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# Results for Mild solution of fractional coupled hybrid boundary value problems

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**Abstract:** The study of coupled system of hybrid fractional differential equations (HFDEs) needs the attention of scientists for the exploration of its different important aspects. Our aim in this paper is to study the existence and uniqueness of mild solution (EUMS) of a coupled system of HFDEs. The novelty of this work is the study of a coupled system of fractional order hybrid boundary value problems (HBVP) with  $n$  initial and boundary hybrid conditions. For this purpose, we are utilizing some classical results, Leray–Schauder Alternative (LSA) and Banach Contraction Principle (BCP). Some examples are given for the illustration of applications of our results.

**Keywords:** Hybrid fractional differential equations, Existence and uniqueness of Mild solution, Leray–Schauder Alternative, Banach Contraction Principle

**MSC:** 34A08, 74H20

## 1 Introduction

There are many scientific contributions to the applications of fractional differential equations (FDEs) in different scientific fields including economics, polymer rheology, chemistry, mechanics, control theory, aerodynamics, biophysics, regular variation in thermodynamics, etc [2, 3, 6, 10]. These applications play a vital role in popularity of FDEs in particular and fractional calculus in general. Scientist are interested in getting highly precised and accurate results in their research models. For this reason they utilize different mathematical tools by the help of which they receive exact solutions of their models as well as numerical approximations. For the exact solutions of different classes of FDEs, we refer to valuable efforts of Y. J. Yang et al. regarding Laplace equation [11]. In [12], C. G. Zhao et al. produced exact solutions of many initial value problems of local FDEs. Besides the exact solutions, we also have valuable efforts of scientist for numerical approximations of FDEs. For example, in [8] H. Khan et al. produced the approximate solution of factional order Logistic equations by the help of operational matrices of Bernstein polynomials.

Various aspects of FDEs have been considered. Recently, a valuable interest has been observed on the existence, uniqueness, multiplicity of results, infinite solutions and no solutions of different classes of FDEs [7, 9]. In this paper, our interest is in the EUMS of coupled systems of HFDEs with  $n$  hybrid initial and boundary conditions. This area of FDEs is one of important problems drawing the attention of many scientists. For some important studies in

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this area we refer to B. Ahmad et al. in [1] and M. A. E. Herzallah and D. Baleanu in [5]. We were influenced by the cited work for the study of EUMS of the coupled system of HFDEs of the type

$$\begin{aligned}
 \mathcal{D}^{\omega_1} \left( \frac{\Psi}{\mathcal{H}(z, \Psi, \Lambda)} \right) &= -\mathcal{T}_1(z, \Psi, \Lambda), \quad \omega \in (n - 1, n], \\
 \mathcal{D}^{\omega_2} \left( \frac{\Lambda}{\mathcal{G}(z, \Psi, \Lambda)} \right) &= -\mathcal{T}_2(z, \Psi, \Lambda), \quad \epsilon \in (n - 1, n], \\
 \left( \frac{\Psi}{\mathcal{H}(z, \Psi, \Lambda)} \Big|_{z=0} \right)^{(i)} &= 0 = \left( \frac{\Psi}{\mathcal{H}(z, \Psi, \Lambda)} \Big|_{z=1} \right)^{(1)}, \quad \frac{\Psi}{\mathcal{H}(z, \Psi, \Lambda)} \Big|_{z=\delta_1} = \delta_1 I_1^{\omega_1-1} \mathcal{T}_1(z, \Psi, \Lambda), \\
 \left( \frac{\Lambda}{\mathcal{G}(z, \Psi, \Lambda)} \Big|_{z=0} \right)^{(j)} &= 0 = \left( \frac{\Phi}{\mathcal{G}(z, \Psi, \Lambda)} \Big|_{z=1} \right)^{(1)}, \quad \frac{\Lambda}{\mathcal{G}(z, \Psi, \Lambda)} \Big|_{z=\delta_2} = \delta_2 I_1^{\omega_2-1} \mathcal{T}_2(z, \Psi, \Lambda), \\
 i, j &= 2, 3, \dots, n - 1,
 \end{aligned} \tag{1}$$

where  $D^{\omega_1}, D^{\omega_2}$  are Caputo’s fractional derivatives of orders  $\omega_1, \omega_2$  respectively,  $\mathcal{T}_1, \mathcal{T}_2 \in C([0, 1] \times \mathbb{R}^2, \mathbb{R})$ ,  $\mathcal{H}, \mathcal{G} \in C([0, 1] \times \mathbb{R}^2, \mathbb{R} - \{0\})$  and  $\Psi, \Lambda \in C([0, 1], \mathbb{R})$ . We utilize LSA and BCP.

**Organization of the paper**

This paper is divided into three sections. The first section is the introduction, which is based on the literature review. In the second section we present our main results, namely the existence of mild solution (EMS) of the problem (1) and the uniqueness of mild solution (UMS) of the problem (1). These results are based on Leray–Schauder Alternative (LSA) and Banach contraction principle. The third section is dedicated to the applications of our results. These applications are shown on two illustrative examples.

We give the following definitions and results from the available literature for the readers and the detailed study on this work can be found in [2, 10].

