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RESTRICTIONS OF FOURIER TRANSFORMS ON A^p

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1. Introduction. Throughout this paper G denotes a locally compact abelian group with dual group Γ. We denote by H any closed subgroup of G and by Λ the annihilator of H. Thus if $\hat{G}$ denotes the dual of G, then

$$\hat{G} = \Gamma\,, \quad (G/H)^\wedge = \Lambda \quad \text{and} \quad \hat{H} = \Gamma/\Lambda\,.$$

Denote by dx the Haar measure of a group K in each indicated integration. We designate by $A^p(G)$ the algebra of all functions f in $L^1(G)$ whose Fourier tranforms $\hat{f}$ are in $L^p(\Gamma)$. Supply the norm in $A^p(G)$ by

$$\|f\|^p = \max\,(\|f\|_1, \|\hat{f}\|_p) \qquad 1 \leqq p < \infty\,,$$

which is equivalent to the sum norm $\|f\|_1 + \|\hat{f}\|_p$. It is known that $A^p(G)$ is a regular, semi-simple commutative Banach algebra with convolution as the multiplication and for $1 \leqq p < \infty$, $A^p(G)$ form an increasing chain of dense ideals in $L^1(G)$. Let $\widehat{A^p(G)} = \hat{A}^p(\hat{G}) = \hat{A}^p(\Gamma)$ be the Fourier algebras of $A^p(G)$ for $1 \leqq p < \infty$ and supply the norm in $\hat{A}^p(\Gamma)$ as same as $A^p(G)$;

$$\|\hat{f}\| = \|f\|^p \quad \text{for} \quad f \in A^p(G)\,, \quad \hat{f} \in \hat{A}^p(\Gamma)\,.$$

We denote also by $A(\Gamma)$ and $B(\Gamma)$ the algebras of Fourier transforms and Fourier Stieltjes transforms on Γ. As ordinary the norms of $A(\Gamma)$ and $B(\Gamma)$ are given by $L^1(G)$-norm and $M(G)$-norm, where $M(G)$ is the bounded regular Borel measures on G.

In this paper we investigate that the restriction map of Fourier algebra $\Phi\colon \hat{A}^p(\Gamma) \to \hat{A}^p(\Lambda)$ is a bounded linear mapping, and ask that does there exists a linear lifting $\lambda\colon \hat{A}^p(\Lambda) \to \hat{A}^p(\Gamma)$ such that $\Phi\circ\lambda = Id.$? We give the affirmative answer in some situations. Evidently if a lifting λ exists, then Φ is onto mapping. Concerning liftings, restrictions and their relationship, Herz [7] has investigated in some stages of group algebras. (Note that in his discussion, the groups are general locally

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compact groups and the Fourier algebras $A_p(G)$ in Eymard [2], Herz [6], and [7] are different from the sense in our $A^p(G)$.)

As an application, in final section, we turn to discuss the functions which operate in $A^p(G)$-algebras. That is the converse of Wiener-Lévy's theorem. Many authors investigated such problem in various stages of group algebras. In this note we explore an operating function of the Fourier algebra $\hat{A}^p(\Gamma)$ that can be treated by our reduction theorems to reduce to the cases of [5] for $A(\Gamma)$ and $B(\Gamma)$.

2. Relations between $\hat{A}^p(R^n)$ and $\hat{A}^p(T^n)$. Let R^n be n-dimensional Euclidean space, Z^n be the group of all lattice points in R^n and $\hat{Z}^n = T^n$ be the n-dimensional torus. We give the following theorem to show the relations between $\hat{A}^p(R^n)$ and $\hat{A}^p(T^n)$.

