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Research Article

Results for Two-Level Designs with General Minimum Lower-Order Confounding

Zhi Ming Li[1] and Run Chu Zhang[2,3]

[1] *School of Mathematical Sciences, Xinjiang University, Urumqi 830046, China*
[2] *KLAS and School of Mathematics, Northeast Normal University, Changchun 130024, China*
[3] *LPMC and School of Mathematical Sciences, Nankai University, Tianjin 300071, China*

Correspondence should be addressed to Zhi Ming Li; zhimingli525@sina.com

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The general minimum lower-order confounding (GMC) criterion for two-level design not only reveals the confounding information of factor effects but also provides a good way to select the optimal design, which was proposed by Zhang et al. (2008). The criterion is based on the aliased effect-number pattern (AENP). Therefore, it is very important to study properties of AENP for two-level GMC design. According to the ordering of elements in the AENP, the confounding information between lower-order factor effects is more important than that of higher-order effects. For two-level GMC design, this paper mainly shows the interior principles to calculate the leading elements ${}_1^{\#}C_2$ and ${}_2^{\#}C_2$ in the AENP. Further, their mathematical formulations are obtained for every GMC 2^{n-m} design with $N = 2^{n-m}$ according to two cases: (i) $5N/16 + 1 \le n < N/2$ and (ii) $N/2 \le n \le N - 1$.

1. Introduction

To find optimal designs in a more elaborate and explicit manner under effect hierarchy principle, Zhang et al. [1] first introduced the aliased effect-number pattern (AENP) and proposed a new criterion of general minimum lower-order confounding (GMC) for two-level regular design. Further, they proved that all the classification patterns conducting the existing criteria, such as maximum resolution (MR) criterion [2], minimum aberration (MA) criterion [3], clear effects (CE) criterion [4], and maximum estimation capacity (MEC) criterion [5], can be expressed as different functions of the AENP so that it can be a basis to unify these criteria.

Through the AENP, we can get a deeper understanding of properties of the above criteria and relationships among them. Zhang and Cheng [6] revealed an exact expression of the average minimum lower-order confounding property of MA design. Hu and Zhang [7] obtained an essential statistical equivalence of MEC design and MA design. From the average least confounding property between lower-order effects, MA designs are most suitable for the situation that all the factors in experiments are treated to be equally important, while GMC design has an individual least confounding property between lower-order effects and possesses the maximum numbers of clear main effects and clear two-factor interactions (2fi's). Because of this, GMC designs can be applied to the experiments which the experimenters have some prior information to the order of the importance factors. In practice, the latter situation more often happens than the former one. Therefore, the study for GMC designs should be significantly important in both theory and application.

Now we review some definitions proposed by Zhang et al. [1]. Let D be a 2^{n-m} design with n factors, m independent defining words, and $N = 2^{n-m}$ runs. We denote the factors by $1, 2, \ldots, n$. An ith-order factor effect is said to be aliased with jth-order factor effects at degree k if it is simultaneously aliased with k jth-order factor effects. The 0th-order effect is the grand mean and 1st-order effect is a main effect.

Let ${}_i^{\#}C_j^{(k)}(D)$ (written by ${}_i^{\#}C_j^{(k)}$ for short) be the number of ith-order factor effects that are aliased with k jth-order factor effects. Denote $K_j = \binom{n}{j}$; a set $\{{}_i^{\#}C_j^{(k)}, 0 \le k \le K_j,$

$0 \leq i, j \leq n\}$ is called *the aliased effect-number pattern (AENP)* of the design D. The set reflects the overall confounding between factor effects in the design. Define ${}_i^{\#}C_j = ({}_i^{\#}C_j^{(0)}, {}_i^{\#}C_j^{(1)}, \ldots, {}_i^{\#}C_j^{(K_j)})$ and a design that sequentially maximizes the vector

$$ {}^{\#}C = \left({}_1^{\#}C_2, {}_2^{\#}C_2, {}_1^{\#}C_3, {}_2^{\#}C_3, {}_3^{\#}C_2, {}_3^{\#}C_3, \ldots \right) \tag{1} $$

is called *a GMC design*, where the ordering of ${}_i^{\#}C_j$'s is in accordance with the rule: ${}_i^{\#}C_j$ *is before* ${}_u^{\#}C_v$ *if either* $\max(i, j) < \max(u, v)$, *or* $\max(i, j) = \max(u, v)$, *with* $i < u$, *or* $\max(i, j) = \max(u, v)$ *with* $i = u$ *and* $j < v$. In order to make main effects or 2fi's estimable, we need to give an assumption: *the interactions involving three or more factors are absent*. Thus, we only study the leading terms ${}_1^{\#}C_2$ and ${}_2^{\#}C_2$ of AENP for two-level GMC design in this paper.

