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Restricting linear syzygies: algebra and geometry

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Abstract

Let $X \subset \mathbb{P}^r$ be a closed scheme in projective space whose homogeneous ideal is generated by quadrics. We say that X (or its ideal I_X) satisfies the condition $\mathbf{N}_{2,p}$ if the syzygies of I_X are linear for p steps. We show that if X satisfies $\mathbf{N}_{2,p}$ then a zero-dimensional or one-dimensional intersection of X with a plane of dimension $\leqslant p$ is 2-regular. This extends a result of Green and Lazarsfeld. We give conditions when the syzygies of X restrict to the syzygies of the intersection. Many of our results also work for ideals generated by forms of higher degree. As applications, we bound the p for which some well-known projective varieties satisfy $\mathbf{N}_{2,p}$. Another application, carried out by us in a different paper, is a step in the classification of 2-regular reduced projective schemes. Extending a result of Fröberg, we determine which monomial ideals satisfy $\mathbf{N}_{2,p}$. We also apply Green's 'linear syzygy theorem' to deduce a relation between the resolutions of I_X and $I_{X\cup\Gamma}$ for a scheme Γ, and apply the result to bound the number of intersection points of certain pairs of varieties such as rational normal scrolls.

Introduction

Let V be a vector space of dimension $r+1$ over an algebraically closed field k with basis $x_0, \dots, x_r$. If $X \subset \mathbb{P}^r_k = \mathbb{P}(V)$ is a nondegenerate closed subscheme we write $\mathcal{I}_X$ for the ideal sheaf and I_X for the homogeneous ideal of X in the homogeneous coordinate ring $S = \mathrm{Sym}(V) = k[x_0, x_1, \dots, x_r]$ of $\mathbb{P}(V)$. Suppose that I_L is an ideal generated by linear forms, that is the ideal of a linear space L. In general, there is no strong connection between the minimal free resolution of I_X and the minimal free resolution of $I_X + I_L$ or of its saturation. The goal of this paper is to exhibit some cases where an interesting connection of this kind exists.

We say that a projective subscheme $X \subset \mathbb{P}^r$ *satisfies the condition* $\mathbf{N}_{d,p}$, for some $d \geqslant 2$, if $\mathrm{Tor}^S_t(I_X, k)$ is concentrated in degrees $\leqslant d+t$ for all $t \leqslant p-1$.

For example, X satisfies condition $\mathbf{N}_{d,1}$ if I_X is generated in degrees $\leqslant d$ or, equivalently, if the truncation $(I_X)_{\geqslant d} = \bigoplus_{e\geqslant d} H^0(\mathcal{I}_X(e))$ of I_X in degrees $\geqslant d$ is generated in degree d. On the other hand, if $p \geqslant r+1$ then X satisfies $\mathbf{N}_{d,p}$ if and only if $\mathcal{I}_X$ is d-regular in the sense of Castelnuovo–Mumford. In general, it is easy to show that X satisfies $\mathbf{N}_{d,p}$ if and only if X satisfies $\mathbf{N}_{d,1}$ and the first p steps of the minimal free resolution

$$\cdots \to F_t \xrightarrow{\phi_t} F_{t-1} \xrightarrow{\phi_{t-1}} \cdots \xrightarrow{\phi_1} F_0 \to (I_X)_{\geqslant d} \to 0$$

of $(I_X)_{\geqslant d}$ are linear, in the sense that ϕ_t is represented by a matrix of linear forms for all $1 \leqslant t \leqslant p-1$.

Our notation comes from the notation $\mathbf{N}_p$ of Green and Lazarsfeld [GL84, GL85] (see also [EL93]); but we do not insist that X be projectively normal, which is their condition $\mathbf{N}_0$ and is included in their condition $\mathbf{N}_p$.

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Theorem 1.1 shows that if $X \subset \mathbb{P}^r$ satisfies $\mathbf{N}_{d,p}$ then the same is true of $\Lambda \cap X$ for any linear subspace Λ such that $\dim \Lambda \cap X \leqslant 1$ and $\dim \Lambda \leqslant p$. Theorems 1.2 and 1.3 refine this statement to show that, under slightly stronger hypotheses, some of the restriction maps between the minimal free resolutions of I_X and $I_{\Lambda \cap X}$ are surjective. As an application we recover a version of a result of Vermeire [Ver01] on the linear system of quadrics through a variety satisfying property $\mathbf{N}_2$. We give examples showing that these results are sharp in various senses.

Theorem 1.1 is also the starting point for our classification of reduced 2-regular projective schemes in [EGHP04].

A converse to Theorem 1.1 would say that a subscheme $X \subset \mathbb{P}^r$ satisfies $\mathbf{N}_{2,p}$ if and only if every linear section $\Lambda \cap X$ of dimension zero satisfies $\deg(\Lambda \cap X) \leqslant 1 + \dim \Lambda$, whenever $\dim \Lambda \leqslant p$. In Corollary 2.4 we show that this converse holds when X is defined by a monomial ideal. Nevertheless it is false in general, as for example in the case of a double structure on a line in $\mathbb{P}^3$, or the case of the plane with embedded point in Example 1.4.

However, there are other cases when such a converse is true: in [EGHP04] we prove it, with $p = \infty$, for any reduced scheme. Since the hypothesis of Theorem 1.1 does not require the full strength of 2-linearity, this gives an unexpected rigidity result (Corollary 1.8): if $X \subset \mathbb{P}^r$ is a reduced subscheme satisfying property $\mathbf{N}_{2,p}$ for $p = \operatorname{codim}(X, \mathbb{P}^r)$, then X is 2-regular.

Further, Green and Lazarsfeld [GL88, Theorem 2] prove it, for any p, when X is a smooth nonhyperelliptic linearly normal curve of degree $d \geqslant 3\,\mathrm{genus}(X) - 2$. (See also Eisenbud [Eis05] for an exposition and Eisenbud *et al.* [EPSW02, Theorem 4.1] for a different perspective.)

In § 2 we characterize property $\mathbf{N}_{2,p}$ for ideals generated by monomials. In the square-free case, an ideal generated by quadratic square-free monomials comes from a simplicial complex that is the clique complex of a graph G, and the property $\mathbf{N}_{2,p}$ is determined by the length of the shortest cycle in G having no chord (see Theorem 2.1; this result was suggested to us by Serkan Hoşten, Ezra Miller, and Bernd Sturmfels). A special case is Fröberg's result [Fro90] characterizing 2-regular square-free monomial ideals. We relate also the property $\mathbf{N}_{2,p}$ for a monomial ideal to the corresponding property for the largest square-free monomial ideal it contains (Proposition 2.3).

In § 3 we use Theorems 1.1 and 1.2 of § 1 to prove (conjecturally sharp) upper bounds for the property $\mathbf{N}_p$ for Veronese, Segre–Veronese, Plücker or Fano embeddings, as well as for certain embeddings of abelian varieties.

In § 4 we make use of the Eisenbud–Koh–Stillman conjecture (proved by Green [Gre99]) to analyze the intersection of a nondegenerate scheme $X \subset \mathbb{P}^r$ of codimension at least p which satisfies $\mathbf{N}_{2,p}$ with a variety whose pth 2-linear syzygies are understood. For example, we show that X can meet a rational normal curve in at most $2r + 1 - p$ points. Using this technique we give a new proof to Green's syzygetic Castelnuovo lemma and a bound on the length of a zero-dimensional intersection of scrolls or Veronese surfaces (for a modern treatment of the latter see Eisenbud *et al.* [EHP03]).

1. Restricting syzygies to linear subspaces

In this section we show how the condition $\mathbf{N}_{2,p}$ influences low-dimensional linear sections, and give examples where our results are sharp.

Theorem 1.1. *Let $X \subset \mathbb{P}^r$ be a closed subscheme satisfying the property $\mathbf{N}_{d,p}$ with $p \geqslant 1$, and let $\Lambda \subset \mathbb{P}^r$ be a linear subspace of dimension $\leqslant p$. If $\dim X \cap \Lambda \leqslant 1$ then $\mathcal{I}_{X \cap \Lambda, \Lambda}$ is d-regular. In particular, if $d = 2$ and $X \cap \Lambda$ is finite, then* $\operatorname{length} X \cap \Lambda \leqslant \dim \Lambda + 1$.

Theorem 1.1 can be proved by the method introduced by Gruson *et al.* [GLP83] (see also [Laz04, Proposition B.1.2, Example 1.8.18]): restrict a resolution of $\mathcal{I}_X$ to Λ to get a complex with at most

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one-dimensional homology, and chase diagrams to check that the form of the complex, coming from the hypothesis $\mathbf{N}_{d,p}$, yields $H^i(\mathcal{I}_{\Lambda\cap X}(d-i)) = 0$, the condition for d-regularity. This was proven in a different way by Eisenbud *et al.* [EHU04]; a special case was given by Caviglia [Cav03]. It could also be proved by the method we use for the next result, which gives an additional conclusion under an additional hypothesis. Since these proofs are all available, we omit the proof here.

The following results give more precise conclusions under an additional hypotheses.

