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Restriction of stable sheaves and representations of the fundamental group

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Introduction

Let X be a projective smooth variety of dimension n over an algebraically closed field k. Let H be an ample line bundle on X. A torsion free sheaf V on X is said to be stable (respectively, semistable) with respect to the polarisation H if for every proper subsheaf $W \subset V$ we have $\deg W/\operatorname{rk} W < \deg V/\operatorname{rk} V$ (respectively $\leqq$) where $\deg W = c_1(W) \cdot H^{n-1}$, $c_1(W)$ the first chern class and rk = rank (see [7, 13]).

In [10] we proved that the restriction of a semistable sheaf V on X to a complete intersection subvariety of X in general position and of high multidegree is again semistable. We prove here that the restriction of a stable sheaf also remains stable. This has some interesting consequences.

When $k = \mathbb{C}$ and $\dim X = 2$ it follows from the recent results of Donaldson [2] and Kobayashi [6] that any stable vector bundle V with $c_1(V) = 0$ and $c_2(V) = 0$ on the surface X comes from an irreducible unitary representation of the fundamental group $\pi_1(X)$. It follows from this and our restriction theorem that the same result holds for higher dimensional varieties as well. This answers a question of Kobayashi [6, Sect. 4, p. 161].

1. Preliminaries

1.1. We recall some notation from [10]. Let X be a projective smooth variety of dimension $n \geqq 2$ over an algebraically closed field k, and H a very ample line bundle on X. For $\mathbf{m} = (m_1, \ldots, m_t)$, a sequence of positive integers, let $S_{\mathbf{m}}$ be $S_{m_1} \times \ldots \times S_{m_t}$, where $S_{m_i} = \mathbb{P} H^0(X, H^{m_i})$. Let $Z_{\mathbf{m}} \subset X \times S_{\mathbf{m}}$ be the correspondence variety:

$$Z_{\mathbf{m}} = \{(x, s_1, \ldots, s_t) \mid s_i(x) = 0 \ \forall i\}.$$

Let $p_{\mathbf{m}}: Z_{\mathbf{m}} \to S_{\mathbf{m}}$ and $q_{\mathbf{m}}: Z_{\mathbf{m}} \to X$ be the projections.

The fibres of $q_{\mathbf{m}}$ are embedded in X by $p_{\mathbf{m}}$ and we usually identify $q_{\mathbf{m}}^{-1}(s)$ with $p_{\mathbf{m}}q_{\mathbf{m}}^{-1}(s)$. The function field of $S_{\mathbf{m}}$ is denoted by $K_{\mathbf{m}}$ and $Y_{\mathbf{m}}$ denotes the generic fibre $Z_{\mathbf{m}} \times_{K_{\mathbf{m}}} S_{\mathbf{m}}$ of $q_{\mathbf{m}}$. Let $\varphi_{\mathbf{m}}$: $Y_{\mathbf{m}} \to Z_{\mathbf{m}}$ be the inclusion.

We call $Y_{\mathbf{m}}$ the generic complete intersection subvariety of type $\mathbf{m}$. When a property holds for $q_{\mathbf{m}}^{-1}(s)$ for s in a nonempty open subset of $S_{\mathbf{m}}$ we say that it holds for a general s.

1.2. We next recall a few well known facts about vector bundles on curves [7, 13, 16]. Let C be a projective smooth curve over k. Let V be a vector bundle on C. We denote $\deg V/rk V$ by $\mu(V)$. A subsheaf $W \hookrightarrow V$ is called a subbundle of V if V/W is torsion free (and hence, C being a curve, locally free). The *subbundle* $\overline{W}$ *generated* by W is the inverse image in V of the torsion subsheaf of V/W.

