Pacific Journal of Mathematics

RESTRICTIONS OF BANACH FUNCTION SPACES

DONALD RICHARD CHALICE

Vol. 39, No. 3 July 1971

PACIFIC JOURNAL OF MATHEMATICS
Vol. 39, No. 3, 1971

RESTRICTIONS OF BANACH FUNCTION SPACES

DONALD R. CHALICE

Let X be a compact Hausdorfs space. Let $C(X)$ be the space of continuous complex-valued functions on X and A be a function algebra on X, that is a uniformly closed separating subalgebra of $C(X)$ containing the constants. If F is a closed subset of X we say that A interpolates on F if $A \mid F = C(F)$. By a positive measure μ we shall always mean a positive regular bounded Borel measure on X. Let F be a measurable subset of X. We say a subspace S of $L^p(\mu)$ interpolates on F if $S \mid F = L^p(F) = L^p(\mu_F)$, where μ_F is the restriction of μ to F. Let $H^p(\mu)$ be the closure of A in $L^p(\mu)$ where $1 \leqq p < \infty$, and let $H^\infty(\mu) = H^2(\mu) \cap L^\infty(\mu)$. One question we are concerned with here is whether interpolation of the algebra is sufficient to imply interpolation of its associated H^p-spaces. We therefore begin by obtaining necessary and sufficient conditions for a closed subspace of $L^p(\mu)$ to have closed restriction in $L^p(F)$. These condition are analogous to some obtained by Glicksberg for function algebras. Using these results we obtain theorems about interpolation of certain invariant subspaces, and then apply them to H^p-spaces. In particular we show that when A approximates in modulus and μ is any measure which is not a point-mass, $H^p(\mu)$ interpolates only on sets of measure zero. (One sees that A interpolates only on sets of measure zero, so our original question has a trivial answer for these algebras.) For uniformly closed weak-star Dirichlet algebras again the answer to our original question is affirmative. Finally we provide an example of an algebra which interpolates such that $H^\infty(\mu)$ interpolates and the $H^p(\mu)$ do not interpolate for $1 \leqq p < \infty$. I am indebted to a paper of Glicksberg for those techniques which inspired the present effort. Below we show that these techniques apply to the L^p situation and to other "similar" situations.

Glicksberg [3] has given necessary and sufficient conditions for interpolation of a closed subspace of $C(X)$. We show here that analogous theorems hold for subspaces of $L^p(X)$. Let $A \subset B$ be Banach spaces. $A^\perp$ will denote all bounded linear functions functionals on B which annihilate A.

THEOREM 1.1. *Let A, A_1, B all be Banach spaces with $A \subset A_1$ and $R: A_1 \to B$ a nonzero bounded linear transformation. Then $R(A)$ is closed in B if and only if $\exists c \ni : \| h - R(A)^\perp \| \leqq c \| h^* - A^\perp \|$ $\forall h \in B^*$, where $h^* = R^*h$. It follows that $c \geqq 1/\| R \|$.*

Proof. The map $R_1 = R \mid A: A \to R(A)$ induces a map

$$T = \psi \circ R_1^* \circ \phi: B^*/R(A)^\perp \longrightarrow A_1^*/A^\perp$$

where $\psi: A^* \to A_1^*/A^\perp$ and $\phi: B^*/R(A)^\perp \to R(A)^*$ are the natural isometric isomorphisms. Further for $g \in B^*$, $g - R(A)^\perp$ is taken to $g^* - A^\perp$ by T, so T is $1-1$. Now the range of R_1 is closed if and only if the range of R_1^* is closed if and only if the range of T is closed [1]. The latter fact is equivalent to: $\exists c \ni : \| h - R(A)^\perp \| \leqq c \| h^* - A^\perp \|$ for all $g \in B^*$ by the open mapping theorem. Further, $\| h^* - A^\perp \| \leqq \| R \| \, \| h - R(A)^\perp \|$ so applying the above inequality gives $c \geqq 1/\| R \|$.

The statement of the above theorem is slightly more general than those of other similar theorems appearing the literature. The proof is virtually the same as that in [3] albeit in a more general setting. See also [2]. The next corollary follows as in [3].