Theorem 1.2. *Let $X \subset \mathbb{P}^r$ be a closed subscheme satisfying the property $\mathbf{N}_{d,p}$ with $p \geqslant 1$, and let $\Lambda \subset \mathbb{P}^r$ be a linear subspace of dimension $\leqslant p-1$. If $\dim X \cap \Lambda = 0$, then the natural restriction $H^0(\mathcal{I}_X(d)) \to H^0(\mathcal{I}_{X\cap\Lambda,\Lambda}(d))$ is surjective.*

Theorem 1.3. *Let $X \subset \mathbb{P}^r$ be a closed subscheme satisfying the property $\mathbf{N}_{2,p}$ with $p \geqslant 1$, and let $\Lambda \subset \mathbb{P}^r$ be a linear subspace of dimension $\leqslant p$. If X is linearly normal, $X \cap \Lambda$ is zero-dimensional and $X \cap \Lambda$ spans Λ, then the natural restriction from the minimal free resolution of $\mathcal{I}_X$ to the minimal free resolution of $\mathcal{I}_{X\cap\Lambda,\Lambda}$ surjects on the first $p-1$ steps.*

We give some examples where Theorems 1.1, 1.2 and 1.3 are sharp.

Example 1.4. The ideal $I \subset k[x_0, \dots, x_4]$ of 2×2 minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & 0 & x_2 \\ 0 & x_0 & x_1 & x_3 \end{pmatrix}$$

is saturated and defines a scheme $Y \subset \mathbb{P}^4$ consisting of a 2-plane with an embedded point of multiplicity 3. The scheme Y is a linear section of a 2-regular variety $X \subset \mathbb{P}^8$, the cone over the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$, which is 2-regular and thus satisfies $\mathbf{N}_{2,p}$ for every $p \geqslant 1$. If Y were at most one-dimensional then we would conclude from Theorem 1.1 that I was 2-regular. However, I is not even linearly presented. This shows that the hypothesis $\dim(X \cap \Lambda) \leqslant 1$ in Theorem 1.1 cannot be weakened.

Although X is not 2-regular, we can apply Theorem 1.1 to the generic determinantal ideal to conclude that every zero-dimensional plane section of X is 2-regular. Thus, the converse of Theorem 1.1, described in the introduction, does not hold for X.

Example 1.5. The intersection of Y with the hyperplane $H = \{x_4 = 0\}$ is one-dimensional, and thus 2-regular by Theorem 1.1. If Y were zero-dimensional we could conclude from Theorem 1.2 that the quadrics on H vanishing on $Y \cap H$ were all restrictions of quadrics on $\mathbb{P}^7$ vanishing on X. However, the saturation J of $I + (x_4)/(x_4)$ has an extra quadratic generator. Thus, the hypothesis $\dim(X \cap \Lambda) \leqslant 0$ in Theorem 1.2 cannot be dropped.

Example 1.6. The homogeneous ideal

$$I = (x_0^2,\ x_0x_1,\ x_0x_2 - x_1x_4,\ x_0x_4,\ x_1x_2 - x_1x_4,\ x_2^2,\ x_2x_4) \subset k[x_0, \dots, x_4]$$

is saturated and satisfies condition $\mathbf{N}_{2,2}$. It defines a scheme $X \subset \mathbb{P}^4$ consisting of two lines meeting in a point and having an embedded component there. The linear subspace $\Lambda = \{x_3 = x_4 = 0\}$ meets X in a simple point, so $\Lambda \cap X$ does not span Λ. The truncation J in degrees $\geqslant 2$ of the saturation of $I + (x_3, x_4)/(x_3, x_4)$ is thus 2-regular, but the natural restriction of linear syzygies between the minimal free resolutions of I and J is not surjective on Tor_1. Thus, the hypothesis that $X \cap \Lambda$ spans Λ in Theorem 1.3 cannot be weakened.

Example 1.7. The homogeneous ideal

$$I = (x_0^2,\ x_0x_1 - x_2x_4,\ x_0x_2 - x_2x_4,\ x_0x_3,\ x_0x_4,\ x_3x_4,\ x_4^2) \subset k[x_0, \dots, x_4]$$

is the saturated ideal of a 2-regular scheme $X \subset \mathbb{P}^4$ consisting of a 2-plane Π with two embedded points. Its restriction to the hyperplane $\{x_4 = 0\}$ (which contains the 2-plane) is a nonsaturated

ideal defining Π. Its saturation and truncation in degrees $\geqslant 2$ is a 2-regular ideal $J \subset k[x_0, \ldots, x_3]$, but the restriction map from the minimal free resolution of I to that of J is not onto. This shows that Theorem 1.3 is sharp.

For the proofs we use the hypercohomology spectral sequences. To fix notations we recall that if

$$\mathcal{F}^\bullet : \quad \cdots \to \mathcal{F}^{-m} \to \mathcal{F}^{1-m} \to \cdots \to \mathcal{F}^{-1} \to \mathcal{F}^0$$

is a complex on Λ, then its hypercohomology $\mathbf{H}(\mathcal{F}^\bullet)$ is computed by two spectral sequences associated to a Cartan–Eilenberg resolution (double complex) of $\mathcal{F}^\bullet$. The filtration by columns of the double complex induces a first spectral sequence with E_2 terms

$$'E_2^{i,j} = H^i(H^j(\Lambda, \mathcal{F}^\bullet)) \implies \mathbf{H}^{i+j}(\mathcal{F}^\bullet),$$

while the filtration by rows induces a second spectral sequence with

$$''E_2^{i,j} = H^j(\Lambda, \mathcal{H}^i(\mathcal{F}^\bullet)) \implies \mathbf{H}^{i+j}(\mathcal{F}^\bullet),$$

where $\mathcal{H}^m(\mathcal{F}^\bullet)$ denotes the mth cohomology sheaf of the complex $\mathcal{F}^\bullet$.

Proof of Theorem 1.2. Let

$$\cdots \to \mathcal{E}^{-n} \to \mathcal{E}^{-n+1} \to \cdots \to \mathcal{E}^{-1} \to \mathcal{E}^0 \to \mathcal{I}_X \to 0$$

be the sheafification of a minimal free resolution of the homogeneous ideal of X. We apply the spectral sequences above to the complex $\mathcal{F}^\bullet := \mathcal{E}^\bullet \otimes \mathcal{O}_\Lambda(d)$ obtained by restricting the resolution to Λ.

Using the fact that $X \cap \Lambda$ is zero-dimensional we first show that $\mathbf{H}^0(\mathcal{F}^\bullet) = H^0(\mathcal{I}_X \otimes \mathcal{O}_\Lambda(d))$. Since $\mathcal{E}^\bullet$ is a resolution, the sheaves $\mathcal{H}^i(\mathcal{F}^\bullet)$ for $i \leqslant -1$ have support on the zero-dimensional scheme $X \cap \Lambda$. Hence, $H^j(\Lambda, \mathcal{H}^{-i}(\mathcal{F}^\bullet)) = 0$ for all $j \geqslant 1$ and $i \leqslant -1$. Thus, the second hypercohomology spectral sequence degenerates at $''E_2$ and $''E_2^{i,-i} = 0$ for all $i \leqslant 0$. This shows that $\mathbf{H}^0(\mathcal{F}^\bullet) = ''E_\infty^{0,0} = ''E_2^{0,0}$. However, $''E_2^{0,0} = H^0(\mathcal{I}_X \otimes \mathcal{O}_\Lambda(d))$ as required since $\mathcal{E}^\bullet$ is a resolution of $\mathcal{I}_X$.

We next use the hypothesis that X satisfies $\mathbf{N}_{d,p}$ to show that the natural restriction map from $H^0(\mathcal{I}_X(d))$ surjects onto $\mathbf{H}^0(\mathcal{F}^\bullet)$, which by the result of the previous paragraph is $H^0(\mathcal{I}_X \otimes \mathcal{O}_\Lambda(d))$. Consider for this the other spectral sequence. By hypothesis $\mathcal{F}^i$ is a direct sum of copies of $\mathcal{O}_\Lambda(i)$ for all $1 - p \leqslant i \leqslant 0$. Since $\dim \Lambda \leqslant p - 1$

$$'E_1^{i,j} = H^j(\Lambda, \mathcal{F}^i) = 0 \quad \text{for } j \geqslant 1 \text{ and } -\dim \Lambda \leqslant i \leqslant 0.$$

In particular $\mathbf{H}^0(\mathcal{F}^\bullet) = {}'E_\infty^{0,0}$. As $\mathcal{F}^i \neq 0$ only for $i \leqslant 0$, we see that $'E_1^{0,0}$ surjects via the natural map onto $'E_\infty^{0,0}$. On the other hand, $'E_1^{0,0} = H^0(\Lambda, \mathcal{F}^0) = H^0(\mathcal{I}_X(d))$ since $\mathcal{E}^\bullet$ is the sheafification of the minimal free resolution of the homogeneous ideal of X. Combining these maps gives the desired surjection.