Suppose V is semistable. Then

i) Any subsheaf $W \hookrightarrow V$ with $\mu(W) = \mu(V)$ is a subbundle and is semistable.

ii) Any homomorphism $W \to V$, where W and V are semistable with $\mu(W) = \mu(V)$, is of constant rank. In particular, if L is a line bundle with $\deg L = \mu(V)$ then any nonzero map $L \to V$ makes L a subbundle of V.

iii) If W, V are stable with $\mu(W) = \mu(V)$ then any nonzero map $W \to V$ is an isomorphism.

iv) Any semistable bundle V has a filtration $0 = V_0 \subset V_1 \subset \ldots \subset V_r = V$ with V_i/V_{i-1} stable and $\mu(V_i/V_{i-1}) = \mu(V)$. We call such a filtration a *stable filtration*. It is not unique. However, the associated graded $\mathrm{gr} V = \bigoplus_{i=1}^{r} V_i/V_{i-1}$ depends only on V. (See [16, Sect. 3]).

v) If V is stable then V is simple i.e. End V = scalars (This is true more generally for higher dimensional base as well).

2. Canonical subbundles of semistable bundles

In this section C will denote a projective smooth curve over k.

2.1. Definition. *Let V be a semistable vector bundle on C. The subbundle of V given by the sum of all stable subbundles W of V with $\mu(W) = \mu(V)$ is called the* socle *of V*

2.2. Lemma. *Let V be a semistable vector bundle on C. Then*

i) *The socle of V is a direct sum of certain stable subbundles of V, each with the same μ as V.*

ii) *If V is semistable but not a direct sum of stable bundles then the socle of V is a proper subbundle of V.*

iii) *The socle of V is invariant under all automorphisms of V.*

iv) *If C and V are defined over a field K and the socle of V (more precisely the inclusion of the socle of V in V) is defined over a Galois extension L of K then the socle of V is invariant under the Galois group* Gal(L/K) *and hence defined over K.*

Proof. Clear.

2.3. To get rationality of the subbundle over inseparable field extensions we introduce the following notion. Let V be a semistable bundle and consider the following two conditions on subbundles $W \hookrightarrow V$ a) $\mu(W)=\mu(V)$ b) each component of $\operatorname{gr} W$ (see Sect. 1.2. iv) is isomorphic to a subbundle of V. If two subbundles W_1 and W_2 satisfy both a) and b) then clearly so does their sum $W_1 + W_2$. Therefore there is a unique maximal subbundle satisfying a) and b).

2.4. Definition. *For a semistable vector bundle V we call the maximal subbundle satisfying the above conditions* a) *and* b) *the* extended socle of V.

2.5. Lemma. *Let V be a semistable vector bundle over C and W its extended socle.*

i) *W contains the socle of V.*

ii) *If further V is simple (i.e.* $\operatorname{End} V =$ *scalars) and not stable then W is a proper subbundle.*

iii) *W is invariant under all automorphisms of V. Moreover* $\operatorname{Hom}(W, V/W) = 0$.

iv) *If C and V are defined over a field K then so is $W \hookrightarrow V$.*

Proof. i) Clear.

ii) Let $V_0 \subset \ldots \subset V_{r-1} \subset V$ be a stable filtration of V (see 1.2. iv). If $W = V$ then from the definition of extended socle V/V_{r-1} is isomorphic to a subbundle W' of V. Then the composite $V \to V/V_{r-1} \approx W' \to V$ gives a nonscalar endomorphism of V.

iii) Suppose σ: $W \to V/W$ is a nontrivial homomorphism. Let $W_0 \subset \ldots \subset W_i \subset \ldots \subset W$ be a stable filtration of W. Then for some i, σ maps W_i/W_{i-1} nontrivially and hence injectively. Then the inverse image of $\sigma(W_i/W_{i-1})$ under $V \to V/W$ satisfies conditions a) and b) of Sect. 2.3 (use Sect. 1.2). Since it is also strictly bigger than W this contradicts the maximality of W.

iv) The extended socle is always defined over the algebraic closure $\bar{K}$ of K (cf. [7]) and hence over a finite extension of K. If $W \to V$ is defined over a Galois extension L/K then it is $\operatorname{Gal}(L/K)$ invariant and hence descends to K. If $W \to V$ is defined over a purely inseparable extension L/K of exponent one we can apply Jacobson descent: $\operatorname{Hom}(W, V/W)$ being zero W is invariant under all derivations of L/K (cf. [7]).