THEOREM 1. *There exists a bounded linear mapping $\Phi: \hat{A}^p(R^n) \to \hat{A}^p(T^n)$, and also a bounded linear mapping $\Psi: \hat{A}^p(T^n) \to \hat{A}^p(R^n)$. Precisely, for any $f \in \hat{A}^p(R^n)$ there exists a function $g \in \hat{A}^p(T^n)$ such that $f(x) = g(x)$ for $|x| = (\sum_{i=1}^n |x_i|^2)^{1/2} \leqq \pi - \delta$, $0 < \delta < \pi$ and $\|g\|_{\hat{A}^p(T^n)} \leqq C_1 \|f\|_{\hat{A}^p(R^n)}$; conversely, for any $g \in \hat{A}^p(T^n)$, there exists a function $f \in \hat{A}^p(R^n)$ such that $f|_{T^n} = g$, and $\|f\|_{\hat{A}^p(R^n)} \leqq C_2 \|g\|_{\hat{A}^p(T^n)}$. Here C_1, C_2 are some positive constants.*

PROOF. Let h be a function on R^n with continuously partial derivative of order $\geqq 2$ such that $0 \leqq h \leqq 1$ and for $0 < \delta < \pi$,

$$h(x_1, x_2, \cdots, x_n) = 1 \quad \text{if} \quad |x| = \Big(\sum_{i=1}^n |x_i|^2\Big)^{1/2} \leqq \pi - \delta$$
$$= 0 \quad \text{if} \quad |x| \geqq \pi .$$

The Fourier transform of $(\partial^2/\partial x_i^2)h(x)$ is $-y_i^2 (i = 1, 2, \cdots, n)$ and since the Fourier transform of

$$h(x) - \frac{\partial^2}{\partial x_i^2} h(x) \qquad (i = 1, 2, \cdots, n)$$

is bounded continuous, it follows that

$$\text{(a)} \qquad |\hat{h}(y)| \leqq \frac{C}{\prod_{i=1}^n (1 + y_i^2)} \leqq \frac{C}{1 + |y|^2},$$

where $y = (y_1, y_2, \cdots, y_n) \in R^n$, C is a positive constant and hence $\hat{h} \in L^1(R^n)$. By inverse theorem and the compact support of h, we see that $\hat{h} \in A^p(R^n)$.

If $f \in \hat{A}^1(R^n) (\subset \hat{A}^p(R^n)$ for $p \geqq 1)$, then f is bounded continuous and belongs to $L^2(R^n)$. Define $g = fh = \Phi f$ (evidently $h \in L^2(R^n)$). Then

$$\hat{g}(k) = \frac{1}{(2\pi)^n}\int_{T^n} g(x)e^{-i\langle k,x\rangle}dx \quad \text{for} \quad k \in Z^n$$
$$= \frac{1}{(2\pi)^n}\int_{R^n} f(x)h(x)e^{-i\langle k,x\rangle}dx$$
$$= \int_{R^n} \hat{f}(y)\hat{h}(k-y)dy \qquad \text{(by Parseval formula)}.$$

By (a) the series $\sum_{k\in Z^n}|\hat{h}(k-y)|$ converges uniformly with value $\leqq$ a constant C_1, we have

$$\|\hat{g}\|_1 = \sum_{k\in Z^n}|\hat{g}(k)| \leqq \int_{R^n}|\hat{f}(y)|\sum_{k\in Z^n}|\hat{h}(k-y)|\,dy \leqq C_1\|\hat{f}\|_1\,,$$

and $\|g\|_p = \|fh\|_p \leqq \|f\|_p$, hence

$$\|g\|_{\hat{A}^p(T^n)} \leqq C_1\|f\|_{\hat{A}^p(R^n)}\,,$$

for some positive constant C_1. Since $\hat{A}^1(R^n)$ is dense in $\hat{A}^p(R^n)$, Φ is defined to a bounded linear mapping of $\hat{A}^p(R^n)$ into $\hat{A}^p(T^n)$.

Conversely if $g \in \hat{A}^p(T^n)$, we associate a function

$$f^*(x_1, x_2, \cdots, x_n) = g(e^{ix_1}, \cdots, e^{ix_n})\,, \qquad x = (x_1, \cdots, x_n) \in R^n\,.$$

The function f^* is then bounded continuous in R^n having period 2π in each of the variables $x_1, x_2, \cdots, x_n$ and hence $f^* \in B(R^n)$, $\|f^*\|_{B(R^n)} = \|g\|_{A(T^n)}$. Since T^n is compact in R^n, it follows from Lai [9; Theorem 3], that there is a $h_1 \in \hat{A}^p(R^n)$ like as h above and an open set $U \supset T^n$ with Haar measure not larger than $1 + \varepsilon^p/_{\|g\|^p}$ (i.e., $|U - T^n| < \varepsilon^p/|_{\|g\|^p}$) for any $\varepsilon > 0$ such that $0 \leqq h_1 \leqq 1$ and

$$h_1(x) = 1 \quad \text{on} \quad T^n$$
$$= 0 \quad \text{outside } U \text{ in } R^n\,.$$