**Definition 1.1.** Fractional integral for  $\omega > 0$  of a function  $\mathcal{J} : (0, \infty) \rightarrow R$  is given by

$$I_{0+}^{\omega} \mathcal{J}(z) = \frac{1}{\Gamma(\omega)} \int_0^z (z - x)^{\omega-1} \mathcal{J}(x) dx,$$

provided the integral converges.

**Definition 1.2.** For  $\mathcal{J} \in C^k[0, 1]$ , the Caputo fractional derivative for  $\omega$  order is defined by

$$\mathcal{D}^{\omega} \mathcal{J}(z) = \frac{1}{\Gamma(k - \omega)} \int_0^z (z - y)^{k-\omega-1} \mathcal{J}^{(k)}(y) dy,$$

provided that the integral on  $(0, \infty)$  is defined.

**Lemma 1.3** ([10]). For  $\epsilon \geq \omega > 0$  and  $\mathcal{J}(t) \in L_1[a, b]$ , we have the following:

$$D^{\omega} I_{a+}^{\epsilon} \mathcal{J}(z) = I_{a+}^{\epsilon-\omega} \mathcal{J}(z)$$

on the interval  $[a, b]$ , if  $\mathcal{J} \in C[a, b]$ .

**Lemma 1.4** ([10]). For  $\mathcal{J}(z) \in C(0, 1)$ , the solution of homogenous FDE  $D_{0+}^{\omega} \mathcal{J}(z) = 0$  is

$$\mathcal{J}(z) = k_1 + k_2 z + k_3 z^2 + \dots + k_n z^{n-1}, \quad k_i \in R, i = 1, 2, 3, \dots, n. \tag{2}$$

**Lemma 1.5** ([1], LSA). Let  $\mathcal{F} : \mathcal{V} \rightarrow \mathcal{V}$  be a completely continuous operator (i.e., a map that is restricted to any bounded set in  $\mathcal{V}$ ). Let  $\mathcal{Y} = \{x \in \mathcal{V} : x = \mu \mathcal{F}x \text{ for some } \mu \in (0, 1)\}$ . Then either  $\mathcal{Y}$  is unbounded or  $\mathcal{F}$  has at least one fixed point.

**Lemma 1.6.** Let  $\mathcal{J}(z)$  be a continuous function then the mild solution of

$$\mathcal{D}^{\omega_1} \left( \frac{\Psi}{\mathcal{H}(z, \Psi, \Lambda)} \right) = -\mathcal{J}(z), \quad \omega_1 \in (n - 1, n], \tag{3}$$

with

$$\left( \frac{\Psi}{\mathcal{H}(z, \Psi, \Lambda)} \Big|_{z=0} \right)^{(i)} = 0 = \left( \frac{\Psi(z)}{\mathcal{H}(z, \Psi, \Lambda)} \Big|_{t=1} \right)^{(1)}, \quad \frac{\Psi}{\mathcal{H}(z, \Psi, \Lambda)} \Big|_{z=\delta_1} = \delta_1 I^{\omega_1-1} \mathcal{J}(1), \tag{4}$$

for  $i = 2, \dots, n - 1$ , is given by

$$\Psi(z) = \mathcal{H}(z, \Psi, \Lambda) \left[ - \int_0^z \frac{(z - y)^{\omega_1-1}}{\Gamma(\omega_1)} \mathcal{J}(y) dy + \int_0^\delta \frac{(\delta - y)^{\omega_1-1}}{\Gamma(\omega_1)} \mathcal{J}(y) dy + z \int_0^1 \frac{(1 - y)^{\omega_1-2}}{\Gamma(\omega_1 - 1)} \mathcal{J}(y) dy \right]. \tag{5}$$

*Proof.* Applying the operator  $I_0^{\omega_1}$  on (3) and using Lemma 1.4, we obtain

$$\frac{\Psi(z)}{\mathcal{H}(z, \Psi, \Lambda)} = -I^{\omega_1} \mathcal{J}(z) + c_1 + c_2 z + c_3 z^2 + \dots + c_n z^{n-1} \tag{6}$$

initial conditions  $\left( \frac{\Psi}{\mathcal{H}(z, \Psi, \Lambda)} \Big|_{z=0} \right)^{(i)} = 0$ , for  $i = 2, \dots, n - 1$  imply  $c_3 = \dots = c_n = 0$ , and (6) gets the form

$$\frac{\Psi(z)}{\mathcal{H}(z, \Psi, \Lambda)} = -I^{\omega_1} \mathcal{J}(z) + c_1 + c_2 z. \tag{7}$$

Applying the boundary condition  $\left( \frac{\Psi(z)}{\mathcal{H}(z, \Psi, \Lambda)} \Big|_{z=1} \right)^{(1)} = 0$ , on (7), we have  $c_2 = I^{\omega-1} \mathcal{J}(1)$  and (7) gets the form

$$\frac{\Psi(z)}{\mathcal{H}(z, \Psi, \Lambda)} = -I^{\omega_1} \mathcal{J}(z) + c_1 + z I^{\omega-1} \mathcal{J}(1). \tag{8}$$