COROLLARY 1.2. *Let X be locally compact and A a uniformly closed subspace of $C_0(X)$. Let F be a locally compact subset of X and suppose $A \mid F \subset C_0(F)$. Then*

(i) *$A \mid F$ is uniformly closed in $C_0(F)$ if and only if $\exists c \ni :$ $\| \mu - (A \mid F)^\perp \| \leqq c \| \mu - A^\perp \|$ $\forall$ regular bounded Borel measure μ on F.*

(ii) *$A \mid F = C_0(F)$ if and only if $\exists c \ni : \| \mu_F \| \leqq c \| \mu_{F'} \|$ $\forall \mu \in A^\perp$.*

We now apply 1.1 to get the analogous conclusion for subspaces of L^p-spaces.

DEFINITION. Let μ be a fixed positive measure on X and F a measurable subset of X. Set $L^p(F) = L^p(\mu_F)$, $1 \leqq p \leqq \infty$ where μ_F is the restriction of μ to F. For $f \in L^q(F)$ let $\tilde{f}$ be the function which is f on F and 0 on F'. Note that if R is the restriction map $L^p(X) \to L^p(F)$, then $\tilde{f} = f^*$. For a subspace S of $L^p(X)$, $(S \mid F)^\perp = \{g \in L^q(F) \mid g^\perp S \mid F\}$. Clearly $\{\tilde{f} \mid f \in (S \mid F)^\perp\} \subset S^\perp$.

THEOREM 1.3. *Let S be a closed subspace of $L^p(X)$, $1 \leqq p < \infty$, and F a measurable subset of X. Then:*

(i) *$S \mid F$ is closed in $L^p(F)$ if and only if*

$$\exists c \ni : \| g - (S \mid F)^\perp \| \leqq c \| \tilde{g} - S^\perp \| \ \forall g \in L^q(F) ; \tag{1}$$

(ii) *$S \mid F = L^p(F)$ if and only if*

$$\exists c \ni : \| g \mid F \|_q \leqq c \| g \mid F' \|_q \ \forall g \in S^\perp . \tag{2}$$

If F has positive measure it follows that $c \geqq 1$. If $p = \infty$ then the *"only if" parts of* (i) *and* (ii) *hold for* $g \in L^1(E)$ *and* $L^1(X) \cap S^\perp$ *respectively.*

Proof. (i) follows by applying 1.1 to the restriction map R. As $\|R\| \leqq 1$, we have $c \geqq 1$. If $S \mid F = L^p(F)$, then (1) becomes $\|g\| \leqq c\|g - S^\perp\| \ \forall\, g \in L^q(F)$. In particular if $g \in S^\perp$,

$$\|g \mid F\| \leqq c\|\widetilde{g \mid F} - g\| = c\|g \mid F'\| .$$

This shows the "only if" part of (ii). For the "if" part of (ii) we shall use a concavity property of the q-norm; namely, if $\alpha, \beta \geqq 0$, $\alpha + \beta \leqq 1$, then $\|f\|_q \geqq \alpha\|f \mid F\|_q + \beta\|f \mid F'\|_q$. Now taking $g \in (S \mid F)^\perp$, and applying (2) to g shows that $(S \mid F)^\perp = 0$, so $S \mid F$ is dense in $L^p(F)$. Thus we need only show that $S \mid F$ is closed. Here (1) reduces to $\|g\|_q \leqq c'\|\tilde{g} - S^\perp\|_q \ \forall\, g \in L^q(F)$. But if $g \in L^q(F)$ and $h \in S^\perp$, then $\|\tilde{g} - h\|_q \geqq \alpha\|(g-h) \mid F\|_q + \beta\|h \mid F'\|_p$ if $\alpha, \beta \geqq 0$ and $\alpha + \beta \leqq 1$. Now choose n so that $c/n + c^2/n \leqq 1$ and let $\alpha = c/n$, $\beta = c^2/n$. Then

$$\|\tilde{g} - h\|_q \geqq c/n\|g \mid F\|_q - c/n\|h \mid F\|_q + c^2/n\|h \mid F'\|_q \geqq c/n\|g \mid F\|_q$$

after applying (2). Thus setting $c' = n/c$ gives $S \mid F$ is closed and thus $S \mid F = L^p(F)$. The latter part of the conclusion is clear from the above arguments.

