# RESTRICTIONS OF FOURIER TRANSFORMS TO CURVES ${ }^{(*)}$ 

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## Introduction.

Given a smooth curve in $\mathbf{R}^{n}$ and a smooth measure $\sigma$ on the curve one may ask for which $a$ and $b$ does the restriction estimate $\left\{\int|\hat{f}(x)|^{b} d \sigma(x)\right\}^{1 / b} \leqslant C\|f\|_{a} \quad(f \in \mathscr{S})$ hold. Such an estimate implies that for $f$ in $\mathrm{L}^{a}\left(\widehat{\mathrm{R}}^{n}\right)$ the restriction of $\hat{f}$ to the curve "makes sense". We refer the reader to [1] and [2] for general information about restriction theorems. The object of this article is to extend the restriction theorem of Prestini [3] to the full range of exponents.

Since $\mathscr{\Im} \mathrm{L}^{a}$ is an affinely invariant space (that is invariant under the group of affine motions) we will consider only affine invariants of the curve. For a discussion of these invariants the reader may consult Guggenheimer [4] pp. 170-173. For the sake of simplicity in laying out the basic idea of this paper we will restrict attention to the special case of the non-compact curve $x(t)=\left(t, \frac{1}{2} t^{2}, \frac{1}{6} t^{3}\right)$ in $\mathbf{R}^{3}$. This is essentially the unique curve for which the first and second affine curvatures vanish and the affine arc length measure is just $d t$.

Theorem 1. - Let $1 \leqslant a<\frac{7}{6}$, let $a^{\prime}=6 b\left(\right.$ so that $\left.\frac{7}{6}<b \leqslant \infty\right)$ and let $x(t)=\left(t, \frac{1}{2} t^{2}, \frac{1}{6} t^{3}\right)$. Then

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$\left\{\int|\hat{f}(x(t))|^{b} d t\right\}^{1 / b} \leqslant \mathrm{C}_{a}\|f\|_{a}$ for all $f \in \circlearrowleft\left(\mathbf{R}^{3}\right)$.
The same techniques also yield the corresponding result in higher dimensions.

Theorem $1^{\prime}$. - Let $\quad n \geqslant 2, \quad 1 \leqslant a<\left(n^{2}+n+2\right) /\left(n^{2}+n\right)$, $a^{\prime}=\frac{1}{2} n(n+1) b$ and let $x(t)=\left(t, \frac{1}{2} t^{2}, \ldots, \frac{1}{n!} t^{n}\right)$. Then

$$
\left\{\int|\hat{f}(x(t))|^{b} d t\right\}^{1 / b} \leqslant \mathrm{C}_{a}\|f\|_{a}
$$

for all $f \in \mathscr{S}\left(\mathbf{R}^{n}\right)$.
It is well known that the ranges of $a$ and $b$ in the above theorems are optimal at least for $n=2$ and 3. The theorem is well known in case $n=2$ (Zygmund [5]).

Our methods can also be used to establish a local result. For this we demand that the curve possess an affine arc length parametrization, that is a parametrization $x(t)$ such that

$$
\operatorname{det}\left(x^{(1)}(t), x^{(2)}(t), \ldots, x^{(n)}(t)\right)=1
$$

for all $t$. Here $x^{(k)}$ denotes the $k t h$ derivative of $x$ viewed as a column vector.

ThEOREM 2.-Let $n \geqslant 2$ and let $x(t)$ be a $C^{(n)}$ curve in $\mathbf{R}^{n}$ defined for $\alpha<t<\beta$ and such that $t$ is the affine arc length. Then for $1 \leqslant a<\left(n^{2}+n+2\right) /\left(n^{2}+n\right) \quad$ and $\quad a^{\prime} \leqslant \frac{1}{2} n(n+1) b$ we have $\quad\left\{\int_{\alpha^{\prime}}^{\beta^{\prime}}|\dot{f}(x(t))|^{b} d t\right\}^{1 / b} \leqslant \mathrm{C}_{\alpha^{\prime}, \beta^{\prime}, a}\|f\|_{a} \quad(f \in \mathscr{X}) \quad$ for every compact subinterval $\left[\alpha^{\prime}, \beta^{\prime}\right]$ of $(\alpha, \beta)$.