This h_1 satisfies the inequality (a). Observe that if we define $f = f^*h_1 = \Psi g$ then $f \in \hat{A}^p(R^n)$. In fact

$$\|f\|_p^p = \int_U |f^*h_1|^p dx$$
$$< \int_{U-T^n}|f^*|^p dx + \|g\|_p^p$$
$$< \varepsilon^p + \|g\|_p^p\,.$$

Hence $\|f\|_p < \varepsilon + \|g\|_p$.

On the other hand, it is clear that $f \in L^1(R^n)$. It follows from inversion theorem that

$$\|\hat{f}\|_1 = \|f\|_{A(R^n)} = \|f^* h_1\|_{A(R^n)}$$
$$\leqq \|f^*\|_{B(R^n)} \|h_1\|_{A(R^n)} = \|g_{A(T^n)}\| \|\hat{h}_1\|_1 < \|\hat{g}\|_1 C .$$

Consequently, $\|f\| \leqq C_2 \|g\|$ for some constant $C_2 > 0$. Evidently $f|_{T^n} = g$.
q.e.d.

3. Restriction of functions in $\hat{A}^p(\Gamma)$ to $\hat{A}^p(\Lambda)$. Let Λ be any closed subgroup of $\Gamma = \hat{G}$ and H be its annihilator group in G. Applying Rudin [14; 2.7.4], the following theorem is not hard to show.

THEOREM 2. *For any $f \in A^p(G)$, there is $g \in A^p(G/H)$ such that $\hat{f}|_\Lambda = \hat{g}$ and $\|\hat{g}\|_{\hat{A}^p(\Lambda)} \leqq \|\hat{f}\|_{\hat{A}^p(\Gamma)}$.*

PROOF. Since the set of all continuous functions in $A^p(G)$ with compact supports is dense in $A^p(G)$, it suffices to take $f \in C_c(G)$ in $A^p(G)$ such that the Weil's formula

$$\int_G f(x)dx = \int_{G/H} \int_H f(x+y)dy d\xi = \int_{G/H} g(\xi)d\xi$$

holds where $d\xi$ is normalized so that $dy_H d\xi_{G/H} = dx_G$ and

$$g(\xi) = g\circ\pi_H(x) = \int_H f(x+y)dy$$

where π_H denotes the cannonical map of $G \to G/H$. It is evident that $\|g\|_{L^1(G/H)} \leqq \|f\|_{L^1(G)}$. Furthermore, for any $\eta \in \Lambda$, $\hat{g}(\eta) = \hat{f}(\eta)$, and by Weil's formula, we have

$$\|\hat{g}\|_{L^p(\Lambda)} = \|\hat{f}\|_{L^p(\Lambda)} \leqq \|\hat{f}\|_{L^p(\Gamma)} .$$

Therefore

$$\|\hat{g}\|_{\hat{A}^p(\Lambda)} \leqq \|\hat{f}\|_{\hat{A}^p(\Gamma)} .$$ q.e.d.

Note that all of the discussions in $\hat{A}^p(\Gamma)$ and $\hat{A}^p(\Lambda)$, it is essential dealing to the spaces $L^p(\Gamma)$ and $L^p(\Lambda)$. If Λ is open or compact subgroup, then there exists a linear lifting λ: $\hat{A}^p(\Lambda) \to \hat{A}^p(\Gamma)$, and hence the restriction in Theorem 2 is an onto linear mapping such that $\mathrm{Res}\circ\lambda = Id$. (cf. Herz [7]).