Zhang et al. [1] listed all two-level GMC designs of 16 and 32 runs, a number of 64-run GMC designs, and obtained the values of ${}_1^{\#}C_2$ and ${}_2^{\#}C_2$ by computer algorithm. However, the method is not suitable for designs with larger runs. Zhang and Cheng [6] and Chen and Liu [8] provided an important theory for constructing GMC designs. Cheng and Zhang [9] and Li et al. [10] finished the construction of GMC 2^{n-m} designs with $N/4 + 1 \leq n \leq N - 1$. However, there are few articles that pay attention to calculating the values of elements in the AENP, especially, the confounding information between main effects and 2fi's, or among 2fi's of two-level GMC design.

This paper mainly reveals the interior principles for calculating the values of ${}_1^{\#}C_2$ and ${}_2^{\#}C_2$ for two-level GMC design. In Section 2, we introduce some notations and obtain useful lemmas to study the lower-order confounding information of two-level GMC designs. Section 3 and Section 4, respectively, obtain values of ${}_1^{\#}C_2$ and ${}_2^{\#}C_2$ for GMC 2^{n-m} design with resolution $R \geq III$, for $5N/16 + 1 \leq n < N/2$ and $N/2 \leq n \leq N - 1$. Concluding remarks are given in Section 5.

2. Some Notations and Lemmas

Denote $q = n - m$ and $1, 2, \ldots, q$ stand for q independent factors. Let H_q be the set containing all main effects $1, 2, \ldots, q$ and all interactions among them, formed by

$$ H_1 = \{1\}, \qquad H_q = \left\{H_{q-1}, q, qH_{q-1}\right\}, \tag{2} $$

where $qH_{q-1} = \{qd: d \in H_{q-1}\}$. By Theorem 2.7.1 of Mukerjee and Wu [11], any 2^{n-m} design D can be represented by an n-subset of H_q; that is, $D \subset H_q$.

Let $T_1 = H_1$ and $T_r = \{r, rH_{r-1}\}$ for $1 < r \leq q$. Evidently, $H_q = \cup_{r=1}^{q} T_r$. For $5N/16 + 1 \leq n \leq N - 1$, Li et al. [10] have gotten that every GMC 2^{n-m} design is constructed by the last n columns of H_q. Therefore, GMC 2^{n-m} designs with $5N/16 + 1 \leq n < N/2$ are directly formed by the last n columns of T_q. Denote $S_{qr} = H_q \setminus H_r$ with $1 \leq r < q$. For $N/2 \leq n \leq N - 1$, there exists a number $r\ (< q)$ so that GMC 2^{n-m} design is formed by the last n columns of $T_r \cup S_{qr}$. Thus, the GMC design can be written by $D_0 \cup S_{qr}$, where D_0 consists of the last $n - (N - 2^r)$ columns of T_r. To get the lower-order confounding information of two-level GMC design, we need to study structure of last n_0 columns of T_r for $r \leq q$ and $n_0 \leq n$.

Suppose D_0 consists of the last n_0 columns of T_r $(r \leq q)$, where $n_0 = \#\{D_0\}$ and $\#A$ denotes the cardinality of a set A. The following example illustrates the structure of D_0.

Example 1. Consider $r = 7$; we select the last n_0 columns of T_7 to construct D_0. Clearly, there are 64 choices besides $D_0 \equiv T_r$. For $1 \leq n_0 \leq 63$, D_0 is one of the following six forms.

(i) $(u + 1) \cdots 7H_u$ for $1 \leq u < 7$.

(ii) $(u + 1) \cdots 7(H_u \setminus H_v)$ for $1 \leq v < u < 7$.

(iii) $(u+1) \cdots 7((s+1) \cdots vH_s \cup (H_u \setminus H_v))$ for $1 \leq s < v < u < 7$.

(iv) $(u + 1) \cdots 7((s + 1) \cdots v(H_s \setminus H_t) \cup (H_u \setminus H_v))$ for $1 \leq t < s < v < u < 7$.

(v) $(u + 1) \cdots 7((s + 1) \cdots v((w + 1) \cdots tH_w \cup (H_s \setminus H_t)) \cup (H_u \setminus H_v))$ for $1 \leq w < t < s < v < u < 7$.

(vi) $(u + 1) \cdots 7((s + 1) \cdots v((w + 1) \cdots t(H_w \setminus H_z) \cup (H_s \setminus H_t)) \cup (H_u \setminus H_v))$ for $1 \leq z < w < t < s < v < u < 7$.