To complete the proof of the theorem we still need to show that the natural restriction map

$$H^0(\mathcal{I}_X \otimes \mathcal{O}_\Lambda(d)) \to H^0(\mathcal{I}_{X\cap\Lambda,\Lambda}(d))$$

is surjective. Consider the short exact sequence

$$0 \to \mathcal{I}_X \cap \mathcal{I}_\Lambda / \mathcal{I}_X \cdot \mathcal{I}_\Lambda \to \mathcal{I}_X \otimes \mathcal{O}_\Lambda \to \mathcal{I}_{X\cap\Lambda,\Lambda} \to 0. \tag{$*$}$$

The kernel $\mathcal{K} := (\mathcal{I}_X \cap \mathcal{I}_\Lambda / \mathcal{I}_X \cdot \mathcal{I}_\Lambda)$ has support on the zero-dimensional scheme $X \cap \Lambda$ so $H^1(\mathcal{K}(d)) = 0$ and the surjectivity follows. □

Proof of Theorem 1.1. The claim follows by applying the above spectral sequences to the complexes $\mathcal{E}^\bullet \otimes \mathcal{O}_\Lambda(d - l)$ for $l \geqslant 1$, and then using the short exact sequence $(*)$ twisted by $-l$. We omit the details. □

Proof of Theorem 1.3. This time we use the spectral sequences on the complex $\mathcal{F}^\bullet = \mathcal{E} \otimes \Omega_\Lambda^{m+1}(m+2)$, with $0 \leqslant m \leqslant p-1$. Recall from Green [Gre84], Green and Lazarsfeld [GL88], Lazarsfeld [Laz89], or Eisenbud [Eis05] that if $Y \subset \mathbb{P}^m$ is a scheme with $H^1(\mathcal{I}_Y(1)) = 0$, then for all $m \geqslant 0$ we have

$$\operatorname{Tor}_m^S(I_Y, k)_{m+2} = H^1(\mathcal{I}_Y \otimes \Omega_{\mathbb{P}^m}^{m+1}(m+2))$$

where $S = S_{\mathbb{P}^m}$ is the homogeneous coordinate ring of $\mathbb{P}^m$. Since we have assumed that X is linearly normal we can apply this to $X \subset \mathbb{P}^r$. Since $X \cap \Lambda$ is 2-regular by Theorem 1.1, and $X \cap \Lambda$ spans Λ, we can also apply this with $Y = X \cap \Lambda$ and $\mathbb{P}^m = \Lambda$. This gives

$$\operatorname{Tor}_m^{S_\Lambda}(I_{X\cap\Lambda,\Lambda}, k)_{m+2} = H^1(\mathcal{I}_{X\cap\Lambda,\Lambda} \otimes \Omega_\Lambda^{m+1}(m+2)).$$

In the sequence $(*)$ the sheaf $\mathcal{K}$ has zero-dimensional support, and we deduce that $H^1(\mathcal{I}_{X\cap\Lambda,\Lambda} \otimes \Omega_\Lambda^{m+1}(m+2)) = H^1(\mathcal{I}_X \otimes \Omega_\Lambda^{m+1}(m+2))$.

Now consider the spectral sequence $''E$. We have $''E_2^{i,j} = 0$ when $i < 0$ and $j > 0$. On the other hand, we have

$$''E_2^{0,1} = \operatorname{Tor}_m^{S_\Lambda}(I_{X\cap\Lambda,\Lambda}, k)_{m+2}$$

by the argument above. For any $q \geqslant 2$ we have

$$''E_2^{0,q} = H^q(\Lambda, \mathcal{I}_X \otimes \Omega_\Lambda^{m+1}(m+2)).$$

These terms are equal to zero because the map

$$\mathcal{I}_X \otimes \Omega_\Lambda^{m+1}(m+2) \to \mathcal{O}_{\mathbb{P}^r} \otimes \Omega_\Lambda^{m+1}(m+2)$$

has zero-dimensional kernel and cokernel, and $H^q(\mathcal{O}_{\mathbb{P}^r} \otimes \Omega_\Lambda^{m+1}(m+2)) = 0$. This shows that

$$\mathbf{H}^1(\mathcal{F}^\bullet) = \operatorname{Tor}_m^{S_\Lambda}(I_{X\cap\Lambda,\Lambda}, k)_{m+2}.$$

Next we turn to $'E$. We have $'E_1^{i,j} = H^j(\Lambda, \mathcal{E}^i \otimes \Omega_\Lambda^{m+1}(m+2))$. If $0 < j < \dim\Lambda$, then Bott's formula gives $'E_1^{i,j} = 0$ unless $j = m+1$ and $i = -m$. As X satisfies property $\mathbf{N}_{2,p}$ and $m \leqslant p-1$, we get $\mathcal{E}^{-m} = \operatorname{Tor}_m^S(I_X, k)_{m+2} \otimes \mathcal{O}_{\mathbb{P}^r}(-m-2)$ so

$$'E_1^{-m,m+1} = H^{m+1}(\Lambda, \mathcal{E}^{-m} \otimes \Omega_\Lambda^{m+1}(m+2)) = \operatorname{Tor}_m^S(I_X, k)_{m+2}.$$

On the other hand, if $i \geqslant -\dim\Lambda + 1$ then $'E_1^{i,\dim\Lambda} = H^{\dim\Lambda}(\Lambda, \mathcal{E}^i \otimes \Omega_\Lambda^{m+1}(m+2)) = 0$. Thus, $'E_1^{-m,m+1}$ surjects onto

$$'E_\infty^{-m,m+1} = \mathbf{H}^1(\mathcal{F}^\bullet) = \operatorname{Tor}_m^{S_\Lambda}(I_{X\cap\Lambda,\Lambda}, k)_{m+2}.$$

This is the natural map induced by the surjection $I_X \to I_{X\cap\Lambda,\Lambda}$. □

In [EGHP04] we prove that 2-regularity for a reduced projective scheme X is equivalent to the condition that every zero-dimensional linear section of X imposes independent conditions on linear forms (equivalently every zero-dimensional linear section of X is 2-regular). Thus, Theorem 1.1 implies the following 'rigidity'.

Corollary 1.8. *If $X \subset \mathbb{P}^r$ is a reduced subscheme satisfying property $\mathbf{N}_{2,p}$ for $p = \operatorname{codim}(X, \mathbb{P}^r)$, then X is 2-regular.*

Theorem 1.2 gives a new proof of a result of Vermeire [Ver01] on rational mappings of projective space.

Corollary 1.9. *If $X \subset \mathbb{P}^r$ satisfies $\mathbf{N}_{2,2}$ and $\operatorname{Sec}(X) \neq \mathbb{P}^r$, then the linear system $|H^0(\mathcal{I}_X(2))|$ on $\mathbb{P}^r$ is one-to-one outside of $\operatorname{Sec}(X)$.*

Proof. Let $x_1, x_2 \in \mathbb{P}^r \setminus X$ be a pair of points imposing only one condition on the quadrics of $|H^0(\mathcal{I}_X(2))|$ and let $\Lambda = \overline{x_1, x_2}$ be the line they span. By Theorem 1.2 the restriction map $H^0(\mathcal{I}_X(2)) \to H^0(\mathcal{I}_{X\cap\Lambda,\Lambda}(2))$ is surjective and thus Λ must be a secant line of $X \subset \mathbb{P}^r$. □

The following corollary may be regarded as a generalization of Corollary 1.9 for the case when the property $\mathbf{N}_{2,p}$, holds for some $p \geqslant 2$.

Corollary 1.10. *Let $X \subset \mathbb{P}^r$ be a closed subscheme satisfying the property $\mathbf{N}_{2,p}$ for some $p \geqslant 2$, and let $x_1, \ldots, x_p \in \mathbb{P}^r \setminus X$ be points in linearly general position which fail to impose independent conditions on the quadrics containing X. Let $\Lambda \cong \mathbb{P}^{p-1}$ be the linear span of $\{x_1, \ldots, x_p\}$ and assume that $\Lambda \cap X$ is zero-dimensional and reduced. Then for some $2 \leqslant q \leqslant p$ there exist subsets $Z_1 \subset \{x_1, \ldots, x_p\}$ and $Z_2 \subset \Lambda \cap X$, both of cardinality q, such that $Z_1 \cup Z_2$ spans a $\mathbb{P}^{q-1}$ and fails (exactly by one) to impose independent conditions on quadrics in $\mathbb{P}^{q-1}$ (in other words, $Z_1 \cup Z_2$ is self-associated).*

See Eisenbud and Popescu [EP00] for the connection with self-association (i.e. self duality under the Gale transform) and the Gorenstein property.

Proof. By Theorem 1.2 the restriction map $H^0(\mathcal{I}_X(2)) \to H^0(\mathcal{I}_{X\cap\Lambda,\Lambda}(2))$ is surjective, so the hypothesis means that the points $x_1, \ldots, x_p \in \Lambda$ fail to impose independent conditions on the quadrics in $|H^0(\mathcal{I}_{X\cap\Lambda,\Lambda}(2))|$. On the other hand, by Theorem 1.1 we know that $\deg(X \cap \Lambda) \leqslant p$. The conclusion follows now from a result of Dolgachev and Ortland [DO88, Lemma 3, p. 45] and Shokurov [Sho71] which implies that every subscheme of $\Gamma := (\Lambda \cap X) \cup \{x_1, \ldots, x_p\} \subset \Lambda$ of degree $\leqslant 2p$ does impose independent conditions on quadrics in Λ if no subset of $2s + 2 < 2p + 2$ points of Γ is contained in a $\mathbb{P}^s$. □

2. Monomial ideals satisfying $\mathbf{N}_{2,p}$

In this section we analyze the conditions $\mathbf{N}_{2,p}$ for monomial ideals. We shall see that in the saturated case (and somewhat more generally) Theorem 1.1 provides a criterion to decide which of these conditions are satisfied.