Applying the condition  $\frac{\Psi}{\mathcal{H}(z, \Psi, \Lambda)} \Big|_{z=\delta_1} = \delta_1 I^{\omega_1-1} \mathcal{J}(1)$ , on (8) we get  $c_0 = I^{\omega_1} \mathcal{J}(\delta_1)$ . Putting the values of  $c_0, c_1, \dots, c_n$  in (6), we get

$$\frac{\Psi(z)}{\mathcal{H}(z, \Psi, \Lambda)} = -I^{\omega_1} \mathcal{J}(z) + I^{\omega_1} \mathcal{J}(\delta) + z I^{\omega_1-1} \mathcal{J}(1) \tag{9}$$

and the integral form of the mild solution  $\Psi(z)$ , is given by

$$\Psi(z) = \mathcal{H}(z, \Psi, \Lambda) \left[ - \int_0^z \frac{(z - y)^{\omega_1-1}}{\Gamma(\omega_1)} \mathcal{J}(y) dy + \int_0^\delta \frac{(\delta - y)^{\omega_1-1}}{\Gamma(\omega_1)} \mathcal{J}(y) dy + z \int_0^1 \frac{(1 - y)^{\omega_1-2}}{\Gamma(\omega_1 - 1)} \mathcal{J}(y) dy \right] \tag{10}$$

thus, the proof is completed. □

## 2 Results for EUMS of the system (1)

In this section we focus on two lemmas for the EMS and UMS of the coupled system (1). For this purpose, we utilize LSA and BCP. With respect to these two lemmas we divided this section into two subsections. From [4] we consider the Banach space  $\mathcal{Y} = \{\Psi(z) : \Psi(z) \in C^1[0, 1]\}$  with norm  $\|\Psi(z)\| = \max\{|\Psi(z)| \text{ for } z \in [0, 1]\}$  and  $(\mathcal{Y} \times \mathcal{Y}, \|(\cdot, \cdot)\|)$  with norm  $\|(\Psi, \Lambda)(z)\| = \|\Psi(z)\| + \|\Lambda(z)\|$ . We define an operator  $\mathcal{F} : \mathcal{Y} \times \mathcal{Y} \rightarrow \mathcal{Y} \times \mathcal{Y}$  by

$$\mathcal{F}(\Psi, \Lambda)(z) = (\mathcal{F}_1(\Psi, \Lambda)(z), \mathcal{F}_2(\Psi, \Lambda)(z)), \tag{11}$$

where

$$\begin{aligned} \mathcal{F}_1(\Psi, \Lambda)(z) = & \mathcal{H}(z, \Psi, \Lambda) \left[ - \int_0^z \frac{(z - y)^{\omega_1-1}}{\Gamma(\omega)} \mathcal{T}_1(y, \Psi, \Lambda) dy + \int_0^{\delta_1} \frac{(\delta_1 - y)^{\omega_1-1}}{\Gamma(\omega_1)} \mathcal{T}_1(y, \Psi, \Lambda) dy \right. \\ & \left. + z \int_0^1 \frac{(1 - y)^{\omega_1-2}}{\Gamma(\omega_1 - 1)} \mathcal{T}_1(y, \Psi, \Lambda) dy \right], \end{aligned}$$

and

$$\begin{aligned} \mathcal{F}_2(\Psi, \Lambda)(z) = & \mathcal{G}(z, \Psi, \Lambda) \left[ - \int_0^z \frac{(z-y)^{\omega_2-1}}{\Gamma(\omega_2)} \mathcal{T}_2(y, \Psi, \Lambda) dy + \int_0^{\delta_2} \frac{(\delta_2-y)^{\omega_2-1}}{\Gamma(\omega_2)} \mathcal{T}_2(y, \Psi, \Lambda) dy \right. \\ & \left. + z \int_0^1 \frac{(1-y)^{\omega_2-2}}{\Gamma(\omega_2-1)} \mathcal{T}_2(y, \Psi, \Lambda) dy \right], \end{aligned}$$

We define the following terms to simplify our calculations:

$$\mathcal{N}_1 = \frac{1}{\Gamma(\omega_1+1)} + \frac{\delta_1^{\omega_1}}{\Gamma(\omega_1+1)} + \frac{1}{\Gamma(\omega_1)}, \tag{12}$$

$$\mathcal{N}_2 = \frac{1}{\Gamma(\omega_2+1)} + \frac{\delta_2^{\omega_2}}{\Gamma(\omega_2+1)} + \frac{1}{\Gamma(\omega_2)}. \tag{13}$$

### 2.1 EMS for the System (1)

In this subsection we focus on the EMS of HFDEs (1). For this, we assume the following two conditions:

- (A<sub>1</sub>) The functions  $\mathcal{H}(z, \Psi, \Lambda), \mathcal{G}(z, \Psi, \Lambda)$  are continuous and bounded such that  $|\mathcal{H}(z, \Psi, \Lambda)| \leq \mu_1, |\mathcal{G}(z, \Psi, \Lambda)| \leq \mu_2$  for all  $(z, \Psi, \Lambda) \in ([0, 1] \times \mathbb{R} \times \mathbb{R})$  and  $\mu_1, \mu_2 \in \mathbb{R}$ ;
- (A<sub>2</sub>) There exist real constants  $\lambda_j, \eta_j, \xi_j > 0$  for  $j = 1, 2$  such that  $|\mathcal{T}_1(z, \Psi, \Lambda)| \leq \lambda_1 + \eta_1|\Psi| + \xi_1|\Lambda|, |\mathcal{T}_2(z, \Psi, \Lambda)| \leq \lambda_2 + \eta_2|\Psi| + \xi_2|\Lambda|$  for all  $\Psi, \Lambda \in \mathbb{R}$  and  $z \in [0, 1]$  also  $\mu_1\mathcal{N}_1(\lambda_1 + \eta_1|\Psi| + \xi_1|\Lambda|) + \mu_2\mathcal{N}_2(\lambda_2 + \eta_2|\Psi| + \xi_2|\Lambda|) < +\infty$ .

**Theorem 2.1.** Assume that (A<sub>1</sub>), (A<sub>2</sub>) are satisfied and there exist real constants  $\mathcal{M}_1, \mathcal{M}_2 \in \mathbb{R}$ , such that

$$|\mathcal{T}_1(z, \Psi, \Lambda)| \leq \mathcal{M}_1, \quad |\mathcal{T}_2(z, \Psi, \Lambda)| \leq \mathcal{M}_2,$$

for all  $z \in [0, 1]$ , then the system (1) has a mild solution.

*Proof.* We give the proof in the following three steps.

**Step 1:** In this step, we prove that the operator  $\mathcal{F}$  maps bounded subset of  $\mathcal{Y} \times \mathcal{Y}$  into a bounded. For this we consider  $(\Psi, \Lambda)(z)$ , then for  $z \in [0, 1]$ , we have

$$\begin{aligned} |\mathcal{F}_1(\Psi, \Lambda)(z)| = & |\mathcal{H}(z, \Psi, \Lambda)| \left[ \left| - \int_0^z \frac{(z-y)^{\omega_1-1}}{\Gamma(\omega_1)} \mathcal{T}_1(y, \Psi, \Lambda) dy + \int_0^{\delta_1} \frac{(\delta_1-y)^{\omega_1-1}}{\Gamma(\omega_1)} \mathcal{T}_1(y, \Psi, \Lambda) dy \right. \right. \\ & \left. \left. + z \int_0^1 \frac{(1-y)^{\omega_1-2}}{\Gamma(\omega_1-1)} \mathcal{T}_1(y, \Psi, \Lambda) dy \right| \right], \tag{14} \\ \leq & \mu_1 \left[ \frac{1}{\Gamma(\omega_1+1)} + \frac{\delta_1^{\omega_1}}{\Gamma(\omega_1+1)} + \frac{1}{\Gamma(\omega_1)} \right] \mathcal{M}_1 = \mu_1 \mathcal{N}_1 \mathcal{M}_1 \end{aligned}$$

similarly,

$$|\mathcal{F}_2(\Psi, \Lambda)(z)| \leq \mu_2 \left[ \frac{1}{\Gamma(\omega_2+1)} + \frac{\delta_2^{\omega_2}}{\Gamma(\omega_2+1)} + \frac{1}{\Gamma(\omega_2)} \right] \mathcal{M}_2 = \mu_2 \mathcal{N}_2 \mathcal{M}_2 \tag{15}$$

from (14) and (15), we have

$$\|\mathcal{F}(\Psi, \Lambda)(z)\| \leq \mu_1 \mathcal{N}_1 \mathcal{M}_1 + \mu_2 \mathcal{N}_2 \mathcal{M}_2 < \infty \tag{16}$$

thus,  $\mathcal{F}$  is uniformly bounded.

**Step 2:** Here we show that  $\mathcal{F}$  is an equicontinuous operator. For this, let  $(\Psi, \Lambda)(z) \in \mathcal{S}_\rho$  and  $z_1, z_2 \in [0, 1]$  such that  $z_1 < z_2$ . Then we have the following estimates

$$\begin{aligned} |\mathcal{F}_1(\Psi, \Lambda)(z_2) - \mathcal{F}_1(\Psi, \Lambda)(z_1)| &= \mu_1 \left[ \int_0^{z_2} \frac{(z_2 - y)^{\omega_1 - 1}}{\Gamma(\omega_1)} \mathcal{T}_1(y, \Psi, \Lambda) dy \right. \\ &\quad \left. - \int_0^{z_1} \frac{(z_1 - y)^{\omega_1 - 1}}{\Gamma(\omega_1)} \mathcal{T}_1(y, \Psi, \Lambda) dy \right. \\ &\quad \left. + (z_2 - z_1) \int_0^1 \frac{(1 - y)^{\omega_1 - 2}}{\Gamma(\omega_1 - 1)} \mathcal{T}_1(y, \Psi, \Lambda) dy \right], \\ &\leq \mu_1 \left[ \frac{z_2^{\omega_1} - z_1^{\omega_1}}{\Gamma(\omega_1 + 1)} + \frac{z_2 - z_1}{\Gamma(\omega_1)} \right] \mathcal{M}_1. \end{aligned} \tag{17}$$