THEOREM 3. *If Λ is an open subgroup of Γ, then there exists a linear lifting λ: $\hat{A}^p(\Lambda) \to \hat{A}^p(\Gamma)$, and $\|\lambda\| = 1$.*

PROOF. For any $\hat{g} \in \hat{A}^p(\Lambda)$, we define λ: $\hat{A}^p(\Lambda) \to \hat{A}^p(\Gamma)$ by

$$\lambda\hat{g}(\eta) = \hat{f}(\eta) = \begin{cases} \hat{g}(\eta) & \text{for } \eta \in \Lambda \\ 0 & \text{for } \eta \notin \Lambda . \end{cases}$$

Since Λ is an open subgroup of Γ, Γ/Λ is discrete and then by Weil's formula, we have

$$\|\hat{f}\|_{L^p(\Gamma)} = \Big(\sum_{\zeta\in\Gamma/\Lambda}\int_\Lambda |\hat{f}(\gamma+\eta)|^p\,d\eta\Big)^{1/p}$$
$$= \Big(\int_\Lambda |\hat{f}(\eta)|^p d\eta\Big)^{1/p} = \|\hat{g}\|_{L^p(\Lambda)}$$

since $\hat{f}(\eta) = 0$ outside of Λ.

On the other hand, the annihilator H of Λ is a compact subgroup in G since Λ is open subgroup of Γ, we normalize the Haar measure of H such that $\int_H dy = 1$. Thus if we define

$$f_1(x) = g\circ\pi_H(x)$$

where π_H is the cannonical map of $G\to G/H$, then $\|f_1\|_{L^1(G)} = \|g\|_{L^1(G/H)}$. We have to show that $\hat{f}_1 = \hat{f}$. In fact,

$$\hat{f}_1(\eta) = \int_G f_1(x)(-x,\eta)dx = \int_{G/H}\int_H g\circ\pi_H(x+y)(-x-y,\eta)dyd\xi$$
$$= \int_{G/H} g(\xi)(-\xi,\eta)\int_H(-y,\eta)dyd\xi\ ,$$

if $\eta\in\Lambda$, $(-y,\eta)=1$ for $y\in H$ and $\int_H dy = 1$, then

$$\hat{f}_1(\eta) = \hat{g}(\eta)$$

if $\eta\notin\Lambda$, $\int_H(-y,\eta)dy = 0$, then

$$\hat{f}_1(\eta) = 0\ .$$

Therefore $\hat{f}_1 = \hat{f}$ and $\|\hat{f}\|_{\hat{A}^p(\Gamma)} = \|\hat{g}\|_{\hat{A}^p(\Lambda)}$. q.e.d.

THEOREM 4. *If Λ is a compact subgroup of Γ, then there exists a linear lifting $\lambda: \hat{A}^p(\Lambda)\to\hat{A}^p(\Gamma)$ with $\|\lambda\|\leqq 1+\varepsilon$.*

PROOF. If Λ is compact in Γ, then there exists $h\in A^p(G)$ and an open set U containing Λ with Haar measure $<1+\varepsilon^p$ for any $\varepsilon>0$, such that

$$\hat{h} = 1 \quad\text{on}\quad \Lambda$$
$$= 0 \quad\text{outside}\quad U\,(0\leqq\hat{h}\leqq 1)\ ,$$

where Λ is normalized so that the Haar measure of Λ is equal to 1. This h can be chosen to be $\|h\|_1\leqq 1+\varepsilon$.

For any $\hat{g}\in\hat{A}^p(\Lambda)$, the Fourier series expansion gives

$$\hat{g} = \sum_{\chi\in G/H} g(\chi)\chi\ ,$$

where χ means the character function $(\chi,\cdot)$ on Λ, then $\|\hat{g}\| = \|g\|_1$. We define

$$\hat{h}_\chi = \hat{h}\cdot\chi \quad \text{on} \quad \Lambda$$
$$= \hat{h} \qquad \text{outside} \quad \Lambda \quad \text{in} \quad \Gamma .$$

Then $\hat{h}_\chi \in L^1(\Gamma)$. By inverse theorem, $h_\chi \in L^1(G)$ and $\hat{h}_\chi \in \hat{A}^p(\Gamma)$, $\|\hat{h}_\chi\|_p < 1 + \varepsilon$ and $\|\hat{h}_\chi\|_{A(\Gamma)} = \|h_\chi\|_1 \leqq \|h\|_1 \leqq 1 + \varepsilon$. Now for any $\hat{g} \in \hat{A}^p(\Lambda)$, define