Proofs of the theorems. - We now seek to prove Theorem 1. We will adopt the dual formulation of the problem. Thus we will prove that

$$
\begin{equation*}
\left\|(\varphi \cdot \sigma)^{\wedge}\right\|_{q} \leqslant \mathrm{C}\|\varphi\|_{p} \tag{1}
\end{equation*}
$$

for $1 \leqslant p<7$ and $p^{-1}+6 q^{-1}=1$. Here $\sigma$ denotes the affine arc length measure on the curve and $\varphi$ is a function in $L^{p}(\sigma)$. We will prove this result by induction on the exponent $p$. Therefore we shall assume that equation (1) holds in the range $1 \leqslant p \leqslant p_{0}$ for some fixed $p_{0}<7$.

Because of the special geometry of the situation there is a 1 -parameter group of affine motions of $\mathbf{R}^{\mathbf{3}}$ given by

$$
\alpha_{s}(x, y, z)=\left(x+s, y+s x+\frac{1}{2} s^{2}, z+s y+\frac{1}{2} s^{2} x+\frac{1}{6} s^{3}\right)
$$

which fix our curve and act on it by translation of the parameter $t$. The orbits of this action are curves affinely equivalent to the initial one. In fact let us parametrize the curves by $y$ and $z$ (taking $x=0$ ) so that the corresponding curve is

$$
t \longrightarrow\left(t, y+\frac{1}{2} t^{2}, z+t y+\frac{1}{6} t^{3}\right)
$$

By affine equivalence our induction hypothesis applies equally well to each of the orbits. For a function $f \in \mathrm{~L}_{\text {loc }}^{1}\left(\mathbf{R}^{3}\right)$ we introduce the auxiliary function $F$ by defining

$$
\begin{equation*}
\mathrm{F}(y, z ; t)=f\left(t, y+\frac{1}{2} t^{2}, z+t y+\frac{1}{6} t^{3}\right) \tag{2}
\end{equation*}
$$

By disintegrating the function $f$ on the family of orbits and applying the induction hypothesis on each orbit we have

Lemma 1. - Let $1 \leqslant p<p_{0}$ and $p^{-1}+6 q^{-1}=1$. Then

$$
\|\hat{f}\|_{q} \leqslant C_{p}\|F\|_{L^{1}\left(L^{p}\right)}
$$

Here the mixed norm space is $\mathbf{L}^{\mathbf{1}}\left(\mathbf{R}_{y, z}^{\mathbf{2}}, \mathbf{L}^{p}\left(\mathbf{R}_{t}\right)\right)$.
A simple change of variable and an application of the Plancherel Theorem also yield $\|f\|_{2}=\|f\|_{2}=\|\mathrm{F}\|_{L^{2}\left(L^{2}\right)}$ : Thus by a routine interpolation argument (Benedeck and Panzone [6]) we have

Lemma 2. - For $\left(a^{-1}, b^{-1}\right)$ in the triangle with vertices $(1,1),\left(1, p_{0}^{-1}\right),\left(\frac{1}{2}, \frac{1}{2}\right)$ and $c$ defined by $5 a^{-1}+b^{-1}+6 c^{-1}=6$ we have $\|\hat{f}\|_{c} \leqslant \mathrm{C}_{a, b}\|\mathrm{~F}\|_{\mathrm{L}^{a}(\mathrm{~L} b)}$.

This lemma may be viewed as a substitute for the HausdorffYoung Theorem.

We now follow the method of Prestini. Let $\varphi$ be a function on $\quad \mathbf{R}$ satisfying $|\varphi| \leqslant I_{\mathrm{E}}$ and meas $(\mathrm{E})=m$. We consider
$(\varphi \cdot \sigma) *(\varphi \cdot \sigma) *(\varphi \cdot \sigma)$ and scale the resulting measure by a factor of $\frac{1}{3}$ so as to adapt it to the original curve. The scaled measure is given by a locally integrable density $f$ such that

$$
\begin{equation*}
\left((\varphi \cdot \sigma)^{\wedge}(3 u)\right)^{3}=\hat{f}(u) \tag{3}
\end{equation*}
$$