The above example provides a way to construct D_0. Generally, for any r $(r \leq q)$, we consider the construction of D_0 in T_r. Define

$$ D_t = H_{i_t} \setminus H_{j_t}, \qquad a_{i_t} = (i_t + 1)(i_t + 2) \cdots j_{t-1}, \quad 1 \leq t \leq m, \tag{3} $$

where $1 \leq j_m < i_m < j_{m-1} < i_{m-1} < \cdots < j_t < i_t < \cdots < j_1 < i_1 < j_0 = r$. Then, D_0 can be constructed by either of the following cases.

Case 1. One has $D_0 = a_{i_1}(a_{i_2}(\cdots(a_{i_{m-1}}(a_{i_m}D_m \cup D_{m-1}) \cdots) \cup D_2) \cup D_1)$.

Case 2. One has $D_0 = a_{i_1}(a_{i_2}(\cdots(a_{i_{m-1}}(a_{i_m}H_{i_m} \cup D_{m-1}) \cdots) \cup D_2) \cup D_1)$.

In Case 1, the number of elements in D_0 is even since $\#\{D_0\} = \sum_{t=1}^{m}(2^{i_k} - 2^{j_k})$. However, that of D_0 in Case 2 is odd because of $\#\{D_0\} = \sum_{t=1}^{m-1}(2^{i_k} - 2^{j_k}) + 2^{i_m} - 1$.

Consider $D \subset H_q$ and any $\gamma \in H_q$; define

$$ B_2(D, \gamma) = \#\{(d_1, d_2): d_1, d_2 \in D, d_1 d_2 = \gamma\}, \tag{4} $$

which is the number of 2fi's in D aliased with γ. By the definition of ${}_i^{\#}C_j^{(k)}(D)$, it can be easily obtained that

$$ {}_1^{\#}C_2^{(k)}(D) = \#\{\gamma: \gamma \in D, B_2(D, \gamma) = k\}, \tag{5} $$

$$ {}_2^{\#}C_2^{(k)}(D) = (k + 1)\#\left\{\gamma: \gamma \in H_q, B_2(D, \gamma) = k + 1\right\}, \tag{6} $$

where $k = 0, 1, \ldots, K_2$. In order to get the lower-order confounding of D_0 in the above cases, we need to study $B_2(D_t, \gamma)$ for $t \geq 1$.

Lemma 2. *Let D_t be defined in (3) for $t \geq 1$. Then*

$$B_2(D_t, \gamma) = \begin{cases} 2^{i_t-1} - 2^{j_t-1}, & \gamma \in H_{j_t}, \\ 2^{i_t-1} - 2^{j_t}, & \gamma \in H_{i_t} \setminus H_{j_t}, \\ 0, & \gamma \in H_q \setminus H_{i_t}. \end{cases} \quad (7)$$

Proof. For $\gamma \in H_q \setminus H_{i_t}$, we have $B_2(D_t, \gamma) = 0$. If $\gamma \in H_{j_t}$, then

$$B_2(D, \gamma) = \frac{\#\{H_{i_t} \setminus H_{j_t}\}}{2} = 2^{i_t-1} - 2^{j_t-1}. \quad (8)$$

For $\gamma \in H_{i_t} \setminus H_{j_t}$, there are $2^{i_t-1} - 1$ pairs of factors in H_{i_t} so that their interactions are aliased with γ. Among these pairs, there are $2^{j_t} - 1$ pairs with one factor from H_{j_t} and another from $H_{i_t} \setminus H_{j_t}$. Thus,

$$B_2(D_t, \gamma) = (2^{i_t-1} - 1) - (2^{j_t} - 1) = 2^{i_t-1} - 2^{j_t}. \quad (9)$$

This completes the proof. □

Next we analyze Case 1 of D_0. For convenience, by (3), denote

$$\mathscr{D}(t) = a_{i_t}\left(\cdots\left(a_{i_{m-1}}\left(a_{i_m} D_m \cup D_{m-1}\right)\cdots\right) \cup D_t\right) \quad (10)$$

for $1 < t \leq m$. Evidently, $\mathscr{D}(t) \subset H_{j_{t-1}}$ and $\mathscr{D}(1) = D_0$ in Case 1. When $d_1 \in \mathscr{D}(t)$ and $d_2 \in D_{t-1}$, we have $d_1 d_2 \in D_{t-1}$. Thus,

$$\#\{(d_1, d_2) : d_1 \in \mathscr{D}(t), d_2 \in D_{t-1}, d_1 d_2 = \gamma\} = \sum_{l=t}^{m} \#\{D_l\} \quad (11)$$

for $\gamma \in D_{t-1}$. Otherwise, the value is zero. Then

$$\#\{(d_1, d_2) : d_1 \in \mathscr{D}(t), d_2 \in D_{t-1}, d_1 d_2 = \gamma\} = \begin{cases} \sum_{l=t}^{m} (2^{i_l} - 2^{j_l}), & \gamma \in D_{t-1}, \\ 0, & \gamma \notin D_{t-1}. \end{cases} \quad (12)$$

Based on Lemma 2 and (12), we can get the following result for Case 1.