We begin with the case of square-free monomial ideals. Using the Stanley–Reisner correspondence, a square-free monomial ideal $I \subset S = k[x_0, \ldots, x_r]$ corresponds to a simplicial complex $\Delta(I)$ with vertices the variables of the ring S (see, for instance, Stanley [Sta96] for details). We will denote by I_Δ the Stanley–Reisner ideal corresponding to a simplicial complex Δ, and for simplicity we will assume that no variable x_i is among the minimal generators of I_Δ.

Recall that if G is a graph, then a *clique* of G is a subset T of vertices of G such that G contains every edge joining two vertices of T. The *clique complex* or *flag complex* of G is the simplicial complex $\Delta(G)$ whose faces are the cliques of G; the graph G can be recovered as the 1-skeleton of $\Delta(G)$. For example, the order complex of a poset P (the complex whose faces are the chains of P) is a clique complex (of the comparability graph of P). In particular, the barycentric subdivision of any simplicial complex is a clique complex.

It is easy to see that a simplicial complex Δ is a clique complex if and only if every minimal non-face of Δ consists of two vertices. Thus, Δ is a clique complex if and only if I_Δ is generated by quadratic monomials.

A *cycle* C in G of length q is a sequence of distinct vertices $v_1, \ldots, v_q$ such that each of the pairs $(v_1, v_2), (v_2, v_3), \ldots, (v_q, v_1)$ is an edge of G. We say that the cycle C of length > 3 has a *chord* if some further edge (v_i, v_j) belongs to G. We say that the cycle is *minimal* if $q > 3$ and C has no chord. The first homology group of $\Delta(G)$ is generated by minimal cycles. The graph G is called *chordal* if every cycle of length > 3 has a chord.

The main theorem of Fröberg [Fro90] asserts that a square-free monomial ideal I_Δ is 2-regular if and only if Δ is the clique complex of a chordal graph. The following refinement was suggested to us by Serkan Hoşten, Ezra Miller, and Bernd Sturmfels.

Theorem 2.1. *Let $\Delta = \Delta(G)$ be the clique complex of a graph G, and let $I = I_\Delta$ be the corresponding ideal generated by quadratic square-free monomials. The ideal I satisfies the condition $\mathbf{N}_{2,p}$, $p \geqslant 1$, if and only if every cycle of G with length at most $p+2$ has a chord.*

Example 2.2. For example, let Δ be the simplicial complex with $d+1$ vertices and $d+1$ edges forming a simple cycle. By Theorem 2.1 the ideal I_Δ satisfies $\mathbf{N}_{2,d-2}$, but not $\mathbf{N}_{2,d-1}$. In fact, the minimal free resolution of S/I_Δ has the form

$$0 \to S(-d-1) \to S(-d+1)^{\beta_{d-2}} \to \cdots \to S(-2)^{\beta_1} \to S$$

as one can see by direct computation or by using Reisner's theorem, quoted in the proof of Theorem 2.1 below. The algebraic set $X \subset \mathbb{P}^d$ corresponding to I_Δ is the union of d lines forming a cycle, a curve of degree $d+1$ and arithmetic genus one: a degenerate *elliptic normal curve* in $\mathbb{P}^d$. If Λ is a hyperplane not containing any components of X, then $\Lambda \cap X$ is a set of $d+1$ points in a $(d-1)$-dimensional plane, and is thus not 2-regular. Thus, Theorem 1.1 also implies that I_Δ does not satisfy condition $\mathbf{N}_{2,d-1}$.

Proof of Theorem 2.1. We use Reisner's theorem (see, for example, Hochster [Hoc77], or Stanley [Sta96]). If $I_\Delta \subset S = k[x_0, \ldots, x_r]$ is a square-free monomial ideal corresponding to the simplicial complex Δ, then $\operatorname{Tor}_i^S(I_\Delta, k)$ is a $\mathbb{Z}^{r+1}$-graded vector space which is nonzero only in degrees corresponding to square-free monomials m and

$$\operatorname{Tor}_i^S(I_\Delta, k)_m = \widetilde{H}_{\deg(m)-i-2}(|m|, k),$$

where $\widetilde{H}_i(|m|, k)$ denotes the ith reduced homology of the full subcomplex $|m|$ of Δ whose vertices correspond to the variables dividing m.

Let $x_0, \ldots, x_r$ be the vertices of G, and write $S = k[x_0, \ldots, x_r]$ for the ambient polynomial ring. Let X be the algebraic set defined by I_Δ in $\mathbb{P}^r$.

First assume that G has a minimal cycle C of length $p+2 > 3$. Let J be the ideal generated by the variables not in the support of C, and let Λ be the projective linear subspace in $\mathbb{P}^r$ defined by J. The plane section $\Lambda \cap X \subset \Lambda$ has homogeneous coordinate ring $S/(I_\Delta + J) = S'/I_C$ where $S' = S/J$. As we have shown in the example above, the ideal I_C is not 2-regular. By Theorem 1.1, the ideal I_Δ does not satisfy $\mathbf{N}_{2,p}$. (Of course the same result may be proven by applying Reisner's theorem directly to Δ, by taking $|m| = C$.)

Conversely, suppose that I does not satisfy the condition $\mathbf{N}_{2,p}$, and take $p > 1$ minimal with this property. We must show that Δ contains a minimal $(p+2)$-cycle.

By Reisner's theorem there exists a square-free monomial m of minimal degree $\deg(m) \geqslant p+2$ such that $\widetilde{H}_{\deg(m)-p-1}(|m|, k) \neq 0$, while $\widetilde{H}_{\deg(m')-i-2}(|m'|, k) = 0$ for all $0 \leqslant i \leqslant \min(p-1, \deg(m')-3)$ and all $m'|m$ with $m' \neq m$. If $\deg(m) = p+2$, then $\widetilde{H}_1(|m|, k) \neq 0$ or equivalently the edge-path group of the simplicial complex $|m|$ is not trivial. Since m is of minimal degree with the above property, the simplicial complex $|m|$ must be connected, and again minimality and the fact that Δ is a clique complex imply that $|m|$ consists of a cycle of length $p+2$ in G, and this cycle is minimal (see also Spanier [Spa66, Theorem 3, p. 140] for a description by generators and relation of the edge-path group). This is exactly the claim of the theorem.

If, however, $\deg(m) > p+2$, let $m'|m$ be a square-free monomial with $\deg(m') = \deg(m) - 1$ and denote by x the extra variable in the support of m. There is a long exact sequence

$$\ldots\ \widetilde{H}_i(|m'|, k) \to \widetilde{H}_i(|m|, k) \to \widetilde{H}_{i-1}(\operatorname{link}(x, |m|), k) \to \widetilde{H}_{i-1}(|m'|, k)\ \ldots$$

which is obtained from the long exact homology sequence of the pair $(|m|, |m'|)$ and the isomorphisms

$$\widetilde{H}_i(|m|, |m'|, k) \cong \widetilde{H}_i(\mathrm{star}(x, |m|), \mathrm{link}(x, |m|), k) \cong \widetilde{H}_{i-1}(\mathrm{link}(x, |m|), k)$$

for all i. The last isomorphism comes from the long exact sequence of the second pair which breaks up into isomorphisms since $\mathrm{star}(x, |m|)$ is contractible.

Since $\widetilde{H}_{\deg(m)-p-1}(|m|, k) \neq 0$ while $\widetilde{H}_{\deg(m)-p-1}(|m'|, k) = 0$, we deduce from the long exact sequence that $\widetilde{H}_{\deg(m)-p-2}(\mathrm{link}(x, |m|), k) \neq 0$, with $\deg(m)-p-2 \geqslant 1$. On the other hand, the simplicial complex $\mathrm{link}(x, |m|)$ is a full (strict) subcomplex of $|m|$ and thus of Δ. Indeed if $x_{i_1}, \ldots, x_{i_s} \in \mathrm{link}(x, |m|)$ are vertices such that $\{x_{i_1}, \ldots, x_{i_s}\} \in |m| \subseteq \Delta$, then obviously $\{x_{i_a}, x_{i_b}\} \in \Delta$ for all $a \neq b$, and also $\{x, x_{i_a}\} \in \Delta$ by the definition of the link. Since Δ is a clique complex it follows that $\{x, x_{i_1}, \ldots, x_{i_s}\}$ must also be a face of Δ with support in $|m|$. However, this means that we have found a full subcomplex $|m''| = \mathrm{link}(x, |m|)$ of Δ, with $\deg(m'') < \deg(m)$, such that $\widetilde{H}_j(|m''|, k) \neq 0$ for some $j \geqslant 1$, which contradicts the fact that I_Δ satisfies property $\mathbf{N}_{2,p-1}$. This concludes the proof of the theorem. □

As Fröberg remarks, the case of a general ideal $I \subset S = k[x_0, \ldots, x_r]$ generated by quadratic monomials may be reduced, by the process of polarization, to the square-free case. However, we can give in the following a more explicit result. Let I be any ideal generated by quadratic monomials. We write I in the form $I = I_\Delta + I_s$ where Δ is a clique complex with vertices $x_0, \ldots, x_r$, and $I_s = (\{x_i^2 \mid x_i^2 \in I\})$. We will refer to the vertices x of Δ such that $x^2 \in I$ as the *square vertices for I*.