2.6. Remark. Suppose C and the vector bundle V are defined over K. If char $K = 0$ suppose $V \otimes \bar{K}$ is not a direct sum of stable bundles. If char $K = p > 0$ suppose moreover $V \otimes \bar{K}$ (or equivalently V) is simple. Then if every subbundle W of V defined over K satisfies $\mu(W) < \mu(V)$ then it is satisfied for all subbundles and hence V is stable. In other words for such V the notion of stability is invariant under change of base field. This follows easily from Lemmas 2.2 and 2.5. We have restricted ourselves to algebraic extension fields. These assertions also hold for arbitrary extensions (cf. [7]).

2.7. Lemma. *Let V be a semistable vector bundle on C. Then the set* $\{\det W \mid W$ *a subbundle of V with $\mu(W)=\mu(V)\}$ of isomorphism classes of line bundles is finite.*

Proof. Let $\operatorname{gr} V \cong V_1 \oplus \ldots \oplus V_r$, V_i stable with $\mu(V_i)=\mu(V)$. Let W be a subbundle of V with $\mu(W)=\mu(V)$. We can then start with a stable filtration of W and complete it to one of V:

$$W_0 \subset \ldots \subset W \subset \ldots \subset \ldots V.$$

Therefore $\operatorname{gr} V$ being independent of the filtration we see that $\det W$ has to be isomorphic to one of $\det V_{i_1} \otimes \ldots \otimes \det V_{i_j}$.

2.8. Lemma. *Let V be a semistable vector bundle on C. Let $G=Gr(r, V)$ be the bundle of Grassmanians of r dimensional subspaces of fibres of V. We have $G \subset \mathbb{P}(\bigwedge^r V)$. Let $\hat{G} \subset \bigwedge^r V$ be the cone over G. Then the set of isomorphism classes of line bundles L satisfying* i) $\deg L = r \cdot \mu(V)$ *and* ii) $\operatorname{Hom}(L, \hat{G}) \neq 0$ *i.e. there is a nonzero $s \in \operatorname{Hom}(L, \bigwedge^r V)$ such that $s(L) \subset \hat{G}$, is finite.*

Proof. Let $\bar{L}$ be the line subbundle generated by the image $s(L)$ of L in $\bigwedge^r V$. Then we have the natural maps $L \to s(L) \subset \bar{L}$ which are isomorphisms over the set of $x \in C$ with $s(x) \neq 0$. Hence $\deg L \leqq \deg \bar{L}$. Since $s(L) \subset \hat{G}$ we have $\bar{L} \subset \hat{G}$. Therefore the subbundle $\bar{L}$ gives rise to a section of $G \subset \mathbb{P}(\bigwedge^r V)$ which in turn gives a rank r subbundle W of V with $\det W = \bar{L}$, by the definition of Grassmanian. Since V is semistable $\mu(W) \leqq \mu(V)$ i.e. $\frac{\deg \bar{L}}{r} \leqq \mu(V)$. Bur $\frac{\deg \bar{L}}{r} \geqq \frac{\deg L}{r} = \mu(V)$. Therefore $\deg \bar{L} = \deg L$ and hence $\bar{L} = L$. Thus $L = \det W$, with W a subbundle of V with $\mu(W)=\mu(V)$. Now apply Lemma 2.7.

2.9. Remark. In the situation of the above lemma suppose further C and V are defined over a field K. Then the set of $L \in \operatorname{Pic}(C/K)$ satisfying the conditions i) and ii) of the above lemma is also finite since the natural map $\operatorname{Pic}(C/K) \to \operatorname{Pic}(\bar{C}/\bar{K})$ is injective where $\bar{K}$ is the algebraic closure of K and $\bar{C} = C \otimes_K \bar{K}$.