Lemma 3. *Let $D_0 = a_{i_1}(a_{i_2}(\cdots(a_{i_{m-1}}(a_{i_m} D_m \cup D_{m-1})\cdots) \cup D_2) \cup D_1)$. Then*

$$B_2(D_0, \gamma) = \begin{cases} \frac{c(m+1)}{2}, & \gamma \in H_{j_m}, \\ c(m+1) - \frac{(c(t) + 2^{i_t})}{2}, & \gamma \in D_t,\ t = 1, \ldots, m, \\ \frac{c(t)}{2}, & \gamma \in H_{j_{t-1}} \setminus H_{i_t},\ t = 2, \ldots, m, \\ 0, & \gamma \in H_q \setminus H_{i_1}, \end{cases} \quad (13)$$

where

$$c(1) = 0, \quad c(t) = \sum_{l=1}^{t-1} (2^{i_l} - 2^{j_l}), \quad t > 1. \quad (14)$$

Proof. For $1 < l \leq m$, by (10), we have

$$\begin{aligned} B_2(a_{i_l}(\mathscr{D}(l) \cup D_{l-1}), \gamma) &= B_2(\mathscr{D}(l) \cup D_{l-1}, \gamma) \\ &= B_2(\mathscr{D}(l), \gamma) + B_2(D_{l-1}, \gamma) \\ &\quad + \#\{(d_1, d_2) : d_1 \in \mathscr{D}(l), d_2 \in D_{l-1}, d_1 d_2 = \gamma\}. \end{aligned} \quad (15)$$

Hence,

$$\begin{aligned} B_2(D_0, \gamma) &= \sum_{l=1}^{m-2} B_2(D_l, \gamma) + B_2(a_{i_m} D_m \cup D_{m-1}, \gamma) \\ &\quad + \sum_{l=2}^{m-1} \#\{(d_1, d_2) : d_1 \in \mathscr{D}(l), d_2 \in D_{l-1}, d_1 d_2 = \gamma\} \\ &= \sum_{l=1}^{m} B_2(D_l, \gamma) \\ &\quad + \sum_{l=2}^{m} \#\{(d_1, d_2) : d_1 \in \mathscr{D}(l), d_2 \in D_{l-1}, d_1 d_2 = \gamma\}. \end{aligned} \quad (16)$$

Put H_r into $2m + 1$ incompatible parts: H_{j_m}, D_{l+1}, and $H_{j_l} \setminus H_{i_{l+1}}$ for $l = 0, 1, \ldots, m-1$. Clearly, if $\gamma \in H_q \setminus H_{i_1}$, then $B_2(D_0, \gamma) = 0$. By Lemma 2 and (12), we, respectively, discuss the following cases.

(i) If $\gamma \in H_{j_m}$, then

$$\#\{(d_1, d_2) : d_1 \in \mathscr{D}(l), d_2 \in D_{l-1}, d_1 d_2 = \gamma\} = 0 \quad (17)$$

for $1 < l \leq m - 1$. Thus,

$$B_2(D_0, \gamma) = \sum_{l=1}^{m} B_2(D_t, \gamma) = \sum_{l=1}^{m} (2^{i_l-1} - 2^{j_l-1}). \quad (18)$$

(ii) If $\gamma \in D_t$ with $1 < t \leq m$, one has

$$\begin{aligned} B_2(D_0, \gamma) &= \sum_{l=1}^{t-1} B_2(D_l, \gamma) + B_2(D_t, \gamma) \\ &\quad + \#\{(d_1, d_2) : d_1 \in \mathscr{D}(t+1), d_2 \in D_t, d_1 d_2 = \gamma\} \\ &= \sum_{l=1}^{t-1} (2^{i_l-1} - 2^{j_l-1}) + (2^{i_t-1} - 2^{j_t}) \\ &\quad + \sum_{l=t+1}^{m} (2^{i_l} - 2^{j_l}) \\ &= \sum_{l=1}^{m} (2^{i_l} - 2^{j_l}) - \sum_{l=1}^{t-1} (2^{i_l-1} - 2^{j_l-1}) - 2^{i_t-1}. \end{aligned} \quad (19)$$

(iii) If $\gamma \in H_{j_{t-1}} \setminus H_{i_t}$ for $t > 1$, then

$$B_2(D_0, \gamma) = \sum_{l=1}^{t-1} B_2(D_l, \gamma) = \sum_{l=1}^{t-1} \left(2^{i_l - 1} - 2^{j_l - 1}\right). \tag{20}$$