COROLLARY 1.4. *If S is a closed subspace of $L^p(X)$, $1 \leqq p < \infty$ and $S^\perp \mid F \subset (S \mid F)^\perp$ then $S \mid F$ is closed in $L^p(F)$.*

Proof. $(\widetilde{S \mid F})^\perp \subset S^\perp$ so in fact $S^\perp \mid F = (S \mid F)^\perp$. Taking $g \in L^q(F)$, and $h \in S^\perp$ we have

$$\|g - S^\perp\|_q \geqq \|g - S^\perp \mid F\|_q = \|\tilde{g} - (S \mid F)^\perp\|_q$$

and (1) applies.

2. Restrictions of invariant subspaces. Let X be a topological space and μ a positive measure on X. Throughout this section A will be a subalgebra of $L^\infty(\mu)$, and S will be a closed subspace of $L^p(X)$ for some $1 \leqq p < \infty$. We assume that S is invariant under multiplication by elements of A. A *separates in modulus* (SM) if $\forall\, \varepsilon > 0$, E, F disjoint closed sets in X, $\exists\, f \in A$ such that $|f| < \varepsilon$ a.e., on E and $|1 - |f|| < \varepsilon$ a.e., on F. Call f a separating function. A *boundedly separates in modulus* (*BSM*) if $\exists\, M \ni : \forall\, \varepsilon > 0$, E, F disjoint closed sets, $\exists$ a separating function $f \in A$ with $\|f\|_\infty < M$. We say that A *boundedly separates in modulus by invertible func-*

tions (BSMI) if A is BSM and the bounded separating functions can be chosen to be invertible. If A is a function algebra on X and the a.e., condition can be left out of the above then we say that A is BSM or BSMI *on* X. For example, if A approximates in modulus then A is BSM on X and if A is logmodular then A is BSMI on X. If A is weak-star-Dirichlet [7] then A may not even be BSM, but H^∞ must be BSMI because $\log V = L^\infty_R$ where V is the set of invertible elements in H^∞. This includes the case where μ is a unique representing measure on X, or more generally, is "minimal" in the sense of [7, pg. 238]. Thus BSM, etc., "localize" the separation properties to the support of the measure in question.

THEOREM 2.1. *Let F be a mesurable set in X. If A is* BSM *then $S \mid F = L^p(F)$ if and only if $g \in S^\perp \Rightarrow g \mid F = 0$. In particular, this holds if A approximates in modulus.*

Proof. 1.4 implies the "if" part. Conversely, suppose $S \mid F = L^p(F)$. Then $\exists\ c$ such that $g \in S^\perp \Rightarrow \|g \mid F\|_q \leqq c \|g \mid F'\|_q$. Choose $\varepsilon > 0$. Find K_n compact $\subset F \subset V_n$ open such that $\mu(V_n \sim K_n) < 1/n$. We can assume that the K_n are monotone. Suppose M is the bounding constant for the separating functions in A. Find $k \in A$ such that $\|k\|_\infty \leqq M$ and $\| k \mid -1 \mid < \varepsilon$ on K_n and $\mid k \mid < \varepsilon$ on V_n' a.e. Then for fixed $g \in S^\perp$,

$$\begin{aligned}(1 - \varepsilon) \|g \mid K_n\|_q &\leqq \|kg \mid K_n\|_q \leqq \|kg \mid F\|_q \leqq c \|kg \mid F'\|_q \\ &\leqq c \|kg \mid F' \cap V_n\|_q + c \|kg \mid V_n'\|_q \\ &\leqq cM \|g \mid F' \cap V_n\|_q + c\ \varepsilon \|g\|_q\ .\end{aligned}$$

Letting $\varepsilon \to 0$, we have $\|g \mid K_n\|_q \leqq cM \|g \mid F' \cap V_n\|_q$. Letting $n \to \infty$, we have $g \mid F = 0$.

COROLLARY 2.2. *Let A be* BSM. *Suppose that F_i are mesurable subsets of X and $F_0 = \bigcup_{i=1}^\infty F_i$. If $S \mid F_i = L^p(F_i)$ for $i = 1, 2, \cdots$ then $S \mid F_0 = L^p(F_0)$.*

Proof. Let $g \in S^\perp$. Then $g \mid F_i = 0$ a.e. for $i = 1, 2, \cdots$ and thus $g \mid F_0 = 0$ a.e.