and

$$
f\left(\frac{1}{3}\left(x\left(t_{1}\right)+x\left(t_{2}\right)+x\left(t_{3}\right)\right)\right)=c v^{-1} \varphi\left(t_{1}\right) \varphi\left(t_{2}\right) \varphi\left(t_{3}\right)
$$

where $v$ stands for the Vandermonde $\left|\left(t_{2}-t_{3}\right)\left(t_{3}-t_{1}\right)\left(t_{1}-t_{2}\right)\right|$ and $c$ is an absolute constant. A calculation now leads to

$$
\begin{equation*}
\|\mathrm{F}\|_{\mathrm{L}^{a(\mathrm{~L} b)}}=c\left\{\int v^{-(a-1)}\left\{\Phi\left(h_{1}, h_{2}\right)\right\}^{a b^{-1}} d h_{1} d h_{2}\right\}^{a^{-1}} \tag{4}
\end{equation*}
$$

with $v$ the Vandermonde $\left|h_{1} h_{2}\left(h_{1}-h_{2}\right)\right|$ and

$$
\Phi\left(h_{1}, h_{2}\right)=\int\left|\varphi(t) \varphi\left(t+h_{1}\right) \varphi\left(t+h_{2}\right)\right|^{b} d t
$$

Clearly $\quad\left|\Phi\left(h_{1}, h_{2}\right)\right| \leqslant m \quad$ and $\quad \int\left|\Phi\left(h_{1}, h_{2}\right)\right| d h_{1} d h_{2} \leqslant m^{3}$. Combining these estimates gives

$$
\begin{equation*}
\left\|\Phi^{a b^{-1}}\right\|_{L_{s, 1}} \leqslant C_{s, a, b} m^{a b^{-1}+2 s^{-1}} \tag{5}
\end{equation*}
$$

where $\mathrm{L}_{s, 1}$ denotes the Lorentz space $\mathrm{L}_{s, 1}\left(d h_{1} d h_{2}\right)$ (see Stein and Weiss [7] or Hunt [8]) and where $1 \leqslant b a^{-1}<s<\infty$. On the other hand routine calculations show that $v^{-(a-1)}$ lies in the dual Lorentz space $L_{s^{\prime}, \infty}\left(d h_{1} d h_{2}\right)$ for $2=3(a-1) s^{\prime}$ and $1<a<\frac{5}{3}$. Thus we obtain from (4) and (5) that for we have, $\quad 1<a<\frac{5}{3}, 5 a^{-1}-2 b^{-1}<3, a \leqslant b$,

$$
\|\mathrm{F}\|_{\mathrm{L} a(\mathrm{~L} b)} \leqslant \mathrm{C}_{a, b} m^{5 a^{-1}+b^{-1}-3}
$$

We are now in a position to apply lemma 2 for $\left(a^{-1}, b^{-1}\right)$ in the quadrilateral defined by $a^{-1} \geqslant b^{-1}, a^{-1}>\frac{3}{5}, 5 a^{-1}-2 b^{-1}<3$ and $\left(p_{0}-2\right) a^{-1}+p_{0} b^{-1} \geqslant p_{0}-1$. Thus there exists a number $a_{0}$ depending only on $p_{0}$ with $a_{0}<\frac{5}{3}$ so that lemma 2 can be applied
in case $a_{0}<a<\frac{5}{3}$ and $b$ is given by

$$
\left(p_{0}-2\right) a^{-1}+p_{0} b^{-1}=p_{0}-1 .
$$

We conclude that for suitable $c_{0}$ we have $\|\hat{f}\|_{c} \leqslant \mathrm{C}_{c} m^{3-6 c^{-1}}$ for all $c$ in the range $30 p_{0}\left(13 p_{0}-1\right)^{-1}<c<c_{0}$. Thus by (3)

$$
\left\|(\varphi \cdot \sigma)^{\wedge}\right\|_{q} \leqslant C_{q} m^{1-6 q^{-1}}
$$

for all $q$ in the range $90 p_{0}\left(13 p_{0}-1\right)^{-1}<q<3 c_{0}$. Routine interpolation arguments now yield $\left\|\varphi \cdot \sigma^{\wedge}\right\|_{q} \leqslant \mathrm{C}_{p}\|\varphi\|_{p}$ for $p^{-1}+6 q^{-1}=1$ and $1 \leqslant p<15 p_{0}\left(2 p_{0}+1\right)^{-1}$. This completes the induction step.