Similarly, we can get

$$|\mathcal{F}_2(\Psi, \Lambda)(z_2) - \mathcal{F}_2(\Psi, \Lambda)(z_1)| \leq \mu_2 \left[ \frac{z_2^{\omega_2} - z_1^{\omega_2}}{\Gamma(\omega_2 + 1)} + \frac{z_2 - z_1}{\Gamma(\omega_2)} \right] \mathcal{M}_2 \tag{18}$$

by the help of (17) and (18), we have

$$|\mathcal{F}(\Psi, \Lambda)(z_2) - \mathcal{F}(\Psi, \Lambda)(z_1)| \leq \mu_1 \left[ \frac{z_2^{\omega_1} - z_1^{\omega_1}}{\Gamma(\omega_1 + 1)} + \frac{z_2 - z_1}{\Gamma(\omega_1)} \right] \mathcal{M}_1 + \mu_2 \left[ \frac{z_2^{\omega_2} - z_1^{\omega_2}}{\Gamma(\omega_2 + 1)} + \frac{z_2 - z_1}{\Gamma(\omega_2)} \right] \mathcal{M}_2 \tag{19}$$

$\|\mathcal{F}(\Psi, \Lambda)(z_2) - \mathcal{F}(\Psi, \Lambda)(z_1)\| \rightarrow 0$  as  $z_2 \rightarrow z_1$ . By the steps 1 and 2, and Arzela Ascoli Theorem, we conclude that  $\mathcal{F}$  is a completely continuous operator.

**Step 3:** Here we show that the set

$$S^* = \{(\Psi, \Lambda)(z) \in \mathcal{S} \times \mathcal{S} : \mu \mathcal{F}(\Psi, \Lambda)(z) = (\Psi, \Lambda)(z)\}$$

where  $0 < \mu < 1$ , is bounded. From the set  $S^*$ , we have  $\mu \mathcal{F}_1(\Psi, \Lambda)(z) = \Psi(z)$  and  $\mu \mathcal{F}_2(\Psi, \Lambda)(z) = \Lambda(z)$ . Consequently,

$$\begin{aligned} |\Psi(z)| &\leq \mu |\mathcal{F}_1(\Psi, \Lambda)(z)| \\ &\leq |\mathcal{H}(z, \Psi, \Lambda)| \left[ \int_0^z \frac{(z - y)^{\omega_1 - 1}}{\Gamma(\omega_1)} \mathcal{T}_1(y, \Psi, \Lambda) dy + \int_0^{\delta_1} \frac{(\delta_1 - y)^{\omega_1 - 1}}{\Gamma(\omega_1)} \mathcal{T}_1(y, \Psi, \Lambda) dy \right. \\ &\quad \left. + z \int_0^1 \frac{(1 - y)^{\omega_1 - 2}}{\Gamma(\omega_1 - 1)} \mathcal{T}_1(y, \Psi, \Lambda) dy \right], \\ &\leq \mu_1 \left[ \frac{1}{\Gamma(\omega_1 + 1)} + \frac{\delta_1^{\omega_1}}{\Gamma(\omega_1 + 1)} + \frac{1}{\Gamma(\omega_1)} \right] (\lambda_1 + \eta_1 |\Psi| + \xi_1 |\Lambda|) = \mu_1 \mathcal{N}_1 (\lambda_1 + \eta_1 |\Psi| + \xi_1 |\Lambda|) \end{aligned} \tag{20}$$

and

$$\begin{aligned} |\Lambda(z)| &\leq \mu |\mathcal{F}_2(\Psi, \Lambda)(z)| \\ &\leq |\mathcal{G}(z, \Psi, \Lambda)| \left[ \int_0^z \frac{(z - y)^{\omega_2 - 1}}{\Gamma(\omega_2)} \mathcal{T}_2(y, \Psi, \Lambda) dy + \int_0^{\delta_2} \frac{(\delta_2 - y)^{\omega_2 - 1}}{\Gamma(\omega_2)} \mathcal{T}_2(y, \Psi, \Lambda) dy \right. \\ &\quad \left. + z \int_0^1 \frac{(1 - y)^{\omega_2 - 2}}{\Gamma(\omega_2 - 1)} \mathcal{T}_2(y, \Psi, \Lambda) dy \right] \\ &\leq \mu_2 \left( \frac{1}{\Gamma(\omega_2 + 1)} + \frac{\delta_2^{\omega_2}}{\Gamma(\omega_2 + 1)} + \frac{1}{\Gamma(\omega_2)} \right) (\lambda_2 + \eta_2 |\Psi| + \xi_2 |\Lambda|) \leq \mu_2 \mathcal{N}_2 (\lambda_2 + \eta_2 |\Psi| + \xi_2 |\Lambda|). \end{aligned} \tag{21}$$

From (20) and (21) we have

$$\|\Psi(z)\| \leq \mu_1 \mathcal{N}_1(\lambda_1 + \eta_1|\Psi| + \xi_1|\Lambda|), \quad (22)$$

$$\|\Lambda(z)\| \leq \mu_2 \mathcal{N}_2(\lambda_2 + \eta_2|\Psi| + \xi_2|\Lambda|). \quad (23)$$