This completes the proof. □

Lemma 3 shows that the value of $B_2(D_0, \gamma)$ in Case 1 depends on all pairs $\{i_t, j_t\}_{1 \le t \le m}$ which relate to $\#\{D_0\} = \sum_{t=1}^{m}(2^{i_t} - 2^{j_t})$. For instance, take $n_0 = \#\{D_0\} = 42$ that is nearer to the number 2^5 than 2^6; we have

$$n_0 = 2^5 + 2^3 + 2 = \left(2^6 - 2^5\right) + \left(2^4 - 2^3\right) + \left(2^2 - 2\right). \tag{21}$$

Thus $i_1 = 6$, $j_1 = 5$, $i_2 = 4$, $j_2 = 3$, $i_3 = 2$, and $j_3 = 1$. And take $n_0 = 54$ which is closer to the number 2^6 than 2^5; one obtains $n_0 = 2^6 - 2^4 + 6 = 2^6 - 2^4 + 2^3 - 2$. Then $i_1 = 6$, $j_1 = 4$, $i_2 = 3$, and $j_2 = 1$.

Consider Case 2 of D_0. Denote

$$\mathscr{H}(t) = a_{i_t}\left(\cdots\left(a_{i_{m-1}}\left(a_{i_m} H_{i_m} \cup D_{m-1}\right)\cdots\right) \cup D_t\right) \tag{22}$$

for $1 < t \le m$. Clearly, $\mathscr{H}(t) \subset H_{j_{t-1}}$ and $\mathscr{H}(1) = D_0$ in Case 2. For two factors $d_1 \in \mathscr{H}(t)$ and $d_2 \in D_{t-1}$, one has $d_1 d_2 \in D_{t-1}$. Therefore,

$$\#\left\{(d_1, d_2) : d_1 \in \mathscr{H}(t),\ d_2 \in D_{t-1},\ d_1 d_2 = \gamma\right\} = \begin{cases} \displaystyle\sum_{l=t}^{m-1}\left(2^{i_l} - 2^{j_l}\right) + 2^{i_m} - 1, & \gamma \in D_{t-1}, \\ 0, & \gamma \notin D_{t-1}. \end{cases} \tag{23}$$

Specifically, if $t = m$, then

$$\#\left\{(d_1, d_2) : d_1 \in a_{i_m} H_{i_m},\ d_2 \in D_{m-1},\ d_1 d_2 = \gamma\right\} = \begin{cases} 2^{i_m} - 1, & \gamma \in D_{m-1}, \\ 0, & \gamma \notin D_{m-1}. \end{cases} \tag{24}$$

For $m \ge 1$ and $\gamma \in H_{i_m}$, there are $2^{i_m - 1} - 1$ pairs of factors in H_{i_m}, which each interaction is aliased with γ. Then

$$B_2\left(a_{i_m} H_{i_m}, \gamma\right) = B_2\left(H_{i_m}, \gamma\right) = \begin{cases} 2^{i_m - 1} - 1, & \gamma \in H_{i_m}, \\ 0, & \gamma \in H_q \setminus H_{i_m}. \end{cases} \tag{25}$$

Based on the above results, we can obtain the value of $B_2(D_0, \gamma)$ for any $\gamma \in H_q$ in Case 2.

Lemma 4. *Let* $D_0 = a_{i_1}(a_{i_2}(\cdots(a_{i_{m-1}}(a_{i_m} H_{i_m} \cup D_{m-1})\cdots) \cup D_2) \cup D_1)$. *Then*

$$B_2(D_0, \gamma) = \begin{cases} \dfrac{c(m)}{2} + 2^{i_m - 1} - 1, & \gamma \in H_{i_m}, \\ c(m) - \dfrac{\left(c(t) + 2^{i_t}\right)}{2} + 2^{i_m} - 1, & \gamma \in D_t,\ t = 1, \ldots, m-1, \\ \dfrac{c(t)}{2}, & \gamma \in H_{j_{t-1}} \setminus H_{i_t},\ t = 2, \ldots, m, \\ 0, & \gamma \in H_q \setminus H_{i_1}, \end{cases} \tag{26}$$

where $c(t)$ *is defined in (14).*

Proof. By (22), we obtain

$$\begin{aligned} B_2(D_0, \gamma) &= \sum_{l=1}^{m-1} B_2(D_l, \gamma) + B_2\left(a_{i_m} H_{i_m}, \gamma\right) \\ &\quad + \#\left\{(d_1, d_2) : d_1 \in a_{i_m} H_{i_m},\ d_2 \in D_{m-1},\ d_1 d_2 = \gamma\right\} \\ &\quad + \sum_{l=2}^{m-1} \#\left\{(d_1, d_2) : d_1 \in \mathscr{H}(l),\ d_2 \in D_{l-1},\ d_1 d_2 = \gamma\right\}. \end{aligned} \tag{27}$$

For $\gamma \in H_q \setminus H_{i_1}$, we have $B_2(D_0, \gamma) = 0$. By Lemma 2, (25), and (23), analyze the following cases.