3. Semicontinuity for sections with values in a cone

The following proposition was suggested to us by M.S. Narasimhan in another context (see [15, Lemma 4.1])

3.1. Proposition. *Let $p: X \to T$ be a projective flat morphism and let $\pi: V \to X$ be a vector bundle. Let $C \subset V$ be a cone, i.e. a closed subvariety of V invariant under multiplication by scalars. For $t \in T$ let $X_t = p^{-1}(t)$ and $V_t = V | X_t$. Then the set*

$$\{t \in T \mid \exists\, \sigma \in H^0(X_t, V_t) \text{ s.t. } \sigma \neq 0, \sigma(X_t) \subset C\}$$

is a closed subvariety of T.

Proof. First note that the question is really local on T so that we can replace T by suitable open subsets as needed.

Let Sections (-, $V/X/T$) be the functor from the category of T-schemes to sets which associates to a $f: S \to T$ the set of sections of $f^* V \to S \times_T X$ (cf. [4, 5]). It is well known (see [4]) that this functor is representable. In fact in this situation, locally on T, there is a morphism $\varphi: E_1 \to E_2$ of vectorbundles $E_i \to T$ over T such that $M_V = \varphi^{-1}$ (Zero section of E_2) represents the above functor ([12, Chap. II, Sect. 5]). The corresponding functor Sections (-, $C/X/T$) for C is represented by a closed subscheme M_C of M_V (see [4, 5]). Since C is a cone the ideal J of functions on E_1 vanishing on $M_C \subset E_1$ is a homogeneous ideal of $\mathrm{Sym}\, E_1^*$, the symmetric algebra of E_1^* (so that $E_1 = \mathrm{Spec}(\mathrm{Sym}\, E_1^*)$). Then the subset of T in question is the image of the projective morphism $\mathrm{Proj}((\mathrm{Sym}\, E_1^*)/J) \to T$ and hence is closed.

3.2. Corollary. *Let T be an irreducible variety and K its function field. Then $H^0(X_K, C_K) \neq 0$ (with the obvious notation) if and only if there exists a nonempty open subset $U \subset T$ such that $H^0(X_t, C_t) \neq 0$ for every $t \in U$. On the other hand if $H^0(X_t, C_t) = 0$ for a single point of $t \in T$ then $H^0(X_K, C_K) = 0$.*

Proof. Follows immediately from the proposition.

3.3. Corollary. *Let $L \to X$ be a line bundle. Then the set $\{t \in T \mid \exists\, s \in H^0(X_t,$ $\mathrm{Hom}(L_t, V_t))$ s.t. $s \neq 0$, $s(L) \subset C_t\}$ is a closed subvariety of T.*

Proof. Apply the proposition to the cone "$L^* \otimes C$" in $L^* \otimes V$.

4. Stable sheaf restricts to stable sheaf

4.1. We fix once for all a sequence of integers $(\alpha_1, \dots, \alpha_{n-1})$ with each $\alpha_i \geqq 2$. We denote the product $\alpha_1 \dots \alpha_{n-1}$ by α. When considering complete intersection subvarieties of codimension t we let (m) stand for $(\alpha_1^m, \dots, \alpha_t^m)$ see [10, Sect. 5.1].

4.2. Let V be a torsion free sheaf on X. Then for a general smooth complete intersection curve C of type (m) in X we have $\deg(V|C) = \alpha^m \cdot c_1(V) \cdot H^{n-1}$ and $\mathrm{rk}\, V = \mathrm{rk}(V|C)$. Therefore it follows that if $V|C$ is semistable (respectively stable) then V itself is semistable (respectively stable). In [10] we proved the converse: V semistable $\Rightarrow V|C$ semistable for a general C for sufficiently large m. We now prove that V stable $\Rightarrow V|C$ stable for such a C. Note that once we have proved this it follows that the restriction of a stable V to a general complete intersection subvariety Y of any codimension t is also stable, for we can further restrict from Y to a suitable curve in Y (cf. [10, Remark 6.2]).