Proposition 2.3. *Let $I = I_\Delta + I_s$ be an ideal generated by quadratic monomials, decomposed as above.*

(a) *The ideal I satisfies $\mathbf{N}_{2,2}$ if and only if I_Δ satisfies $\mathbf{N}_{2,2}$, no two square vertices are adjacent, and the link of each square vertex is a simplex.*

(b) *If I satisfies $\mathbf{N}_{2,2}$, then I satisfies $\mathbf{N}_{2,p}$ for some $p \geqslant 3$ if and only if I_Δ satisfies $\mathbf{N}_{2,p}$.*

Proof. If $I_s = (0)$ the result is obvious. Otherwise, let x be a square vertex for I, and let $I' = I_\Delta + I'_s$, where $I'_s \subset I_s$ is the ideal generated by the squares of all square vertices for I other than x. The exact sequence

$$0 \to ((I' : x^2)/I')(-2) \to S/I'(-2) \xrightarrow{x^2} S/I' \to S/I \to 0$$

and the observation that $(I' : x^2) = (I' : x)$ yields a short exact sequence

$$0 \to (S/(I' : x))(-2) \xrightarrow{x^2} S/I' \to S/I \to 0.$$

From the long exact sequence in Tors, we see that I satisfies property $\mathbf{N}_{2,2}$ if and only if I' satisfies $\mathbf{N}_{2,2}$ and $(I' : x)$ is generated by linear forms. On the other hand, we have $(I' : x) = I_{\mathrm{link}(x,\Delta)} + I'_s$. This is generated by linear forms if and only if $\mathrm{link}(x, \Delta)$ is a simplex not containing any of the square vertices that appear in I'_s. This proves part (a).

When $(I' : x)$ is generated by linear forms, each $\mathrm{Tor}_i^S(S/(I' : x), k)$ is concentrated in degree i. In this circumstance the long exact sequence in Tors coming from the short exact sequence above shows that I satisfies $\mathbf{N}_{2,p}$ for some $p \geqslant 3$ if and only if I' satisfies $\mathbf{N}_{2,p}$, and we are done by induction. □

Corollary 2.4. *If $I = I_X$ is the ideal of a closed subscheme $X \subset \mathbb{P}^r$, and I is generated by quadratic monomials, then I satisfies $\mathbf{N}_{2,p}$ if and only if the scheme $\Lambda \cap X$ is 2-regular for all planes Λ of dimension $\leqslant p$ having zero-dimensional intersection with X.*

We will need to know when an ideal generated by quadratic monomials is saturated.

Lemma 2.5. *Let $I = I_\Delta + I_s$ be an ideal generated by quadratic monomials, decomposed as above, with I_Δ a square-free quadratic monomial ideal and I_s the ideal generated by the squares of the square vertices for I. Then I is saturated if and only if every maximal face of Δ contains at least one nonsquare vertex for I.*

Proof. If the ideal generated by all the vertices is associated to S/I, it must annihilate a square-free monomial, and this can be taken to be the product of all vertices of some facet of Δ. Such a product is annihilated by the maximal ideal if and only if every vertex in that facet is a square vertex for I. □

Proof of Corollary 2.4. If a linear subspace Λ of dimension $\leqslant p$ meets X in a zero-dimensional scheme $X \cap \Lambda$ that is not 2-regular, then Theorem 1.1 shows that I does not satisfy $\mathbf{N}_{2,p}$.

Conversely, suppose that I does not satisfy $\mathbf{N}_{2,p}$, with $p \geqslant 2$ minimal, and decompose $I = I_\Delta + I_s$ as above. If $p > 2$, then from Proposition 2.3(b) we see that I_Δ does not satisfy $\mathbf{N}_{2,p}$, and thus the 1-skeleton of Δ has a minimal cycle C of length $p+2$. If x is a vertex of such a cycle then $\mathrm{link}(x, \Delta)$ is not a simplex, and it follows that x is not a square vertex for I. If Λ' is the linear subspace spanned by all the vertices in the cycle C, then $X \cap \Lambda' \subset \Lambda'$ is a degenerate 'elliptic normal curve' as in Example 2.2. As remarked in that example, any sufficiently general plane $\Lambda \subset \Lambda'$ of codimension 1 in Λ' is a p-plane that meets X in a zero-dimensional scheme that is not 2-regular.

Finally, suppose that I does not satisfy $\mathbf{N}_{2,2}$. We use the characterization in part (a) of Proposition 2.3. If I_Δ does not satisfy $\mathbf{N}_{2,2}$ then we proceed as before. Otherwise there is a square vertex x for I such that either the link of x in Δ is not a simplex, or the link of x in Δ is a simplex containing another square vertex for I.

Suppose we are in the first case and the link of x contains no other square vertex. We can choose vertices y, z in $\mathrm{link}(x, \Delta)$ such that $yz \in I_\Delta \subseteq I$. Factoring out all the variables except x, y, z we get from I the monomial ideal

$$\overline{I} = (x^2, yz) \subset k[x, y, z].$$

The scheme defined by $\overline{I}$ is obviously not 2-regular.

In the second case, let y be one of the square vertices for I such that $y \in \mathrm{link}(x, \Delta)$. Since I is saturated we may choose a vertex $z \in \mathrm{link}(x, \Delta)$ that is not a square vertex for I. Factoring out all the variables except x, y and z we get from I the saturated ideal $\overline{I} = (x^2, y^2) \subset k[x, y, z]$, which defines a zero-dimensional scheme that is not 2-regular. This concludes the proof of the corollary. □

The graph of facets of the Stanley–Reisner simplicial complex of a square-free, 2-regular monomial ideal is a tree (see [EGHP04] or [HHZ04].) Using this fact and Proposition 2.3 we can describe the primary decomposition of 2-regular monomial ideals completely. It follows, for example, that such an ideal has a primary decomposition for which the primary components have the form

$$(\text{ideal generated by variables}) + (\text{ideal generated by variables})^2.$$

We hope to return to this subject elsewhere.

Corollary 2.6. *The condition that a monomial ideal in $S = k[x_0, \ldots, x_r]$ satisfies property $\mathbf{N}_{2,p}$ for some $p \geqslant 1$, and in particular 2-regularity, is independent of the field k (not necessarily algebraically closed).*

The example (see Reisner [Rei76]) of the minimal triangulation of the projective plane shows that the analogous statement for 3-regularity is false.

Remark 2.7. By a result of Bayer and Stillman [BS87] (or see Eisenbud [Eis95, Theorem 15.20]) a subscheme $X \subset \mathbb{P}^r$ over a field of characteristic zero is 2-regular if and only if it has a Borel-fixed (generic) initial ideal generated by quadratic monomials. Any scheme defined by a monomial

ideal is, moreover, the degeneration by a flat family of linear sections, of a reduced union Y of planes defined by the monomials of a 'polarization' (see, for example, [Eis05]). Thus, each 2-regular projective scheme X over a field of characteristic zero, reduced or not, is associated canonically with an absolutely reduced scheme Y, a union of coordinate planes, that is also 2-regular.

3. Examples and conjectures about $\mathbf{N}_{2,p}$

From an alternative perspective the results in § 1 provide geometric explanations for the failure of property $\mathbf{N}_p$ and thus allow us to test optimality of results of Green, Ein, Lazarsfeld, and many others, mentioned in the introduction.

Perhaps the simplest example (handled by different methods in [OP01]) is the necessity of the conditions in the following.

Conjecture 3.1. Property $\mathbf{N}_p$ holds for the d-uple embedding of $\mathbb{P}^n$ if and only if:

- $n \leqslant 1$; or
- $n \geqslant 2$, $d = 2$, $p \leqslant 5$; or
- $n \geqslant 2$, $d \geqslant 3$, $p \leqslant 3d - 3$.

Jozefiak *et al.* [JPW81] show that in characteristic zero the 2-uple embedding of $\mathbb{P}^n$, $n \geqslant 3$, satisfies property $\mathbf{N}_5$. In the case of the d-uple embedding of $\mathbb{P}^2$ its minimal free resolution restricts to the minimal free resolution of a hyperplane section (a plane curve), and so Green [Gre84] implies that for $d \geqslant 3$, the d-uple embedding of $\mathbb{P}^2$ satisfies property $\mathbf{N}_{3d-3}$. See also [Rub03] for a proof of the fact that the 3-uple embedding of $\mathbb{P}^n$ satisfies property $\mathbf{N}_4$ for all n, and [HSS05] for related results and extensions in a toric setting.

In all other cases the sufficiency of the conditions in Conjecture 3.1 is open as far as we know. On the other hand, Theorem 1.1 yields the necessity of those conditions.