(i) For $\gamma \in H_{i_m}$, we obtain

$$\begin{aligned} B_2(D_0, \gamma) &= \sum_{l=1}^{m-1} B_2(D_l, \gamma) + B_2\left(a_{i_m} H_{i_m}, \gamma\right) \\ &= \sum_{l=1}^{m-1}\left(2^{i_l - 1} - 2^{j_l - 1}\right) + 2^{i_m - 1} - 1. \end{aligned} \tag{28}$$

(ii) For $\gamma \in D_t$ with $1 \le t \le m-1$, one has

$$\begin{aligned} B_2(D_0, \gamma) &= \sum_{l=1}^{t-1} B_2(D_l, \gamma) + B_2(D_t, \gamma) \\ &\quad + \#\left\{(d_1, d_2) : d_1 \in \mathscr{H}(t+1),\ d_2 \in D_t,\ d_1 d_2 = \gamma\right\} \\ &= \sum_{l=1}^{t-1}\left(2^{i_l - 1} - 2^{j_l - 1}\right) \\ &\quad + \sum_{l=t}^{m-1}\left(2^{i_l} - 2^{j_l}\right) - 2^{i_t - 1} + 2^{i_m} - 1 \\ &= c(m) - \frac{\left(c(t) + 2^{i_t}\right)}{2} + 2^{i_m} - 1. \end{aligned} \tag{29}$$

(iii) For $\gamma \in H_{j_{t-1}} \setminus H_{i_t}$ with $2 \le t \le m$, $B_2(D_0, \gamma) = \sum_{l=1}^{t-1}(2^{i_l - 1} - 2^{j_l - 1})$.

□

In Lemma 4, the value of $B_2(D_0, \gamma)$ is relative to these pairs $\{i_t, j_t\}_{1 < t < m}$ and i_m. For example, consider $n_0 = \#\{D_0\} = 21$. Since $n_0 = (2^5 - 2^4) + (2^3 - 2^2) + 2 - 1$, it yields $i_1 = 5$, $j_1 = 4$, $i_2 = 3$, $j_2 = 2$, and $i_3 = 1$. Taking $n_0 = 29$, we have $n_0 = 2^5 - 3 = 2^5 - 2^2 + 1$; thus $i_1 = 5$, $j_1 = 2$, and $i_2 = 1$.

Lemmas 3 and 4, respectively, obtain the value of $B_2(D_0, \gamma)$ that D_0 consists of the last n_0 columns of T_r $(r \le q)$ for two cases. These results play a key role in calculating ${}_1^{\#}C_2$ and ${}_2^{\#}C_2$'s for all GMC 2^{n-m} designs with $5N/16 + 1 \le n \le N - 1$. Next sections will, respectively, discuss two-level GMC designs with the factor number n satisfying (i) $5N/16 + 1 \le n < N/2$ or (ii) $N/2 \le n \le N - 1$.

3. GMC 2^{n-m} Designs with $5N/16+1 \le n<N/2$

Li et al. [10] showed all GMC 2^{n-m} designs with $5N/16+1 \le n < N/2$, constructed by the last n columns of T_q. In Section 2, D_0 is constructed by Case 1 or Case 2, which is the last n_0 columns of T_r $(r \le q)$ for $n_0 = \#\{D_0\}$. Therefore, for any GMC 2^{n-m} design D with $5N/16+1 \le n < N/2$, its construction is similar to that of D_0. In (3), take $j_0 = r = q$.

Theorem 5. *Consider GMC 2^{n-m} design*

$$D = a_{i_1}\left(a_{i_2}\left(\cdots\left(a_{i_{m-1}}\left(a_{i_m}D_m \cup D_{m-1}\right)\cdots\right)\cup D_2\right)\cup D_1\right) \tag{30}$$

with $5N/16+1 \le n < N/2$. Then

(a)

$$ {}_1^{\#}C_2^{(k)}(D) = \begin{cases} n, & k=0, \\ 0, & otherwise, \end{cases} \tag{31}$$

(b)

$$ {}_2^{\#}C_2^{(k)}(D) = \begin{cases} \dfrac{n\left(2^{j_m}-1\right)}{2}, & k = \dfrac{n}{2}-1, \\ \left(n - \dfrac{\left(c(t)+2^{i_t}\right)}{2}\right)\left(2^{i_t}-2^{j_t}\right), & k = n - \dfrac{\left(c(t)+2^{i_t}\right)}{2}-1, \\ & t=1,\dots,m, \\ \dfrac{c(t)\left(2^{j_{t-1}}-2^{i_t}\right)}{2}, & k=\dfrac{c(t)}{2}-1,\ t=2,\dots,m, \\ 0, & otherwise, \end{cases} \tag{32}$$

where $c(t)$ is defined in (14).