4.3. Theorem. *Let V be a stable (respectively semistable) torsion free sheaf on X with respect to the polarisation H. Let $Y_{(m)}$ be the generic complete intersection subvariety of type (m) (see Sects. 4.1 and 1.1 for notation). Then there is an m_0 such that for $m \geqq m_0$ the restriction $V|Y_{(m)}$ (i.e. $\varphi_{(m)}^* p_{(m)}^* V$, see Sect. 1.1) is stable (respectively semistable) with respect to the induced polarisation $H|Y_{(m)}$. Equivalently for $m \geqq m_0$ and for s in a nonempty open subset of $S_{(m)}$, $V|q_{(m)}^{-1}(s)$ is stable (respectively semistable).*

The rest of this section is devoted to the proof of this theorem.

The semistable case has been proved in [10]. Further, as remarked above it is enough to consider the complete intersection curve case. So we assume dim $Y_{(m)} = 1$ and $(m) = (\alpha_1^m, \dots, \alpha_{n-1}^m)$.

4.4. Since V is torsion free there is an open subset $U \subset X$ with codim $(X-U) \geqq 2$ such that $V|U$ is a vector bundle. Therefore $V|Y_{(m)}$ and $V|q_{(m)}^{-1}(s)$ for a general s are vector bundles.

4.5. Lemma. *There is an integer N such that for $m \geqq N$ and for a general s the restriction $V|q_{(m)}^{-1}(s)$ is simple.*

Proof. Let V^{**} be the double dual of V. It is reflexive and the natural map $V \to V^{**}$ is an isomorphism in codimension 2. Therefore V^{**} is also stable and hence simple. Moreover $V|q_{(m)}^{-1}(s) = V^{**}|q_{(m)}^{-1}(s)$ for a general s. We can apply the general Enriques-Severi Lemma ([10, Proposition 3.2]) to the reflexive sheaf V^{**}. Thus there is an integer N such that for $m \geqq N$ End $V^{**} \to \mathrm{End}(V^{**}|q_{(m)}^{-1}(s))$ is surjective for a general s. Since End V^{**} = scalars, this proves the lemma.

4.6. We now fix N sufficiently large so that for $m \geqq N$,

i) $V|q_{(m)}^{-1}(s)$ is simple and semistable for general s (use above lemma and [10, Theorem 6.1]) and

ii) Pic $X \to$ Pic $Y_{(m)}$ is bijective (Weil's Lemma, [10, Proposition 2.1]).

We also choose open subsets U_m of $S_{(m)}$ as follows. Let $W_m \hookrightarrow V|Y_{(m)}$ be the extended socle of $v \mid Y_{(m)}$. Then U_m satisfies

a) $V|q_{(m)}^{-1}(s)$ is a simple semistable vector bundle for every $s \in U_m$ and

b) W_m extends to a subbundle, again denoted by W_m, of $(p_{(m)}^* V)|q_{(m)}^{-1}(U_m)$.

4.7. One uses families of degenerating curves to compare $V|Y_{(l)}$ with $V|Y_{(m)}$, $l > m$ (see [10, Sects 4, 5]).

Let A be a discrete valuation ring over k with quotient field K. Let $D \to \mathrm{Spec}\, A$, $D \times \mathrm{Spec}\, A \to X$, be a flat family of curves in X parametrised by A, such that (a) D is smooth (b) the generic fibre D_K is a smooth curve in U_l (more precisely, $D_K = q_{(l)}^{-1}(s) \subset X \otimes_k K$ for a K-valued point s of U_l) and (c) the special fibre D_k is reduced with nonsingular components D_k^i in U_m, $i = 1, \ldots, \alpha^{l-m}$. Suppose $W \to V|D_K$ is a subbundle. Then W extends uniquely to a subsheaf $\tilde{W}$ of $V|D$, flat over A. Then we have the following facts (see [10, Sect. 4]):

(1) $\tilde{W}$ is a vector bundle (but not necessarily a subbundle of $V|D$).