Proposition 3.2. *Let $n \geqslant 2$ and $d \geqslant 2$ be integers.*

(a) *If $n \geqslant 2$ and $d \geqslant 3$, then the d-uple embedding of $\mathbb{P}^n$ fails property $\mathbf{N}_{3d-2}$.*

(b) *If $n \geqslant 3$, then the 2-uple embedding of $\mathbb{P}^n$ fails property $\mathbf{N}_6$.*

Proof. For all $m < n$ the d-uple embedding of $\mathbb{P}^m$ is a linear section of the d-uple embedding of $\mathbb{P}^n$. Thus, by Theorem 1.1, for the failure of property $\mathbf{N}_p$ it is enough to produce a $(p+2)$-secant p-plane to the d-uple embedding of $\mathbb{P}^m$ for some $m < n$.

To prove part (a) we may assume that $n = 2$. For $d \geqslant 3$, a complete intersection Z of type $(3, d)$ in $\mathbb{P}^2$ is cut out by forms of degree d but fails to impose independent conditions on such forms. This means that the linear span of the d-uple embedding of Z is a $(3d-2)$-plane which is $3d$-secant to the d-uple embedding of $\mathbb{P}^2$, which thus fails property $\mathbf{N}_{3d-2}$ by Theorem 1.1.

Similarly, a complete intersection of three quadrics in $\mathbb{P}^3$ fails to impose independent conditions on quadrics, so the linear span of its 2-uple is a 6-plane which is 8-secant to the 2-uple embedding of $\mathbb{P}^3$. By Theorem 1.1, it follows that the 2-uple Veronese embedding of $\mathbb{P}^3$ fails property $\mathbf{N}_6$. □

The failure of property $\mathbf{N}_{3d-2}$ for the d-uple embedding of $\mathbb{P}^2$ can be accounted for also by the existence of a relatively long strand of linear syzygies in the minimal free resolution of $\omega_{\mathbb{P}^2}(d)$. Namely, with notation as in [EPSW02], we have the following result.

Proposition 3.3. *Let $W = H^0(\omega_{\mathbb{P}^2}(d))$ and set $w = \dim(W)$, let $U = H^0(\omega_{\mathbb{P}^2}^{-1})$, let $V = H^0(\mathcal{O}_{\mathbb{P}^2}(d))$ and $S = \mathrm{Sym}(V)$. If $d \geqslant 3$, the natural multiplication pairing $\mu : W \otimes U \to V$*

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makes $Q = \bigoplus_l (\wedge^{l+1}(W^*) \otimes \mathrm{Sym}^l(U^*))$ *into a graded* $E = \wedge^*(V^*)$*-module such that the maximal irredundant quotient of the linear complex*

$$\mathbf{L}(Q^*) : \ 0 \to \wedge^w W \otimes D_{w-1}(U) \otimes S(-w+1) \to \cdots \to \wedge^2 W \otimes U \otimes S(-1) \to W \otimes S$$

is a linear complex of the same length which injects as a degreewise direct summand into the minimal free resolution of the S*-module* $\bigoplus_{m \geqslant 0} H^0(\omega_{\mathbb{P}^2}(d(m+1)))$*. In particular the* d*-uple embedding of* $\mathbb{P}^2$ *fails property* $\mathbf{N}_{3d-2}$*.*

Proof. The above multiplication pairing μ is obviously geometrically 1-generic so the first part of the claim is a direct application of [EPSW02, Proposition 2.10] with $L = \mathcal{O}_{\mathbb{P}^2}(d-3)$, $L' = \mathcal{O}_{\mathbb{P}^2}(3)$, and $L'' = \mathcal{O}_{\mathbb{P}^2}(d)$, and with W, U, and V as in the statement of the proposition.

For the second claim observe first that the homogeneous ideal I_d of the d-uple embedding of $\mathbb{P}^2$ is generated by quadrics and is 3-regular. Thus, its minimal free resolution has two strands (linear and quadratic). On the other hand, the dual of the maximal irredundant quotient of the linear complex $\mathbf{L}(Q^*)$ has length $\binom{d-1}{2}$ and is a degreewise direct summand of the second strand into the minimal free resolution of I_d. Since the whole resolution of I_d has length $\binom{d+2}{2} - 3$ it follows that the d-uple embedding of $\mathbb{P}^2$ fails property $\mathbf{N}_{3d-2}$. □

The argument used in the proof of Proposition 3.2(a) provides upper bounds for property $\mathbf{N}_p$ for other Fano-type varieties and embeddings. For instance, for embeddings of ruled and Del Pezzo surfaces we obtain the following bounds (where Proposition 3.2(a) is the case where $S = \mathbb{P}^2$).

Proposition 3.4. *Let* S *be a smooth surface and* L *be a very ample line bundle on* S*. If* $|-K_S| \neq \emptyset$*, and* $\mathcal{O}(K_S) \otimes L$ *is globally generated, then the image of* S *via the linear system* $|L|$ *fails property* $\mathbf{N}_{-K_S \cdot L - 2}$*.*

Proof. Let $D \in |-K_S|$, let $C \in |L|$ be a general curve and denote by $Z = D \cap C$ their intersection. The Koszul complex on the sections defining D and C expands to the following commutative diagram

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & \mathcal{O}_C(-D) \arrow[r] \arrow[u] & \mathcal{O}_C \arrow[r] \arrow[u] & \mathcal{O}_Z \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathcal{O}_S(-D) \arrow[r] \arrow[u] & \mathcal{O}_S \arrow[r] \arrow[u] & \mathcal{O}_D \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathcal{O}_S(-D-C) \arrow[r] \arrow[u] & \mathcal{O}_S(-C) \arrow[r] \arrow[u] & \mathcal{O}_D(-C) \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

which we twist by L and take cohomology. From the long exact sequence of the middle row, since $H^1(\mathcal{O}(-D) \otimes L) = H^1(\mathcal{O}(K_S) \otimes L) = 0$ by Kodaira vanishing (in characteristic 0) or by Shepherd-Barron [She91] and Terakawa [Ter99, Theorem 1.6] (in positive characteristic), we deduce that the natural restriction map $H^0(L) \to H^0(L_{|D})$ is surjective. Taking cohomology of the last column

$$\cdots \to H^0(L_{|D}) \to H^0(L_{|Z}) \to H^1(\mathcal{O}_D(-C) \otimes L) \to H^1(L_{|D}) \to \cdots$$

we have $h^1(\mathcal{O}_D(-C) \otimes L) = h^0(\mathcal{O}_D) \geqslant 1$, while $h^1(L_{|D}) = h^0(L_{|D}^{-1}) = 0$ since $\mathcal{O}(K_D) = \mathcal{O}_D$ and L is ample. Putting everything together it follows that the subscheme Z fails to impose independent conditions on the sections of L. Since $\mathcal{O}(K_S) \otimes L$ is globally generated we deduce that $\mathcal{I}_Z \otimes L$

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is also globally generated. Since $\mathrm{length}(Z) = -K_S \cdot L$ the claim follows as before directly from Theorem 1.1. □

Remark 3.5. (1) By adjunction (see [Som79] or [SV87]) the line bundle $\mathcal{O}(K_S) \otimes L$ in Proposition 3.4 is globally generated if and only if (S, L) is not one of the following pairs: $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$, or $(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$ with E a rank 2 vector bundle on a curve.

(2) A similar argument as in Proposition 3.4 shows that if X is a smooth projective surface and L is a very ample divisor on it, then the embedding of X via the linear system $|K_X + (p+3)L|$ fails to satisfy property $\mathbf{N}_{3pL^2-2}$, for $p \geqslant 3$ (or fails to satisfy property $\mathbf{N}_{(2p+2)L^2-2}$ for $p \geqslant 2$, if $(X, \mathcal{O}(L)) \neq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$).

Proposition 3.6. *Let X denote the image of the Segre–Veronese embedding*

$$\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_m} \xhookrightarrow{(d_1, d_2, \ldots, d_m)} \mathbb{P}^{\prod_{i=1}^m \binom{n_i+d_i}{d_i} - 1}.$$

(a) *If $m \geqslant 3$ and $d_i = 1$ for at least three values of $1 \leqslant i \leqslant m$, then X fails property $\mathbf{N}_4$.*

(b) *If $m \geqslant 3$ and $d_i = 1$ for exactly two values of $1 \leqslant i \leqslant m$, then X fails property $\mathbf{N}_{2 \min_{\{i | d_i > 1\}} d_i + 2}$.*

(c) *If $m \geqslant 3$ and $d_i = 1$ for at most one value of $1 \leqslant i \leqslant m$, or if $m \geqslant 2$ and $d_i > 1$ for all $1 \leqslant i \leqslant m$, then X fails property $\mathbf{N}_{2 \min_{\{i \neq j | d_i, d_j > 1\}} (d_i + d_j) - 2}$.*

Proof. We argue as in the proof of Proposition 3.2 and exhibit for suitable p a p-dimensional linear subspace which is $(p+2)$-secant to the Segre–Veronese embedding of a product of $r < m$ factors. Failure of property $\mathbf{N}_p$ follows then from Theorem 1.1.

To prove (a) we may assume that $m = 3$. The linear span of the Segre–Veronese embedding of a complete intersection of type $(1,1,1)^3$ is a 6-secant $\mathbb{P}^4$, thus X fails property $\mathbf{N}_4$ in this case.