THEOREM 2.3. *Let F be a closed subset of X. If A is* BSMI *then $S \mid F$ is closed in $L^p(F)$ if and only if $g \in S^\perp \Rightarrow g \mid F \in (S \mid F)^\perp$.*

Proof. "If." Apply Corollary 1.4. Here it is not necessary that F be closed.

"Only if." Find V_n open $\supset F$ such that $\mu(V_n \sim F) < 1/n$. Then

$\exists\ M > 0$ and k_n invertible in A such that $\|k_n\|_\infty \leqq M$, $|1 - |k_n|| < \varepsilon$ a.e. on F and $|k_n| < \varepsilon$ a.e. on V_n'. Now $\exists\ c$ such that 1.3 (1) holds so $g \in S^\perp \Rightarrow \|g \mid F - (S \mid F)^\perp\|_q \leqq c\|g \mid F'\|_q$. The same holds for $k_n g$. Thus

$$\begin{aligned}\|k_n g \mid F - (S \mid F)^\perp\|_q &\leqq c\|k_n g \mid V_n \cap F'\|_q + c\|k_n g \mid V_n'\|_q \\ &\leqq cM\|g \mid V_n \sim F\|_q + c\varepsilon\|g\|_q .\end{aligned}$$

Now since k_n are invertible, $k_n(S \mid F)^\perp = (S \mid F)^\perp$. Thus

$$\begin{aligned}(1 - \varepsilon)\|g \mid F - (S \mid F)^\perp\|_q &\leqq \|k_n g \mid F - (S \mid F)^\perp\|_q \\ &\leqq cM\|g \mid V_n \sim F\|_q + c\varepsilon\|g\|_q .\end{aligned}$$

Letting $\varepsilon \to 0$ and $n \to \infty$ gives $g \mid F \in (S \mid F)^\perp$.

COROLLARY 2.4. *Let A be* BSMI. *Suppose F_i are closed subsets of X and $F = \bigcup_{i=1}^\infty F_i$. If $S \mid F_i$ is closed in $L^p(F_i)$ for each i, then $S \mid F$ is closed in $L^p(F)$.*

Proof. Take $g \in S^\perp$. Then $g \mid F_i \in (S \mid F_i)^\perp$, and by the dominated convergence theorem, it follows that $g \mid F \in (S \mid F)^\perp$.

Using the above theorem we also encounter the following phenomenon which is different from that which usually occurs in the function algebra setting.

COROLLARY 2.5. *Let F be a closed subset of X, and let A be* BSMI. *Then $S \mid F$ is closed in $L^p(F) \Rightarrow S \mid F'$ is closed in $L^p(F')$. In particular this happens if A is logmodular.*

Proof. Let $g \in S^\perp$. Then $g \mid F \in (S \mid F)^\perp$. Hence

$$\widetilde{g \mid F'} = g - (\widetilde{g \mid F}) \in S^\perp$$

and thus $g \mid F' \in (S \mid F')^\perp$, so $S \mid F'$ is closed.

The above is explained by the following "splitting lemma" which was pointed out to me by K.B. Laursen.

LEMMA 2.6. *Let S be a closed subspace of $L^p(\mu)$, $1 \leqq p < \infty$, and let F be a measurable subset of X. Then $S = \widetilde{S \mid F} \oplus \widetilde{S \mid F'}$ if and only if $g \in S^\perp \Rightarrow \widetilde{g \mid F} \in S^\perp$.*

REMARKS. The following illustrates 2.5. Let X be the union of two disjoint disks, $\mu = m_1 + m_2$ where m_1 and m_2 are the Lebesgue measures on the two circles comprising the boundary of X, and let A be the algebra of functions continuous on X and analytic on the interior of X. Then $H^1(m_1) + L^1(m_2)$ splits and neither F nor F' have measure 0.

Also it is easy to find examples of closed subspaces of $L^1(-1, 1)$ which are proper and interpolate on $(-1, 0]$ and $(0, 1)$. For example, let S be the set of functions f in $L^1(-1, 1)$ such that $f(x) = f(-x)$ a.e.