Consequently,

$$\|(\Psi, \Lambda)(z)\| \leq \mu_1 \mathcal{N}_1(\lambda_1 + \eta_1|\Psi| + \xi_1|\Lambda|) + \mu_2 \mathcal{N}_2(\lambda_2 + \eta_2|\Psi| + \xi_2|\Lambda|) < +\infty \quad (24)$$

this implies that  $S^*$  is bounded. Thus, by the help of **Steps 1-3** we conclude that  $\mathcal{F}$  is completely continuous and by Lemma 1.5,  $\mathcal{F}$  has fixed the point  $(\Psi, \Lambda)(z)$  and is the mild solution of the coupled system of HFDEs (1).  $\square$

## 2.2 UMS for system the (1)

In this subsection, we are giving a proof for the UMS of the coupled system of HFDEs (1). For this we use the BCP.

**Theorem 2.2.** Assume that  $\mathcal{T}_1, \mathcal{T}_2 \in C([0, 1] \times \mathbb{R}^2, \mathbb{R})$  and there exist positive real constants  $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ , such that for all  $z \in [0, 1]$  and  $\Psi_1, \Psi_2, \Lambda_1, \Lambda_2 \in \mathbb{R}$ ,

$$|\mathcal{T}_1(z, \Psi, \Lambda) - \mathcal{T}_1(z, \Psi_1, \Lambda_1)| \leq \gamma_1|\Psi - \Psi_1| + \gamma_2|\Lambda - \Lambda_1|$$

and

$$|\mathcal{T}_2(z, \Psi, \Lambda) - \mathcal{T}_2(z, \Psi_1, \Lambda_1)| \leq \gamma_3|\Psi - \Psi_1| + \gamma_4|\Lambda - \Lambda_1|,$$

with the condition that

$$\mu_1 \mathcal{N}_1(\gamma_1 + \gamma_2) + \mu_2 \mathcal{N}_2(\gamma_3 + \gamma_4) < 1$$

where  $\mathcal{N}_1, \mathcal{N}_2$  are defined by (12), (13). Then the system (1), has a unique solution.

*Proof.* Assume that  $\sup_{z \in [0, 1]} \mathcal{T}_1(z, 0, 0) = \zeta_1 < \infty$ ,  $\sup_{z \in [0, 1]} \mathcal{T}_2(z, 0, 0) = \zeta_2 < \infty$  and the set  $\bar{S}_v = \{(\Psi, \Lambda)(z) \in \mathcal{Y} \times \mathcal{Y} : \|(\Psi, \Lambda)(z)\| \leq v\}$ , such that

$$v \geq \frac{\mu_1 \mathcal{N}_1 \zeta_1 + \mu_2 \mathcal{N}_2 \zeta_2}{1 - \mu_1 \mathcal{N}_1(\gamma_1 + \gamma_2) - \mu_2 \mathcal{N}_2(\gamma_3 + \gamma_4)} \quad (25)$$

then

$$\begin{aligned} |\mathcal{F}_1(\Psi, \Lambda)(z)| &\leq |\mathcal{H}(z, \Psi, \Lambda)| \left[ \int_0^z \frac{(z-y)^{\omega_1-1}}{\Gamma(\omega_1)} \mathcal{T}_1(y, \Psi, \Lambda) dy + \int_0^{\delta_1} \frac{(\delta_1-y)^{\omega_1-1}}{\Gamma(\omega_1)} \mathcal{T}_1(y, \Psi, \Lambda) dy \right. \\ &\quad \left. + z \int_0^1 \frac{(1-y)^{\omega_1-2}}{\Gamma(\omega_1-1)} \mathcal{T}_1(y, \Psi, \Lambda) dy \right] \\ &\leq \mu_1 \left[ \int_0^z \frac{(z-y)^{\omega_1-1}}{\Gamma(\omega_1)} (|\mathcal{T}_1(y, \Psi, \Lambda) - \mathcal{T}_1(y, 0, 0)| + |\mathcal{T}_1(y, 0, 0)|) dy \right. \\ &\quad \left. + \int_0^{\delta_1} \frac{(\delta_1-y)^{\omega_1-1}}{\Gamma(\omega_1)} (|\mathcal{T}_1(y, \Psi, \Lambda) - \mathcal{T}_1(y, 0, 0)| + |\mathcal{T}_1(y, 0, 0)|) dy \right. \\ &\quad \left. + z \int_0^1 \frac{(1-y)^{\omega_1-2}}{\Gamma(\omega_1-1)} (|\mathcal{T}_1(y, \Psi, \Lambda) - \mathcal{T}_1(y, 0, 0)| + |\mathcal{T}_1(y, 0, 0)|) dy \right] \\ &\leq \mu_1 \left( \frac{1}{\Gamma(\omega_1+1)} + \frac{\delta_1^{\omega_1}}{\Gamma(\omega_1+1)} + \frac{1}{\Gamma(\omega_1)} \right) (\gamma_1|\Psi| + \gamma_2|\Lambda| + \zeta_1) \leq \mu_1 \mathcal{N}_1((\gamma_1 + \gamma_2)v + \zeta_1). \end{aligned} \quad (26)$$