Case (b) is similar: we may assume that $m = 3$ and consider the linear span of the Segre–Veronese embedding of a complete intersection of one hypersurface of multidegree $(1,1,2)$, and two hypersurfaces of multidegree $(1,1,d)$ with $d = \min_{\{i | d_i > 1\}} d_i$.

Finally in case (c) we may assume that $m = 2$ and that both degrees are $\geqslant 2$, in which case the claim follows from Proposition 3.4 for $S = \mathbb{P}^1 \times \mathbb{P}^1$. □

Proposition 3.6 seems to be sharp. We give the results we know below.

Remark 3.7 (Some results require characteristic 0). (1) If $d_1, d_2 \geqslant 2$, then the embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ via the linear system $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(d_1, d_2)|$ satisfies $\mathbf{N}_{2d_1+2d_2-3}$ (see [GP01]), but fails to satisfy $\mathbf{N}_{2d_1+2d_2-2}$ by Proposition 3.6 or Proposition 3.4 above.

(2) Lascoux [Las78] and Pragacz and Weyman [PW85] describe the minimal free resolution of the Segre embedding of $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$. In particular, they show that it satisfies property $\mathbf{N}_p$ if and only if $p \leqslant 3$.

(3) Using simplicial methods, Rubei [Rub02, Rub04] shows that the Segre embedding of $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_m}$ (at least three factors) satisfies property $\mathbf{N}_p$ if and only if $p \leqslant 3$. Corollary 8 in [Rub02] proves part (b) in Proposition 3.6 via a different method. Related results for Segre–Veronese embeddings were recently obtained by [HSS05] as special cases of their estimates on the length of the strand of linear syzygies for toric embeddings.

(4) The resolution of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ as well as a number of other special cases where the resolution is self-dual are investigated by Barcanescu and Manolache [BM81].

Proposition 3.8. *The Plücker embedding of the Grassmannian $\mathrm{Gr}(k,n) \subset \mathbb{P}^{\binom{n}{k}-1}$, where $2 \leqslant k \leqslant n-2$ and $n \geqslant 5$, fails property $\mathbf{N}_3$.*

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Proof. It is enough to observe that for $2 \leqslant k \leqslant n-2$ and $n \geqslant 5$, the Plücker embedding of the Grassmannian $\mathrm{Gr}(k,n) \subset \mathbb{P}^{\binom{n}{k}-1}$ has as linear section the Plücker embedding of $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$. A general codimension three linear section of $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$ is a collection of five points spanning only a $\mathbb{P}^3$, so the conclusion follows now as above from Theorem 1.1. □

Remark 3.9. (1) Jerzy Weyman informed us that property $\mathbf{N}_2$ always holds for the Plücker embedding of any Grassmannian, so Proposition 3.8 is sharp.

(2) Manivel [Man96] proved that if $X = G/P$, where $G = \mathrm{SL}(V)$, V is a complex vector space and P a parabolic subgroup, and L is a very ample line bundle on X, then the embedding defined by the complete linear system $|L^p|$ satisfies property $\mathbf{N}_p$ for all $p \geqslant 1$.

Recall that a complete linear system $|L|$ on a projective variety X is said to be k-very ample if for any zero-dimensional subscheme $Z \subset X$ of length $k+1$ the restriction map

$$H^0(L) \to H^0(L_{|Z})$$

is surjective. In particular 0-very ample is 'base point free' and 1-very ample is 'very ample'.

Pareschi and Popa [Par00, PP03] proved in characteristic 0 that if X is an abelian variety and $L_1, \ldots, L_{p+3}$ are ample line bundles on X then the embedding of $X \subset \mathbb{P}^N$ by the linear system $|L_1 \otimes \cdots \otimes L_{p+3}|$ satisfies property $\mathbf{N}_p$. If Λ a linear subspace of dimension $\leqslant p$ of $\mathbb{P}^N$, then Theorem 1.1 and the classification of small algebraic sets in [EGHP04] show that every positive-dimensional reduced irreducible component of $\Lambda \cap X$ is a variety of minimal degree in its linear span. In particular, the components of $\Lambda \cap X$ are rational. Since abelian varieties do not contain rational positive dimensional subvarieties, Theorem 1.1 implies that $L_1 \otimes \cdots \otimes L_{p+3}$ is $(p+1)$-very ample, which is a special case of [BS97a, Theorem 1].

Observe also that if $X = \prod_{i=1}^{\dim(X)} E_i$ is a product of elliptic curves, each with origin o_{E_i}, and $L := \prod_i p_i^*(\mathcal{O}_{E_i}(o_{E_i}))$ is the canonical principal polarization on X, then L^{p+3} fails to satisfy property $\mathbf{N}_{p+1}$. This is a consequence of Theorem 1.1 and Abel's theorem since one may choose $(p+3)$ points on E_i such that any divisor in the linear system $|(p+3)o_{E_i}|$ containing $(p+2)$ of those points contains also the remaining point.

Gross and Popescu [GP98] conjectured that the general $(1,d)$-polarized abelian surface, for $d \geqslant 10$, satisfies property $\mathbf{N}_{[d/2]-4}$. As above, by Theorem 1.1, this would imply that a $(1,d)$-polarization on a general abelian surface is k-very ample if $d \geqslant 2k+3$ and $d \geqslant 10$ (compare again with [BS97a, Theorem 1] and [BS97b]).

4. Secants and syzygy varieties

In this section we analyze the restriction of linear syzygies to nonlinear varieties with known syzygies such as rational normal curves, rational scrolls and Veronese surfaces.

Theorem 4.1. *Let $X, \Gamma \subset \mathbb{P}^r$ be subschemes such that X is nondegenerate and Γ is reduced with every irreducible component spanning all of $\mathbb{P}^r$. If the natural restriction map*

$$\mathrm{Tor}_p^S(I_{X\cup\Gamma}, k)_{p+2} \to \mathrm{Tor}_p^S(I_X, k)_{p+2}$$

is not surjective, then

$$h^0(\mathcal{I}_{X\cap\Gamma}(2)) > p + h^0(\mathcal{I}_\Gamma(2)).$$

In particular, $h^0(\mathcal{I}_{X\cap\Gamma,\Gamma}(2)) > p$.

Proof. The second statement follows from the first by using the short exact sequence

$$0 \to \mathcal{I}_\Gamma(2) \to \mathcal{I}_{X\cap\Gamma}(2) \to \mathcal{I}_{X\cap\Gamma,\Gamma}(2) \to 0.$$

To prove the first statement we use the Eisenbud–Koh–Stillman conjecture (EKS) (proved by Green [Gre99]; see also [EK91]) which says that if $M = \bigoplus_{i\geqslant 0} M_i$ is a finitely generated graded module over the polynomial ring $S = \mathrm{Sym}(V)$ such that:

(a) $\ker(\wedge^p V \otimes M_0 \to \wedge^{p-1} V \otimes M_1) \neq 0$ for some $p > 0$; and, moreover,

(b) $\dim M_0 \leqslant p$;

then there exist a p-dimensional family of rank one relations (i.e. decomposable tensors) in the kernel of the multiplication map $V \otimes M_0 \to M_1$.

We will not need the full strength of EKS, but just the existence of such rank one relations under the above hypothesis, and we will apply EKS to

$$M = \bigoplus_{i\geqslant 0} \frac{H^0(\mathcal{I}_{X\cap\Gamma}(i+2))}{H^0(\mathcal{I}_{\Gamma}(i+2))}$$

regarded as a finitely generated module over $S = \mathrm{Sym}(V)$, the polynomial ring of the ambient $\mathbb{P}^r$.

There are no rank one relations in the kernel of the multiplication morphism $V \otimes M_0 \to M_1$. Such a rank one relation would amount to the existence of a quadric defined by $Q \in H^0(\mathcal{I}_{X\cap\Gamma}(2))$ not vanishing on Γ and a hyperplane defined by $H \in H^0(\mathcal{O}_{\mathbb{P}^r}(1))$ such that $QH \in H^0(\mathcal{I}_{\Gamma}(3))$, which is impossible since each irreducible component of Γ is assumed to be nondegenerate.

We will relate condition (a) in EKS for the module M to the analogous one for the module

$$P = \bigoplus_{i\geqslant 0} \frac{H^0(\mathcal{I}_{X}(i+2))}{H^0(\mathcal{I}_{X\cup\Gamma}(i+2))}.$$

Expressing as usual the Tor's via Koszul cohomology our hypothesis that

$$\mathrm{Tor}_p^S(I_{X\cup\Gamma}, k)_{p+2} \to \mathrm{Tor}_p^S(I_X, k)_{p+2}$$

is not surjective translates into the existence of an element

$$\alpha \in \ker(\wedge^p V \otimes H^0(\mathcal{I}_X(2)) \to \wedge^{p-1} V \otimes H^0(\mathcal{I}_X(3)))$$

which is not in the image of the natural inclusion morphism

$$\wedge^p V \otimes H^0(\mathcal{I}_{X\cup\Gamma}(2)) \to \wedge^p V \otimes H^0(\mathcal{I}_X(2)).$$

Taking global sections in the first row of the exact diagram of ideal sheaves

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}_{X\cup\Gamma}(2) \arrow[r] \arrow[d, hook] & \mathcal{I}_{X}(2) \arrow[r] \arrow[d, hook] & \mathcal{I}_{X\cap\Gamma,\Gamma}(2) \arrow[d, "\cong"'] & \\
0 \arrow[r] & \mathcal{I}_{\Gamma}(2) \arrow[r] & \mathcal{I}_{X\cap\Gamma}(2) \arrow[r] & \mathcal{I}_{X\cap\Gamma,\Gamma}(2) \arrow[r] & 0
\end{tikzcd}$$

we see that α induces a nontrivial element $\bar{\alpha}$ in

$$\bar{\alpha} \in \ker(\wedge^p V \otimes P_0 \to \wedge^{p-1} V \otimes P_1).$$

On the other hand, twisting and taking global sections in the above diagram yields the inclusion $P \subseteq M$. In particular, we may view $\bar{\alpha}$ as an element of $\ker(\wedge^p V \otimes M_0 \to \wedge^{p-1} V \otimes M_1)$, which is thus nonzero. By EKS, since there are no rank one relations in the kernel of $V \otimes M_0 \to M_1$, we deduce that $\dim M_0 > p$ which finishes the proof of the theorem. □

The following result, with $Y = X \cup \Gamma$, gives us a way to apply Theorem 4.1.