3. Interpolation of H^p-spaces and function algebras. Throughout this section unless it is otherwise stated, we assume that A is a function algebra on a compact space X, μ is a representing measure for A which is not a point-mass and I is the corresponding maximal ideal.

PROPOSITION 3.1. *If I is* SM *in $L^\infty(\mu)$ then the only open sets on which $H^p(\mu)$ interpolates for some $1 \leqq p \leqq \infty$ are those of measure* 0.

Proof. If H^p interpolates on V open and $\mu(V) > 0$ then find K compact in V of positive measure. Find a sequence in I whose moduli converge to 1 on K and 0 on V'. This contradicts 1.3 (ii).

PROPOSITION 3.2. *If I is* BSM *in $L^\infty(\mu)$ then the only measurable sets on which $H^p(\mu)$ interpolates for some $1 \leqq p \leqq \infty$ are those of measure* 0.

Proof. Suppose H^p interpolates on a set F of positive measure. We may assume that F is closed. Since μ is assumed to not be a point-mass F' has positive measure. We can therefore choose K_n compact and monotone in F' so that $\mu(K_n) \to \mu(F')$. Find f_n in I which are uniformly bounded such that $||f_n| - 1| < 1/n$ on F and $|f_n| < 1/n$ on K_n. This contradicts 1.3 (ii).

We wish to study the relation between interpolation of the algebra A and its associated H^p-spaces. As was pointed out in the introduction, if A approximates in modulus then the situation is trivial. For if F is a closed set on which A interpolates then because F is an intersect of peak sets, we must have that $\mu(F) = 0$ by the dominated convergence theorem. So interpolation of the H^p-space follows vacuously. More generally we have the following.

PROPOSITION 3.3. *Let A be* BSM *on X, and F a closed subset of X. If A interpolates on F then $H^p(\mu)$ interpolate on F for any measure μ, and any $1 \leqq p < \infty$.*

Proof. $g \perp H^p \Rightarrow g\, d\mu \perp A \Rightarrow g\, d\mu_F = 0 \Rightarrow g \mid F = 0$ a.e., $\mu \Rightarrow H^p$ interpolates on F.

PROPOSITION 3.4. *If μ is a representing measure for A, and A is* BSM *in $L^\infty(\mu)$, then $H^p(\mu)$ interpolates only on sets of measure* 0 *if $1 \leqq p \leqq \infty$.*

Proof. Suppose for some p, $H^p \mid F = L^p(F)$. Let A_0 be the ideal determined by μ. Then $A_0 \subset (H^p)^\perp$ so by 2.1., $g \in A_0 \Rightarrow g \mid F = 0$ a.e. But if $f \in H^p$, then $f - \int f d\mu$ is a pointwise a.e. limit of a sequence of elements of A_0 and thus $f = \int f d\mu$ a.e. on F, so that all H^p functions are constant a.e. on F. Thus $L^p(F) =$ constants and thus μ_F is a point-mass at some point x. But μ must be continuous, for $\exists\ g \in I$ such that $g(x) \neq 0$ and applying 2.1 gives $\mu\{x\} = 0$.

PROPOSITION 3.5. *Let A be* BSMI *on X, and F a closed subset of X. If $A \mid F$ is closed then $H^p(\mu)$ restricted to F is closed for any measure μ, and any $1 \leqq p < \infty$.*

Proof. $g \perp H^p \Rightarrow g\ d\mu \perp A \Rightarrow g\ d\mu_F \in (A \mid F)^\perp \Rightarrow g\ d\mu_F \in (H^p)^\perp \Rightarrow H^p$ restricted to F is closed by 2.3.

REMARKS. Both 3.3 and 3.5 hold because F is an intersect of peak sets. By the above it is easy to construct examples in which the H^p spaces interpolate on sets of positive measure (where μ is not a representing measure). For another example, let A be the disk algebra on the unit disk, and let $\mu = 1/2\ d\theta + 1/2\ \delta_0$ where δ_0 is the poin-mass at 0. As yet we have boon unable to construct examples which are not of this discrete type when μ is a represeting measure.