Similarly,

$$|\mathcal{F}_2(\Psi, \Lambda)(z)| \leq \mu_2 \mathcal{N}_2((\gamma_3 + \gamma_4)v + \zeta_2) \tag{27}$$

from (25), (26) and (27), we have

$$|\mathcal{F}(\Psi, \Lambda)(z)| \leq v. \tag{28}$$

Let  $(\Psi_1, \Lambda_1)(z), (\Psi_2, \Lambda_2)(z) \in \mathcal{Y} \times \mathcal{Y}$  and for any  $z \in [0, 1]$ , we have

$$\begin{aligned} & |\mathcal{F}_1(\Psi_2, \Lambda_2)(z) - \mathcal{F}_1(\Psi_1, \Lambda_1)(z)| \leq |\mathcal{H}(z, \Psi, \Lambda)| \cdot \\ & \cdot \left[ \int_0^z \frac{(z-y)^{\omega_1-1}}{\Gamma(\omega_1)} (|\mathcal{T}_1(y, \Psi_2, \Lambda_2) - \mathcal{T}_1(y, \Psi_1, \Lambda_1(y))|) dy \right. \\ & + \int_0^{\delta_1} \frac{(\delta_1-y)^{\omega_1-1}}{\Gamma(\omega_1)} (|\mathcal{T}_1(y, \Psi_2, \Lambda_2) - \mathcal{T}_1(y, \Psi_1, \Lambda_1(y))|) dy \\ & \left. + z \int_0^1 \frac{(1-y)^{\omega_1-2}}{\Gamma(\omega_1-1)} (|\mathcal{T}_1(y, \Psi_2, \Lambda_2) - \mathcal{T}_1(y, \Psi_1, \Lambda_1(y))|) dy \right] \\ & \leq \mu_1 \left( \frac{1}{\Gamma(\omega_1+1)} + \frac{\delta_1^{\omega_1}}{\Gamma(\omega_1+1)} + \frac{1}{\Gamma(\omega)} \right) (\gamma_1 |\Psi_2 - \Psi_1| + \gamma_2 |\Lambda_2 - \Lambda_1|) \\ & \leq \mu_1 \mathcal{N}_1(\gamma_1 + \gamma_2)(|\Psi_2 - \Psi_1| + |\Lambda_2 - \Lambda_1|). \end{aligned} \tag{29}$$

Similarly,

$$|\mathcal{F}_2(\Psi_2, \Lambda_2)(z) - \mathcal{F}_2(\Psi_1, \Lambda_1)(z)| \leq \mu_2 \mathcal{N}_2(\gamma_3 + \gamma_4)(|\Psi_2 - \Psi_1| + |\Lambda_2 - \Lambda_1|) \tag{30}$$

therefore from (29), (31) we can write

$$\|\mathcal{F}(\Psi_2, \Lambda_2)(z) - \mathcal{F}(\Psi_1, \Lambda_1)(z)\| \leq (\mu_1 \mathcal{N}_1(\gamma_1 + \gamma_2) + \mu_2 \mathcal{N}_2(\gamma_3 + \gamma_4))(|\Psi_2 - \Psi_1| + |\Lambda_2 - \Lambda_1|) \tag{31}$$

Thus the function  $\mathcal{F}$  is a contraction mapping and, consequently,  $\mathcal{F}$  fixes a point which is the UMS of the coupled system of HFDEs (1). □

### 3 Applications of our results

**Example 3.1.** In this example, we utilize Theorem 2.1, for the EMS of the following Coupled system of HFDEs,

$$\begin{aligned} D^{3.5} \left( \frac{\Psi}{1 + |\sin \Psi| + |\cos \Lambda|} \right) &= -\frac{z + |\sin \Psi| + |\cos \Lambda|}{40(1 + 2z)}, \\ D^{3.5} \left( \frac{\Omega}{1 + |\sin \Psi| + |\cos \Lambda|} \right) &= -\frac{z + |\sin \Psi| + |\cos \Lambda|}{40(1 + 3z)}, \\ \left( \frac{\Psi}{\mathcal{H}(z, \Psi, \Lambda)} \Big|_{z=0} \right)^{(i)} &= 0 = \left( \frac{\Psi}{\mathcal{H}(z, \Psi, \Lambda)} \Big|_{z=1} \right)^{(1)}, \quad \frac{\Psi}{\mathcal{H}(z, \Psi, \Lambda)} \Big|_{z=0.5} = 0.5 I^{2.5} \mathcal{T}_1(z, \Psi, \Lambda) \Big|_{z=1}, \\ \left( \frac{\Omega}{\mathcal{G}(z, \Psi, \Lambda)} \Big|_{z=0} \right)^{(i)} &= 0 = \left( \frac{\Omega}{\mathcal{G}(z, \Psi, \Lambda)} \Big|_{z=1} \right)^{(1)}, \quad \frac{\Omega}{\mathcal{G}(z, \Psi, \Lambda)} \Big|_{z=0.5} = 0.5 I^{2.5} \mathcal{T}_2(z, \Psi, \Lambda) \Big|_{z=1}. \end{aligned} \tag{32}$$