The induction starts trivially with $p_{0}=1$. One step of the induction yields the result for $1 \leqslant p<5$ - that is the result of Prestini and with the same proof. With two steps we have the result for $1 \leqslant p<75 / 11$ and it is clear that for any $p$ with $1 \leqslant p<7$ the result for that $p$ will follow after only finitely many steps.

The proof of theorem $1^{\prime}$ is entirely analogous.
We will leave the detailed proof of theorem 2 to the reader. Some comments however are in order. First of all in general there is no group action preserving the initial curve. Thus a typical curve of our family will be defined by

$$
t \longrightarrow n^{-1} \sum_{k=1}^{n} x\left(t+h_{k}\right)
$$

the family of curves being indexed by the ( $n-1$ )-dimensional manifold of $\left(h_{1}, \ldots, h_{n}\right)$ satisfying $\sum_{k=1}^{n} h_{k}=0$. The inductive nature of the proof then leads in general to further curves of the form

$$
\begin{equation*}
y(t)=\sum_{k=1}^{K} \alpha_{k} x\left(t+\ell_{k}\right): \tag{6}
\end{equation*}
$$

where $\alpha_{k}>0, \sum_{k=1}^{K} \alpha_{k}=1$ and the $\ell^{\prime} s$ are sums of the $h^{\prime} s$.
Let $t_{0}$ be a fixed point $\alpha<t_{0}<\beta$. It will be necessary to establish uniform estimates for the curve (6) on an interval
$t_{0}-\epsilon<t<t_{0}+\epsilon$ and for $\left|\ell_{k}\right|<\epsilon$. Towards this we select convex neighbourhoods $\mathrm{V}_{k}$ of $x^{(k)}\left(t_{0}\right)$ such that

$$
2 \geqslant \operatorname{det}\left(v_{1}, \ldots, v_{n}\right) \geqslant \frac{1}{2}
$$

for $v_{k} \in \mathrm{~V}_{k}(1 \leqslant k \leqslant n)$. It now follows from the fact that the initial curve is $\mathrm{C}^{(n)}$ that there exists a number $\epsilon>0$ such that

$$
2 \geqslant \operatorname{det}\left(y^{(1)}\left(\tau_{1}\right), \ldots, y^{(n)}\left(\tau_{n}\right)\right) \geqslant \frac{1}{2}
$$

for $\left|\tau_{k}-t_{0}\right|<\epsilon$ and $\left|\ell_{k}\right|<\epsilon$. (In particular it follows that the measure $d t$ is uniformly equivalent to the affine arc length measure of (6) for $\left.\left|t-t_{0}\right|<\epsilon, \quad\left|\ell_{k}\right|<\epsilon\right)$. The vital estimate is a lower bound on the absolute value of the Jacobian $J$ of the barycentre map $\left(t_{1}, \ldots, t_{n}\right) \longrightarrow \frac{1}{n} \sum_{k=1}^{n} y\left(t_{k}\right)$. Up to a constant factor this is $\left|\operatorname{det}\left(y^{(1)}\left(t_{1}\right), \ldots, y^{(1)}\left(t_{n}\right)\right)\right|$ and by a generalization of the mean-value theorem (Polya, Szegö [9]. Vol. II, part V, Chap. 1, No. 95) this is equivalent to

$$
\left(\prod_{1<i<j \leqslant n}\left|t_{i}-t_{j}\right|\right) \mid \operatorname{det}\left(y^{(1)}\left(\tau_{1}\right), \ldots, y^{(n)}\left(\tau_{n}\right) \mid\right.
$$

for suitable $\tau_{1}, \ldots, \tau_{n}$. This now yields the uniform estimate

$$
|\mathrm{J}| \geqslant c_{n} \prod_{1 \leqslant i<j \leqslant n}\left|t_{i}-t_{j}\right| \quad\left(c_{n}>0\right)
$$

for $\left|t_{0}-t_{k}\right|<\epsilon(1 \leqslant k \leqslant n),\left|\ell_{k}\right|<\epsilon$. This completes our comments on Theorem 2.

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