We now construct examples in which the algebra and H^∞ interpolate but in which none of the H^p-spaces, $1 \leqq p < \infty$, interpolate. Let $\{r_n\}$ be a nonnegative interpolating sequence in the open unit disk converging to 1. Then $F = \{r_n\} \cup \{1\}$ is an interpolating sequence for the disk algebra on the unit disk [6]. Let μ_n be the Poisson measures for r_n on the unit circle. Choose a sequence $\alpha_n \geqq 0$ such that $\sum_{n=1}^\infty \alpha_n \mu_n < 1/2\, d\theta$ (*). Consider the positive measure $\mu = \sum_{n=1}^\infty \alpha_n(\delta_n - \mu_n) + d\theta$ where δ_n is the point-mass at r_n. Then μ represents 0 for the disk algebra and we claim that $H^\infty(\mu)$ interpolates on F while $H^p(\mu)$ $1 \leqq p < \infty$ do not interpolate on F. To see this we need the following.

LEMMA 3.6. *$H^p(\mu) = H^p \mid F \cup T$ where H^p is the usual H^p-space for the disk algebra* $(1 \leqq p \leqq \infty)$ *on the closed unit disk.*

Proof. If $f \in H^p(d\theta)$ then $\exists\ f_n \in A \ni : f_n \to f$ in $L^p(d\theta)$. If $\hat{f}$ denotes the harmonic extension of f to H^p, then

$$\int |\hat{f}_n - \hat{f}|^p \, d\mu \leqq (1 + \sum 2\alpha_j(1 + r_j)/(1 - r_j)) \int |f_n - f| \, d\theta \longrightarrow 0 .$$

So $H^p | F \cup T \subset H^p(d\mu)$. Conversely, if $f_n \in A$ and $f_n \to f$ in $L^p(\mu)$, then $f_n \to f$ in $L^p(d\theta)$, so $f | T \in H^p(d\theta)$ and therefore extends to $g = \widehat{f | T}$ in H^p. So $g | F \cup T \in H^p(\mu)$ and $g | T = f | T$. But since the functions in $H^p(\mu)$ are determined by their values on T, we have $f = g \in H^p | F \cup T$, and we are done for $1 \leqq p < \infty$. Now

$$H^\infty(d\mu) = H^2(d\mu) \cap L^\infty(d\mu) = [H^2 | F \cup T] \cap L^\infty(\mu)$$
$$= [H^2(d\theta) \cap L^\infty(d\theta)\hat{]} \, | F \cup T = H^\infty | F \cup T ,$$

and this completes the proof.

Now observe that if $f \in H^p(d\mu)$, then

$$| f(r_n) |^p \leqq [(1 + r_n)/(1 - r_n)] \int | f |^p \, d\theta$$

so that $\exists \; c \ni$: the growth condition $| f(r_n) |^p \leqq c(1 + r_n)/(1 - r_n)$ is satisfied. Thus if we choose a (nonnegative) sequence $\{x_n\}$ such that $x_n^p(1 - r_n)/(1 + r_n) \to \infty$ and such that $\Sigma x_n^p(1 + r_n)\alpha_n/(1 - r_n) < \infty$, we obtain an element of $L^p(\mu_F)$ which is not the restriction of a function from $H^p(d\mu)$. Such a sequence can be found for example by finding $\beta_n \geqq 0$ to satisfy (*) and setting $\alpha_n = \beta_n^2$ and $x_n = (\beta_n)^{-1/p}$.

Since H^∞ interpolates on F, we see that $H^\infty(d\mu)$ interpolates on F by 3.6.

Thus one may ask for conditions that will force interpolation of H^p-spaces to follow from interpolation of the algebra. The following is one such condition.

THEOREM 3.7 *Let A be a function algebra on X, μ a representing measure for A, and A_0 the corresponding maximal ideal. Suppose that $H^p(\mu) = H^\alpha(\mu) \cap L^p(\mu)$, $\alpha \leqq p$. If A_0 is weak-star dense in $H^\alpha(\mu)^\perp$, then interpolation of A on a closed set F implies interpolation of $H^p(\mu)$ on F for all $\alpha \leqq p < \infty$ with integer conjugates q.*