In this system of HFDEs we have  $\omega_2 = \omega_1 = 3.5, \delta_1 = \delta_2 = 0.5, \mathcal{H} = \mathcal{G} = 1 + |\sin \Psi| + |\cos \Lambda| \leq 3 = \mu_1 = \mu_2, \mathcal{H}, \mathcal{G} \in \mathbb{R} - \{0\}$  also  $|\mathcal{T}_1| \leq \frac{1}{40} + \frac{|\Psi|}{40} + \frac{|\Lambda|}{40}$  and  $|\mathcal{T}_2| \leq \frac{1}{40} + \frac{|\Psi|}{40} + \frac{|\Lambda|}{40}$  thus  $\lambda_i = \eta_i = \xi_i = \frac{1}{40}$  for  $i = 1, 2$  and  $\mu_1 \mathcal{N}_1(\lambda_1 + \eta_1 |\Psi| + \xi_1 |\Lambda|) + \mu_2 \mathcal{N}_2(\lambda_2 + \eta_2 |\Psi| + \xi_2 |\Lambda|) < +\infty$  for any  $\Psi, \Lambda \in C([0, 1], \mathbb{R})$ . Therefore all the conditions of Theorem 2.1 are satisfied and so the problem (32) has a solution.

**Example 3.2.** In this example, we utilize Theorem 2.2, for the UMS of the following Coupled system of HFDEs,

$$\begin{aligned}
 D^{3.5} \left( \frac{\Psi}{1 + |\sin \Psi| + |\cos \Lambda|} \right) &= -\frac{z + |\sin \Psi| + |\cos \Lambda|}{35(1 + 2z)}, \\
 D^{3.5} \left( \frac{\Omega}{1 + |\sin \Psi| + |\cos \Lambda|} \right) &= -\frac{z + |\sin \Psi| + |\cos \Lambda|}{35(1 + 3z)}, \\
 \left( \frac{\Psi}{\mathcal{H}(z, \Psi, \Lambda)} \Big|_{z=0} \right)^{(i)} &= 0 = \left( \frac{\Psi}{\mathcal{H}(z, \Psi, \Lambda)} \Big|_{z=1} \right)^{(1)}, \quad \frac{\Psi}{\mathcal{H}(z, \Psi, \Lambda)} \Big|_{z=0.5} = 0.5I^{2.5} \mathcal{T}_1(z, \Psi, \Lambda) \Big|_{z=1}, \\
 \left( \frac{\Omega}{\mathcal{G}(z, \Psi, \Lambda)} \Big|_{z=0} \right)^{(i)} &= 0 = \left( \frac{\Omega}{\mathcal{G}(z, \Psi, \Lambda)} \Big|_{z=1} \right)^{(1)}, \quad \frac{\Omega}{\mathcal{G}(z, \Psi, \Lambda)} \Big|_{z=0.5} = 0.5I^{2.5} \mathcal{T}_2(z, \Psi, \Lambda) \Big|_{z=1}.
 \end{aligned} \tag{33}$$

In this system of HFDEs (33), we have  $\omega_1 = \omega_2 = 3.5$ ,  $\delta_1 = \delta_2 = 0.5$ ,  $\mathcal{T}_1(z, 0, 0) \leq \frac{1}{35}$ ,  $\mathcal{T}_2(z, 0, 0) \leq \frac{1}{35}$  and

$$\begin{aligned}
 |\mathcal{T}_1(z, \Psi, \Lambda) - \mathcal{T}_1(z, \Psi_1, \Lambda_1)| &\leq \frac{1}{35} (|\Psi - \Psi_1| + |\Lambda - \Lambda_1|), \\
 |\mathcal{T}_2(z, \Psi, \Lambda) - \mathcal{T}_2(z, \Psi_1, \Lambda_1)| &\leq \frac{1}{35} (|\Psi - \Psi_1| + |\Lambda - \Lambda_1|)
 \end{aligned} \tag{34}$$

From (34), we have  $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = \frac{1}{35}$ ,  $\mathcal{N}_1 = \mathcal{N}_2 = 0.394472$ ,  $\zeta_1 = \zeta_2 = 1$ , ultimately, we get  $v \geq 0.0782001$ , thus, the condition  $(A_1)$  is satisfied and  $\mu_1 \mathcal{N}_1 (\gamma_1 + \gamma_2) + \mu_2 \mathcal{N}_2 (\gamma_3 + \gamma_4) = 0.135248 < 0.2$ . Therefore by Theorem 2.2, there exist a unique solution for the coupled system of HFDEs (32) in  $\tilde{S}_{0.1}(0)$ .

## 4 Conclusion

In this paper, the study of EUMS for a coupled system of HFDEs (1) has been considered. For this purpose we utilized some classical results, Leray–Schauder Alternative and Banach Contraction Principle. We obtained results for the existence of solution in Theorem 2.1 and the uniqueness of solution in Theorem 2.2. For the applications of the results, we have given two examples. In the future, this type of work can get the attention of researchers in q-difference equations,  $p$ -Laplacian fractional BVP and many others.

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