Proof. Evidently, $n = \sum_{l=1}^{m}(2^{i_l}-2^{j_l})$; we have $c(m+1)=n$. By Lemma 3,

$$B_2(D,\gamma) = \begin{cases} \dfrac{n}{2}, & \gamma \in H_{i_m}, \\ n - \dfrac{\left(c(t)+2^{i_t}\right)}{2}, & \gamma \in D_t,\ t=1,\dots,m, \\ \dfrac{c(t)}{2}, & \gamma \in H_{j_{t-1}} \setminus H_{i_t},\ t=2,\dots,m, \\ 0, & \gamma \in H_q \setminus H_{i_1}. \end{cases} \tag{33}$$

(a) Since $D \subset H_q \setminus H_{i_1}$, hence by (33) and (5)

$$ {}_1^{\#}C_2^{(0)}(D) = \#\{\gamma: \gamma \in D, B_2(D,\gamma)=0\} = \#\{D\} = n. \tag{34}$$

Otherwise, ${}_1^{\#}C_2^{(k)}(D) = 0$ for $k \ne 0$.

(b) Following (33) and (6), we obtain

$$\begin{aligned} {}_2^{\#}C_2^{(k)}(D) = {} & (k+1) \\ & \times \Big[\#\left\{\gamma: \gamma \in H_{j_m}, B_2(D,\gamma)=k+1\right\} \\ & + \sum_{t=1}^{m}\#\left\{\gamma: \gamma \in D_t, B_2(D,\gamma)=k+1\right\} \\ & + \sum_{t=2}^{m}\#\left\{\gamma: \gamma \in H_{j_{t-1}} \setminus H_{i_t}, B_2(D,\gamma)=k+1\right\} \\ & + \#\left\{\gamma: \gamma \in H_q \setminus H_{i_1}, B_2(D,\gamma)=k+1\right\}\Big]. \end{aligned} \tag{35}$$

If $k = n/2-1$, then

$$ {}_2^{\#}C_2^{(k)}(D) = \frac{n\#\left\{H_{j_m}\right\}}{2} = \frac{n\left(2^{j_m}-1\right)}{2}. \tag{36}$$

For $k = n-(c(t)+2^{i_t})/2-1$ with $1 \le t \le m$, one has

$$\begin{aligned} {}_2^{\#}C_2^{(k)}(D) &= \left(n - \frac{\left(c(t)+2^{i_t}\right)}{2}\right)\#\{D_t\} \\ &= \left(n - \frac{\left(c(t)+2^{i_t}\right)}{2}\right)\left(2^{i_t}-2^{j_t}\right). \end{aligned} \tag{37}$$

And if $k = c(t)/2-1$ with $1 < t \le m$, then

$$ {}_2^{\#}C_2^{(k)}(D) = \frac{c(t)\#\left\{H_{j_{t-1}} \setminus H_{i_t}\right\}}{2} = \frac{c(t)\left(2^{j_{t-1}}-2^{i_t}\right)}{2}. \tag{38}$$

Otherwise, ${}_2^{\#}C_2^{(k)}(D) = 0$. □

For GMC 2^{n-m} design with $5N/16+1 \le n < N/2$, Theorem 5 reveals that the value of ${}_1^{\#}C_2$ only depends on the factor number n. However, the value of ${}_2^{\#}C_2$ is related to the numbers $\{i_t, j_t\}_{1\le t\le m}$ besides n. We illustrate them via a simple example.

Example 6. Take $q=5$ and $n=10$; consider GMC 2^{10-5} design D. Since $n = 2^3+2 = (2^4-2^3)+(2^2-2)$, clearly, we have $i_1=4$, $j_1=3$, $i_2=2$, and $j_2=1$. Hence, $2^{i_1}-2^{j_1}=8$, $2^{i_2}-2^{j_2}=2$, $2^{j_1}-2^{i_2}=4$, and $c(1)=0$, $c(2)=8$. By Theorem 5, we get

$$\begin{aligned} {}_1^{\#}C_2^{(k)}(D) &= \begin{cases} 10, & k=0, \\ 0, & \text{otherwise}, \end{cases} \\ {}_2^{\#}C_2^{(k)}(D) &= \begin{cases} 16, & k=1, \\ 24, & k=3, \\ 5, & k=4, \\ 0, & \text{otherwise}. \end{cases} \end{aligned} \tag{39}$$

Theorem 5 applies to the case that the factor number n of GMC design is even. If n is odd, similar to the proof of Theorem 5, by Lemma 4, one can get the result below.