Proof. The conclusion deals only with $1 \leqq \alpha \leqq p \leqq 2$. Suppose $1 < \alpha$ and $A | F = C(F)$. Then $\exists \; c \ni$: $\| \mu_F \| \leqq c \| \mu_{F'} \|$ for every $\mu \in A^\perp$. Now choose $g \in A_0$. Then $g^q d\mu \in A^\perp$ so $\int_F | g |^q d\mu \leqq c \int_{F'} | g |^q d\mu$ or (*) $\| g | F \|_q \leqq c^{1/q} \| g | F' \|_q$. Since A_0 is dense in $H^p(\mu)^\perp$ also, we have (*) holds for every $g \in H^p(\mu)^\perp$ and thus $H^p(\mu)$ interpolates on F. Suppose $\alpha = 1$. For $g \perp H^1(\mu)$ we have $\| g | F \|_q \leqq c^{1/q} \| g | F' \|_q$ for $q = 2, 3, \cdots$, and thus letting $q \to \infty$ we have $\| g | F \|_\infty \leqq \| g | F' \|_\infty$ so that $H^1(\mu)$ also interpolates on F.

COROLLARY 3.8. *If A is a function algebra which is weak-star-Dirichlet in $L^\infty(\mu)$ then A interpolates only on sets of μ measure* 0.

Proof. A satisfies the hypotheses of 3.7 [7] and thus H^1 interpolates on F. But H^1 is invariant under H^∞ which is BSMI so that F has μ measure 0 by 3.4.

It is also clear from 3.4 that when A is weak-star-Dirichlet, H^p interpolate only on sets of measure 0 for $1 \leqq p \leqq \infty$. Using the invariant subspace theorem we have the following.

THEOREM 3.9. *Let A be weak-star-Dirichlet. If F is closed and $H^p(\mu)$ restricted to F is closed for some $1 \leqq p < \infty$, then $\mu(F) = 0$, or $\mu(F') = 0$.*

Proof. Since H^p is invariant under H^∞ which is BSMI, applying 2.3 and 2.6 we have $H^p = \widetilde{H^p | F} \oplus \widetilde{H^p | F'}$. Now if F has positive measure, then $\widetilde{H^p | F}$ is a simply invariant subspace of L^p and by the invariant subspace theorem [7, 4.16], $\widetilde{H^p | F} = qH^p$ where $|q| = 1$ a.e. But $q \in \widetilde{H^p | F}$ so we have $\mu(F') = 0$.

The example preceding 3.7 is clearly not weak-star-Dirichlet because the measure μ is not minimal. In addition we have the following.

COROLLARY 3.10. *In the example preceding* 3.7, *A_0 is not weak-star dense in $H^1(\mu)^\perp$.*

Proof. We only need to verify that $H^p(\mu) \supset H^1(\mu) \cap L^p(\mu)$. But if $f \in H^1(\mu) \cap L^p(\mu)$ then $f | T = g | T$ where

$$g \in H^1(d\theta) \cap L^p(d\theta) = H^p(d\theta) .$$

So as $\hat{g} | F \cup T \in H^p(\mu)$, and $\hat{g}$ and f agree on T, we have

$$f = \hat{g} | F \cup T \in H^p(\mu) .$$

Finally we remark that 1.3 should hold for function spaces whose duals restrict in some sense and whose norm satisfies the concavity condition. We hope to consider such examples at a later date.

REFERENCES

1. N. Dunford and J. Schwartz, *Linear operators*, I,Interscience, New York, 1958.
2. T. Gamelin, *Restriction of subspaces of $C(X)$*, Trans. Amer. Math. Soc., **122** (1964), 278-286.

3. I. Glicksberg, *Measures orthogonal to algebras and sets of antisymmetry*, Trans. Amer. Math. Soc., **105** (1962).
4. ———, *The abstract F. and M. Riesz theorem*, J. Functional Analysis, **1** (1967).
5. M. Hasumi, *Interpolation sets for logmodular Banach algebras*, Osaka J. Math., **3** (1966).
6. K. Hoffman, *Banach spaces of analytic function*, Prentice-Hall, Englewood Cliffs, N. J., 1962.
7. T. P. Srinivasan and J. K. Wang, *Weak*-Dirichlet Algebras*, Function algebras, F. T. Birtel (Editor), Scott, Foresman, Chicago, 1966.

Received April 29, 1970. Supported by Western Washington State College Bureau for Faculty Research.

WESTERN WASHINGTON STATE COLLEGE

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. **39**. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is $8.00; single issues, $3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues $1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 39, No. 3 July, 1971