(2) $\tilde{W}|D_k^i \to V|D_k^i$ is injective making $\tilde{W}|D_k^i$ a subsheaf of $V|D_k^i$.

(3) From (1) and (2) it follows that $\mu_K \leqq \sum_i \mu_k^i$ where

$$\mu_K = \max\{\mu(W) | W \text{ subbundle of } V|D_K\} \text{ and}$$
$$\mu_k^i = \max\{\mu(W) | W \text{ subbundle of } V|D_k^i\}.$$

Note also that deg $V|Y_{(l)} = \alpha^l$(deg V on X).

4.8. Lemma. *If for some $m \geqq N$, $V|Y_{(m)}$ is stable then $V|Y_{(l)}$ is stable for all $l \geqq m$.*

Proof. $V|Y_{(l)}$ stable implies that $V|q_{(m)}^{-1}(s)$ is stable for a general s. (This follows, for example, from Remark 2.6. cf [10, Sect. 4.2]). We can then construct a degenerating family of curves $D \to \mathrm{Spec}\, A$ as in Sect. 4.7 above with $V|D_k^i$

stable for all $i=1,\dots,\alpha^{l-m}$ (see [10, Proposition 5.2]). Since $\mu(V|q_{(l)}^{-1}(s))=\alpha^{l-m}\mu(V|q_{(m)}^{-1}(s))$ the lemma follows from 4.7 (3) and Remark 2.6.

4.9. So to prove the theorem we now assume that for all $m\geqq N$, $V|Y_{(m)}$ is not stable and show that this leads to the contradiction that V itself is not stable.

For $m\geqq N$, since we have assumed that $V|Y_{(m)}$ is not stable, by Lemma 4.5 and Lemma 2.5 (ii) it follows that the extended socle $W_m\to V|Y_{(m)}$ is a proper subbundle: $0<\operatorname{rk}W_m<\operatorname{rk}V$. Therefore there is an infinite sequence Q_1 of integers $\geqq N$ such that $\operatorname{rk}W_m$ is a constant, say r, for all $m\in Q_1$. Let $L_m\in\operatorname{Pic}X$ be the unique line bundle on X such that $L_m|Y_{(m)}=\det W_m$ for $m\in Q_1$. We then have $\deg L_m=r\cdot\mu(V)$ for $m\in Q_1$. Let $\hat{G}_m\to U_m$ be the cone over the Grassmann bundle in

$$p_{(m)}^*\bigwedge^r V|q_{(m)}^{-1}(U_m)\to U_m.$$

Now we fix an $m_0\in Q_1$. By Lemma 2.8 and Remark 2.9 the set

$$E=\{L\in\operatorname{Pic}X\,|\deg L=r\cdot\mu(V),\ \operatorname{Hom}(L|Y_{(m_0)},\hat{G}_{m_0}|Y_{(m_0)})\neq 0\}$$

is finite. We then have the following lemma.

4.10. Lemma. *For $l\geqq m_0$ with $l\in Q_1$ we have $L_l\in E$.*

Proof. By Corollaries 3.2 and 3.3 it is enough to prove that $\operatorname{Hom}(L_l|q_{(m_0)}^{-1}(s),\hat{G}_{m_0}|q_{(m_0)}^{-1}(s))\neq 0$ for a general s. (Note that $L_l|q_m^{-1}(s)$ has the right degree).

We construct a degenerating family of curves $D\to\operatorname{Spec}A$ with D_K in U_l and D_k^i in U_{m_0} ([10, Proposition 5.2]). Extend the extended socle $W_l\to V|Y_{(l)}$ to a subsheaf $\tilde{W}_l$ of $V|D$ (see Sect. 4.7). Since $\tilde{W}_l|D_k^i$ is a subsheaf of $V|D_k^i$ and the latter is semistable we have

$$\mu(\tilde{W}_l|D_k^i)\leqq\mu(V|D_k^i)=\alpha^{m_0}\mu(V)\ \forall\, i=1,\dots,\alpha^{l-m_0}. \tag{1}$$

Further (see Sect. 4.7)

$$\mu(W_l|D_K)=\sum_{i=1}^{\alpha^{l-m_0}}\mu(\tilde{W}|D_k^i). \tag{2}$$

Since W_l is the extended socle $\mu(W_l|D_K)=\mu(V|D_K)=\alpha^l\mu(V)$. Therefore (2) implies that equality must hold in (1) for every i. Hence we have $\deg\det(\tilde{W}|D_k^i)=\alpha^{m_0}\mu(V)=\deg(L_l|D_k^i)$.