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Proposition 4.2. *Suppose that $X \subseteq Y \subset \mathbb{P}^r$ are nondegenerate closed subschemes. If X satisfies property $\mathbf{N}_{2,p+1}$ for some $0 \leqslant p < \operatorname{codim} X$ and the natural map*

$$\operatorname{Tor}_p^S(I_Y, k)_{p+2} \to \operatorname{Tor}_p^S(I_X, k)_{p+2}$$

is surjective, then $X = Y$.

Proof. We do induction on p. If $p = 1$, then all of the quadrics in the ideal of X are also in the ideal of Y; since X is defined by quadrics we get $X = Y$.

Now suppose $p > 1$. Let

$$\mathbf{F}_\bullet : \quad F_p \to F_{p-1} \to \cdots \to F_0 \to I_X \to 0$$

be the first p steps of a minimal resolution, so that $F_p = S(-p-2)^{\beta_p}$ for some β_p, and let

$$\mathbf{G}_\bullet : \quad G_p \to G_{p-1} \to \cdots \to G_0 \to I_Y \to 0$$

be the first p steps of the 2-linear part of the resolution of I_Y. The inclusion of ideals induces a map $\mathbf{G}_\bullet \to \mathbf{F}_\bullet$ of complexes, and G_i is a direct summand of F_i for all $i \in \{0, \ldots, p\}$.

Now suppose that

$$\operatorname{Tor}_p^S(I_Y, k)_{p+2} \to \operatorname{Tor}_p^S(I_X, k)_{p+2}$$

is surjective or, equivalently, $F_p = G_p$. The differential $F_p \to F_{p-1}$ maps F_p into the direct summand G_{p-1}. Since $p < \operatorname{codim} X$, the dual of this map is part of the minimal free resolution of $\operatorname{coker}(F_{p-1}^* \to F_p^*)$. If $F_{p-1} \neq G_{p-1}$, then the dual map $F_{p-1}^* \to F_p^*$ would send a nontrivial free summand to 0, a contradiction. Thus, $F_{p-1} = G_{p-1}$ and we are done by induction. □

Combining Theorem 4.1 with Proposition 4.2 we get the first statement of the following result.

Corollary 4.3. *Suppose that $X, \Gamma \subset \mathbb{P}^r$ are nondegenerate closed subschemes and that Γ is reduced and irreducible. If $\Gamma \not\subset X$, and X satisfies property $\mathbf{N}_{2,p+1}$ for some $0 \leqslant p < \operatorname{codim} X$, then*

$$h^0(\mathcal{I}_{X\cap\Gamma,\Gamma}(2)) \geqslant h^0(\mathcal{I}_{X\cap\Gamma}(2)) - h^0(\mathcal{I}_\Gamma(2)) > p.$$

In particular, if Γ is a rational normal curve in $\mathbb{P}^r$, then

$$\operatorname{length}(X \cap \Gamma) < 2r + 1 - p.$$

Proof. The last statement follows because, under the given hypothesis, $\mathcal{I}_{X\cap\Gamma,\Gamma}(2)$ is a line bundle on Γ of degree $2r - \operatorname{length}(X \cap \Gamma)$. □

Remark 4.4. In the special case where both X and Γ are rational normal curves in $\mathbb{P}^r$, Corollary 4.3 yields that X and Γ can meet at most in $2r+1-(r-1) = r+2$ points. Equality can occur: the union of the two rational normal curves is a degeneration of a canonical curve in $\mathbb{P}^r$ (a so-called 'binary' curve). See also Eisenbud and Harris [EH92a, EH92b], or Diaz [Dia86] and Giuffrida [Giu88] for related results.

We can use a different method of checking the hypothesis of Theorem 4.1 to derive a new proof of Green's 'syzygetic Castelnuovo lemma' [Gre84] (see also Ehbauer [Ehb94], Yanagawa [Yan94], and Eisenbud and Popescu [EP99]).

Corollary 4.5. *Let $X \subset \mathbb{P}^r$ be a finite subscheme which contains a subscheme of length $r + 3$ in linearly general position. If $\operatorname{Tor}_{r-2}(I_X, k)_r \neq 0$, then X lies on a (unique) smooth rational normal curve.*

Proof. Corollary 4.3 (or Remark 4.4) shows that two distinct rational normal curves in $\mathbb{P}^r$ can meet in at most $r+2$ points, so there is at most one rational normal curve containing X.

Let $X' \subset X$ be a subscheme of length $r+3$ in linearly general position. By Eisenbud and Harris [EH92a, EH92b] or Eisenbud and Popescu [EP00], there is a rational normal curve Γ containing X'. Suppose that X is not contained in Γ. The quadrics containing $X \cup \Gamma$ must be a proper subset of the quadrics containing Γ. By Ehbauer [Ehb94] or Eisenbud-Popescu [EP99], the syzygy ideal of any $(r-2)$-syzygy of I_Γ is precisely I_Γ, so $\mathrm{Tor}_{r-2}(I_{X\cup\Gamma}, k)_r = 0$. Since $\mathrm{Tor}_{r-2}(I_\Gamma, k)_r \neq 0$ the hypotheses of Theorem 4.1 are thus satisfied, so $h^0(\mathcal{O}_\Gamma(2H - (X \cap \Gamma))) > r-2$, and thus the length of $X \cap \Gamma$ is at most $r+2$, a contradiction. □

Remark 4.6. The hypothesis that Γ is nondegenerate in $\mathbb{P}^r$ cannot be dropped in Corollary 4.3. Similarly the number r in the second statement of Corollary 4.3 cannot be replaced by the dimension of the span of Γ. For example, let X be the cone in $\mathbb{P}^4$ over a twisted cubic curve, say

$$X = \left\{ x \;\middle|\; \mathrm{rank}\begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{pmatrix} \leqslant 1 \right\} \subset \mathbb{P}^4 = \mathbb{P}^4(x_0, x_1, x_2, x_3, x_4)$$

and let $\Lambda = \{x_2 = x_3 = 0\} \subset \mathbb{P}^4$. Then $X \cap \Lambda = \{x_2 = x_3 = x_1x_4 = 0\}$ which is a degenerate conic (union of two lines). If Γ is a smooth conic in Λ, then $h^0(\mathcal{I}_{X\cap\Gamma,\Gamma}(2)) = 1$ and $\mathrm{length}(X \cap \Gamma) = 4$, whereas from Corollary 4.3 with $p = 1$ we would get $h^0(\mathcal{I}_{X\cap\Gamma,\Gamma}(2)) > 1$ and $\mathrm{length}(X \cap \Gamma) < 2 \cdot 2 + 1 - 1 = 4$.

Coble [Cob22] and Conner [Con11] assert that two Veronese surfaces in $\mathbb{P}^5$ can intersect in at most 10 points: see [EHP03] for a modern treatment. Using directly Theorem 4.1 one can prove a (nonoptimal) bound of 12.

We can also get a result for zero-dimensional intersections of scrolls.

Proposition 4.7. *Let X and Γ be two nondegenerate rational scrolls of dimensions m and n, respectively, in $\mathbb{P}^r$ with $m \leqslant n$ and such that $X \cap \Gamma$ is a zero-dimensional scheme. Then* $\mathrm{length}(X \cap \Gamma) \leqslant nr + m - \binom{n}{2} + 1$.

Proof. From Corollary 4.3 we obtain that $h^0(\mathcal{I}_{X\cap\Gamma,\Gamma}(2)) \geqslant r - m$. On the other hand, if we restrict the minimal free resolution of I_X to Γ we obtain a complex which fails to be exact off a zero-dimensional subscheme of Γ. Since I_X is 2-regular and Γ is rational scroll, standard cohomological vanishings and an argument as in Theorem 1.1, or as in Lazarsfeld [Laz04, Proposition B.1.2], show that $h^1(\mathcal{I}_{X\cap\Gamma,\Gamma}(2)) = 0$. We have also $h^0(\mathcal{I}_\Gamma(2)) = \binom{r-n+1}{2}$ (for instance, from the Eagon–Northcott complex) and the claimed bound follows now via direct computation. □

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