$$\hat{f} = \lambda\hat{g} = \sum_{\chi \in G/H} g(\chi)\hat{h}_\chi .$$

This is a function in $\hat{A}^p(\Gamma)$ and

$$\hat{f}\,|_\Lambda = \sum_{\chi \in G/H} g(\chi)\chi = \hat{g} .$$

Furthermore,

$$\|\hat{f}\|_p \leqq \sum_{\chi \in G/H} |g(\chi)|\,\|\hat{h}_\chi\|_p \leqq \|g\|_1\,\|\hat{h}_\chi\|_p < \|\hat{g}\|\,(1 + \varepsilon)$$

and

$$\|f\|_1 = \|\hat{f}\|_{A(\Gamma)} \leqq \sum_{\chi \in G/H} |g(\chi)|\,\|\hat{h}_\chi\|_{A(\Gamma)} < \|\hat{g}\|\,(1 + \varepsilon) .$$

Hence

$$\|\hat{f}\| < \|\hat{g}\|\,(1 + \varepsilon) . \qquad \text{q.e.d.}$$

REMARK 1. It is worthy to remark here that if Λ is any closed subgroup of Γ, then the existence of lifting λ: $\hat{A}^p(\Lambda) \to \hat{A}^p(\Gamma)$ is an open question.

4. **Functions which operate in $A^p(G)$-algebras.** A classical theorem of Wiener-Lévy stated that if $f \in A$, the class of all functions on the unit circle which sums of absolutely convergent trigonometric series, and if F is defined and analytic on the range of f, then $F(f) \in A$. This theorem was extended by Gelfand who showed that it holds for regular semi-simple commutative Banach algebra. Many authors investigated in the converse: Which function F have the property that $F(f) \in A$ whenever $f \in A$? where A denotes certain algebra. We give a definition that a function F operates in a commutative Banach algebra as follows.

DEFINITION. A function F defined in a set D of complex plane operates in a commutative Banach algebra A if $F(\hat{f}) \in \hat{A}$ whenever $f \in A$ and the range of $\hat{f}$ is included in D, where $\hat{f}$ is the Gelfand transformation defined on the character space and range of $\hat{f}$ is the spectrum of f.

We denote by $F\circ f \in A$ to be that $F(\hat{f}) \in \hat{A}$ if F operates in A (some time it is equivalent to say that F is operating in $\hat{A}$). Without loss of generality, throughout we may assume that F is defined in the closed interval $I = [-1, 1]$ and that $F(0) = 0$ (cf. Helson, Kahane, Katznelson and Rudin [5]). In this section, we give an application of the reduction theorems proved in previous sections. Our main theorem in this section is following:

THEOREM 5. *If G is a noncompact locally compact abelian group and if F operates in $A^p(G)$, then F is an analytic function on $I = [-1, 1]$.*

PROOF. Note that if G is noncompact locally compact, then Γ is not discrete. The continuity of F is immediately (cf. [5; 1.1]).

(i) If G is infinite discrete, then $A^p(G) = L^1(G)$ with norm $\|f\|^p = \|f\|_1$ for any $f \in L^1(G)$. Indeed, for $f \in L^1(G)$, $\hat{f} \in L^p(\Gamma)$ for $1 \leqq p < \infty$, we have $\|\hat{f}\|_p \leqq \|\hat{f}\|_\infty \leqq \|f\|_1$ since Γ is compact, then $\|f\|^p = \|f\|_1$. In this case the theorem follows from Helson, Katznelson, and Rudin [5; Theorem 2] that F is analytic on I.

(ii) If G is nondiscrete (and noncompact), then $\Gamma = \hat{G}$ contains an open subgroup $\Gamma_0 = \Lambda \oplus R^n$, the direct sum of compact group Λ and Euclidean space $R^n (n \geqq 0)$. If $n = 0$, $\Gamma_0 = \Lambda$, then by Theorem 3 and Theorem 4 that $F(\hat{f}) \in \hat{A}^p(\Gamma)$ for every $\hat{f}$ in $\hat{A}^p(\Gamma)$ with values in $[-1, 1]$ implies $F(\hat{g}) \in \hat{A}^p(\Gamma_0) = \hat{A}^p(\Lambda)$ for $\hat{g} \in \hat{A}^p(\Lambda)$ where $\hat{g}$ is the restriction of $\hat{f}$ on Λ. It follows from (i) again that F is analytic on I. Hence it is sufficient to consider now that $n > 0$. Again by applying Theorem 3, when the function F is operating in $\hat{A}^p(\Gamma)$, then it reduces to operating in $\hat{A}^p(\Gamma_0)$, where Γ_0 is an open subgroup of Γ. If we consider the subalgebra $\hat{A}^p(\Gamma_0)$ consisting of those f in $\hat{A}^p(\Gamma_0)$ which are constant on the cosets of Λ, then it is sufficient to show that F is operating in $\hat{A}^p(R^n)$, and, using Theorem 1, one can prove easily that the function F is operating in $\hat{A}^p(T^n)$ (cf. Remark 2 in following). Consequently all the proof returns to the case (i) and then F is analytic on I. q.e.d.