Thus the two line bundles $L_l|D$ and $\det\tilde{W}$ on D are isomorphic on D_K and have the same degree on each component D_k^i of the special fibre. Therefore the must be isomorphic on the whole of D. But $\det\tilde{W}|D_k^i$, by its construction, admits a nontrivial homomorphism into $\hat{G}_{m_0}|D_k^i$. Hence so does $L_l|D_k^i$. Since D_k^i can be chosen to be a general $q_{(m_0)}^{-1}(s)$ ([10, Proposition 5.2]) the Lemma is proved.

Now the set E being finite there is an infinite subsequence Q_2 of Q_1 such that for $l\in Q_2\cdot L_l$ is isomorphic to a fixed $L\in E\subset\operatorname{Pic}X$.

We now claim that for sufficiently large $l\in Q_2$ and general $s\in S_{(l)}$ the subbundle $W_l|q_{(l)}^{-1}(s)$ of $V|q_{(l)}^{-1}(s)$ can be lifted to a subsheaf $\tilde{W}$ of V. Then such a $\tilde{W}$ would be a proper subsheaf of V with $\mu(\tilde{W})=\mu(V)$, contradicting the stability of V. Thus it only remains to prove the claim. The proof of this is the same as that of Lemma 6.5.3 of [10]. We sketch it briefly. Firstly note that as

$\phi \in H^0(X, \operatorname{Hom}(L, \overset{r}{\wedge} V))$ varies the subvarieties $\Sigma(\phi)=\{x\in U \mid \phi(x)\in$ cone over the Grassmannian in $\mathbb{P}(\overset{r}{\wedge} V)\}$ form a bounded family. Therefore for l very large, no $\Sigma(\phi)$ with $\Sigma(\phi)\neq U$ can contain a curve of type (l). On the other hand for any sufficiently large $l\in Q_2$, for a suitable ϕ, $\Sigma(\phi)$ does contain a curve of type (l) because we can lift the section corresponding to $W_l|q_{(l)}^{-1}(s)\to V|q_{(l)}^{-1}(s)$ to $\operatorname{Hom}(L, \overset{r}{\wedge} V)$ by Enriques-Severi Lemma (Note that det $W_l = L|Y_{(l)}$). Thus for some ϕ we must have $\Sigma(\phi)=U$ which gives the existence of a subsheaf of V lifting a suitable $W_l|q_{(l)}^{-1}(s)$. This proves the claim and we have completed the proof of Theorem 4.3.

5. Narasimhan-Seshadri Theorem for higher dimensions

The results of Donaldson [2] and Kobayashi [6] extend to surfaces the result of Narasimhan and Seshadri on stable and unitary bundles on curves [13]. We show here how Theorem 4.3. enables one to extend this result to arbitrary dimensions using the results of Donaldson and Kobayashi and the boundedness theorem of Foster-Hirschowitz-Schneider [3] and Maruyama [8].

5.1. Theorem. *Let X be a projective nonsingular variety over $\mathbb{C}$ of dimension n. Let H be an ample line bundle on X. Let V be a vector bundle on X, with $c_1(V)=0$ and $c_2(V)\cdot H^{n-2}=0$. Then V comes from an irreducible unitary representation of the fundamental group $\pi_1(X)$ if and only if V is stable (with respect to H).*

Proof. Suppose V comes from an irreducible unitary representation ρ: $\pi_1(X)\to U(r)\subset GL(r,\mathbb{C})$, i.e. V is associated to the principal $\pi_1(X)$-bundle $\tilde{X}\to X$, $\tilde{X}$ being the universal cover of X, for the action of $\pi_1(X)$ on $\mathbb{C}^r$ through ρ. For a general complete intersection smooth curve $C\subset X$, $\pi_1(C)\to\pi_1(X)$ is surjective [1] and hence $V|C$ comes from an irreducible unitary representation of $\pi_1(C)$. Therefore by the theorem of Narasimhan and Seshadri [13–15] $V|C$ is stable and hence so is V (cf. Sect. 4.2).

