539
0.8	−0.855661	0.514245	−0.0811716	0.223182	−1.2104	−1.72751
1.2	0.0382944	−0.158981	−1.29077	0.332499	−1.81215	−3.58851
1.6	0.17021	−0.878633	−2.74264	0.327786	−1.78911	−3.64828
2.	0.300888	−1.59201	−4.20409	0.325854	−1.77901	−3.65703
2.4	0.431014	−2.30246	−5.66754	0.324865	−1.77374	−3.65981
2.8	0.560832	−3.01127	−7.13174	0.324268	−1.77053	−3.66104
3.2	0.690454	−3.71903	−8.5963	0.323868	−1.76838	−3.66196
3.6	0.819941	−4.42606	−10.0611	0.323582	−1.76684	−3.66207
4.	0.949329	−5.13255	−11.5259	0.323368	−1.76569	−3.66231
4.4	1.07864	−5.83864	−12.9909	0.323201	−1.6478	−3.66248
4.8	1.20789	−6.5444	−14.4559	0.323068	−1.76406	−3.66259
5.2	1.3371	−7.24991	−15.921	0.322959	−1.76347	−3.66268
5.6	1.46626	−7.95519	−17.386	0.322868	−1.76298	−3.66274
6.	1.5954	−8.6603	−18.8521	0.322792	−1.76256	−3.66279
6.4	1.7245	−9.36525	−20.3163	0.322726	−1.76221	−3.66283
6.8	1.85358	−10.0701	−21.7814	0.322669	−1.7619	−3.66287
7.2	1.98263	−10.7748	−23.2466	0.322619	−1.76163	−3.66289
7.6	2.11167	−11.4794	−24.7117	0.322575	−1.76139	−3.66291
8.	2.2407	−12.1839	−26.1769	0.322536	−1.76117	−3.66293

3) Using the initial conditions u_{0j}; $j = 1,2,3,4,5,6$ from step 1 to solve numerically the above differential system of step 2 for u_j; $j = 1,2\cdots,6$ $\forall t_0 \le t \le t_f$, (note that $u_4 \equiv \dot{u}_1, u_5 \equiv \dot{u}_2, u_6 \equiv \dot{u}_3$).

4) Using $u_j; \dot{u}_j$; $j = 1,2,3$ from step 3 and the direct transformations of Equations (7) and (8) to compute numerically x, y, z and $\dot{x}, \dot{y}, \dot{z}$ $\forall t_0 \le t \le t_f$.

5) End.

4.2. Numerical Example

Consider the initial values

$$x_0 = -0.153910449;$$

$$y_0 = 0.886499068;$$

$$z_0 = 0.384340387;$$

$$\dot{x}_0 = -0.00000000017268248;$$

$$\dot{y}_0 = -0.0000000002545393;$$

$$\mu = 0.0121505816;$$

$$t_f = 8.0;$$

$$t_0 = 0.0;$$

applying the above procedure we get the results as displayed in **Tables 1** and **2**.

4.3. Graphical Representations

Figure 1 illustrates the time variations of the two sidereal coordinate systems (x, y, z) (left) and (u_1, u_2, u_3) (right).

5. Conclusion

In this paper of the series, the equations of motion for the spatial circular restricted three-body problem in sidereal spherical coordinates system were established. Initial value procedure that can be used to compute both the spherical and Cartesian sidereal coordinates and velocities was also developed. The application of the procedure

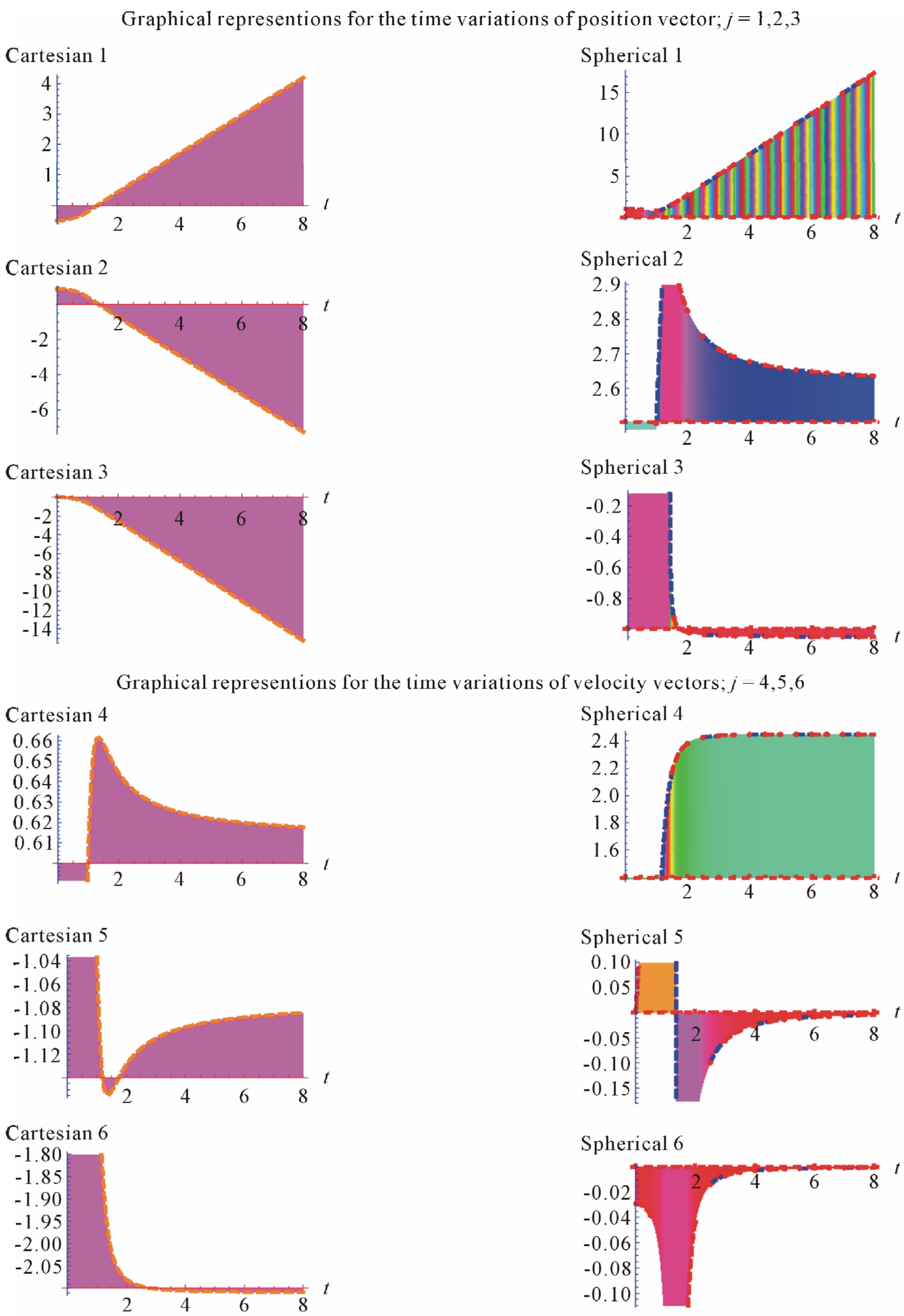

Figure 1. The time variations of the two sidereal coordinate systems (x, y, z) (left) and (u_1, u_2, u_3) (right).

was illustrated by numerical example and graphical representations of the variations of the two sidereal coordinate systems.

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