REMARK 2. It is not hard to show that if $g \in \hat{A}^p(T^n)$ with value $g(e^{ix}) = g(e^{ix_1}, \cdots, e^{ix_n})$ in $[-1, 1]$ and F is operating in $\hat{A}^p(R^n)$, then $F(g) \in \hat{A}^p(T^n)$, i.e., F is operating in $\hat{A}^p(T^n)$.

PROOF. For any $g \in \hat{A}^p(T^n)$, by Theorem 1, there exists a $f \in \hat{A}^p(R^n)$ such that $f|_{T^n} = g$, this means $g(e^{ix}) = f(x)$, $x = (x_1, x_2, \cdots, x_n)$, $|x| \leqq \pi$. Since F is operating in $\hat{A}^p(R^n)$, $F(f) \in \hat{A}^p(R^n)$ and $F(f)|_{T^n} = F(g)$. Setting $\psi(x) = F(f)(x)$, we have $\psi|_{T^n} \equiv \phi_1(x) \equiv \phi(e^{ix}) \equiv F(g(e^{ix}))$ for $|x| \leqq \pi$. Then $\psi \in \hat{A}^p(R^n)$ and we have to show $\phi \in \hat{A}^p(T^n)$. As in the proof of Theorem 1, we can choose a positive function h on R^n with partial derivative of order $\geqq 2$ such that $h = 1$ on T^n and $= 0$ outside of an open set U containing T^n with measure $\leqq 1 + \varepsilon$ for a given $\varepsilon > 0$. Then

$$\|\hat{\phi}\|_1 = \sum_{k \in Z^n} |\hat{\phi}(k)| = \sum_{k \in Z^n} \frac{1}{(2\pi)^n} \left| \int_{T^n} \phi_1(x) e^{i\langle k, x\rangle} dx \right|$$
$$\leqq C \sum_{k \in Z^n} \frac{1}{(2\pi)^n} \left| \int_{R^n} \psi(x) h(x)^2 e^{i\langle k, x\rangle} dx \right|$$

for some constant $C > 0$. Since $\psi(x)h(x) = \psi_h(x) \in L^1 \cap L^2(R^n)$, $h \in L^1 \cap L^2(R^n)$, by Parseval theorem, we have

$$\|\hat{\phi}\|_1 \leqq C \sum_{k \in Z^n} \left| \int_{R^n} \hat{\psi}_h(x)\hat{h}(x-k)dx \right|$$
$$\leqq C \int_{R^n} |\hat{\psi}_h(x)| \sum_{k \in Z^n} |\hat{h}(x-k)|\, dx \leqq C_1 \|\hat{\psi}_h\|_1 < \infty$$

since h has partial derivative of order $\geqq 2$, $\sum_{k \in Z^n} |\hat{h}(x-k)|$ converges uniformly to a constant and $\hat{\psi}_h \in L^1(R^n)$. Therefore

$$\phi \in \hat{A}^p(T^n)\ . \qquad \text{q.e.d.}$$

REMARK 3. If G is infinite compact and $1 \leqq p \leqq 2$, then $\|f\|^p = \|\hat{f}\|_p$ and the function F operating in $\hat{A}^p(\Gamma)$ need not be analytic, for example if we take $F(\hat{f}) = \pm\hat{f}$, then F is only a bounded function. If G is infinite compact and $p > 2$, it seems to be an open question that whether the operating function F in $\hat{A}(\Gamma)$ is analytic or not.

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