Now conversely suppose V is stable. Let S be the set of isomorphism classes of stable vector bundles on X, with all chern classes zero and of the same rank as V. Then by [3, 8] the set S forms a bounded family. Therefore one can find an m sufficiently large such that for a general complete intersection surface Y in X of type (l) with $l\geqq m$, the restriction map $\operatorname{Hom}(V, W)\to\operatorname{Hom}(V|Y, W|Y)$ is surjective for all $W\in S$. (Use Enriques-Severi Lemma. See [10, Proposition 3.2]).

By Theorem 4.3 there is an $l\geqq m$ such that $V|Y$ is a stable vector bundle on Y for a general complete intersection smooth surface Y of type (l). The results of Donaldson [2] and Kobayashi [6, Sect. 4 p. 161] imply that $V|Y$ comes from an irreducible unitary representation ρ of $\pi_1(Y)$. By Lefchetz we know that $\pi_1(Y)\to\pi_1(X)$ is an isomorphism. Thus we get a stable bundle V_ρ on X associated to the representation of $\pi_1(X)$ given by ρ. Since $\operatorname{Hom}(V|Y,$

$V_\rho|Y)\neq 0$, $l \geq m$ and $V_\rho \in S$ we see that $\mathrm{Hom}(V, V_\rho)\neq 0$. But V and V_ρ are stable and the assumption on the chern classes then implies that any nonzero homomorphism $V \to V_\rho$ is an isomorphism. This completes the proof of the theorem.

5.2. Remark. The theorem immediately gives that if $c_1(V)=0$ and $c_2(V)\cdot H^{n-2}=0$ then V stable with respect to H implies that it remains stable with respect to any other polarisation.

5.3. Remark. In the proof of the above theorem we can avoid the use of the boundedness theorems of [3, 8]. For we need only the boundedness of the family of bundles on X coming from representations of $\pi_1(X)$ in ($U(n)$ or) $GL(n)$ and clearly such representations form a closed subvariety of $GL(n)\times \ldots \times GL(n)$, $\pi_1(X)$ being finitely presented.

5.4. Remark. Ramanan has pointed out that the above theorem implies that if V is a stable vector bundle whose projective Chern classes are zero then the corresponding projective bundle $\mathbb{P}(V)$ comes from a unitary representation of $\pi_1(X)$ into the projective unitary group $PU(n)$, $n = rkV$. This can be seen as follows. The assumption on V implies that End V has c_1 and c_2 zero. Since End V is associated to the stable bundle V it follows from [17, Theorem 3.18 and Sect. 4.7] that End V is a direct sum of stable bundles. Each of its stable components has c_1 zero and hence by Bogomolov's inequality the stable components also have c_2 zero. Hence End V being the direct sum of stable bundles with c_1, c_2 zero comes from a unitary representation $\pi_1(X)\to U(n^2)$, by the above theorem. Therefore the bundle $W = \mathrm{Hom}$ (End $V \otimes$ End V, End V) also comes from a unitary representation. Hence the section of W corresponding to the algebra structure on End V is given by a $\pi_1(X)$ invariant element of the corresponding representation space (see [18, Proposition 4.1]). Clearly this implies that the representation $\pi_1(X)\to U(n^2)$ giving End V takes values in $PU(n)$ sitting in $U(n^2)$ via the adjoint representation. This proves that $\mathbb{P}(V)$ comes from a unitary representation $\pi_1(X)\to PU(n)$.

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