Theorem 7. *Consider GMC 2^{n-m} design*

$$D = a_{i_1}\left(a_{i_2}\left(\cdots\left(a_{i_{m-1}}\left(a_{i_m}H_{i_m} \cup D_{m-1}\right)\cdots\right)\cup D_2\right)\cup D_1\right) \tag{40}$$

with $5N/16+1 \le n < N/2$. Then

(a)

$$ {}_1^{\#}C_2^{(k)}(D) = \begin{cases} n, & k = 0, \\ 0, & \text{otherwise}, \end{cases} \tag{41}$$

(b)

$$ {}_2^{\#}C_2^{(k)}(D) = \begin{cases} \dfrac{(n-1)\left(2^{i_m}-1\right)}{2}, & k = \dfrac{(n-1)}{2}-1, \\ \left(n-\dfrac{\left(c(t)+2^{i_t}\right)}{2}\right)\left(2^{i_t}-2^{j_t}\right), & k = n-\dfrac{\left(c(t)+2^{i_t}\right)}{2}-1, \\ & t = 1,\dots,m-1, \\ \dfrac{c(t)\left(2^{j_{t-1}}-2^{i_t}\right)}{2}, & k = \dfrac{c(t)}{2}-1,\ t = 2,\dots,m, \\ 0, & \text{otherwise}, \end{cases} \tag{42}$$

where $c(t)$ is defined in (14).

Proof. Note that $n = \sum_{l=1}^{m-1}(2^{i_l}-2^{j_l}) + 2^{i_m} - 1 = c(m) + 2^{i_m} - 1$. □

Example 8. Let $q = 5$ and $n = 11$; consider GMC 2^{11-6} design D. Here $n = 2^4 - 2^3 + 2^2 - 1$; we have $i_1 = 4$, $j_1 = 3$, and $i_2 = 2$. Thus, $c(2) = 2^{i_1} - 2^{j_1} = 8$ and $2^{j_1} - 2^{i_2} = 4$. Following Theorem 7, it is directly obtained by

$$ {}_1^{\#}C_2^{(k)}(D) = \begin{cases} 11, & k = 0, \\ 0, & \text{otherwise}, \end{cases} \qquad {}_2^{\#}C_2^{(k)}(D) = \begin{cases} 24, & k = 2, \\ 16, & k = 3, \\ 15, & k = 4, \\ 0, & \text{otherwise}. \end{cases} \tag{43}$$

4. GMC 2^{n-m} Designs with $N/2 \le n \le N-1$

In Section 2, we know that any GMC 2^{n-m} design with $N/2 \le n \le N-1$ is constructed by $D_0 \cup S_{qr}$, where D_0 is the last $n-(N-2^r)$ columns of T_r $(r < q)$. Lemmas 3 and 4 have shown the confounding information of D_0. Next we will study a special design $S_{qr} = H_q \setminus H_r$ $(r < q)$, which consists of the last $N-2^r$ columns of H_q. Since $r < q$, the factor number of the design S_{qr} satisfies $N - 2^r \ge N/2$. Hence, the design S_{qr} has GMC. By Lemma 2, we directly give the value of $B_2(S_{qr}, \gamma)$ as follows:

$$ B_2(S_{qr}, \gamma) = \begin{cases} \dfrac{N}{2} - 2^r, & \gamma \in H_q \setminus H_r, \\ \dfrac{N}{2} - 2^{r-1}, & \gamma \in H_r. \end{cases} \tag{44}$$

Next we discuss the values of ${}_1^{\#}C_2$ and ${}_2^{\#}C_2$ for GMC design S_{qr} with $r < q$.

Theorem 9. *Consider any GMC design $S_{qr} = H_q \backslash H_r$ for $r < q$. Then*

(a)

$$ {}_1^{\#}C_2^{(k)}(S_{qr}) = \begin{cases} N - 2^r, & k = \dfrac{N}{2} - 2^r, \\ 0, & \text{otherwise}, \end{cases} \tag{45}$$

(b)

$$ {}_2^{\#}C_2^{(k)}(S_{qr}) = \begin{cases} \left(\dfrac{N}{2} - 2^r\right)(N - 2^r), & k = \dfrac{N}{2} - 2^r - 1, \\ \left(\dfrac{N}{2} - 2^{r-1}\right)(2^r - 1), & k = \dfrac{N}{2} - 2^{r-1} - 1, \\ 0, & \text{otherwise}. \end{cases} \tag{46}$$

Proof. (a) If $k = N/2 - 2^r$, by (44), then

$$ \begin{aligned} {}_1^{\#}C_2^{(k)}(S_{qr}) &= \#\left\{\gamma : \gamma \in S_{qr}, B_2(S_{qr}, \gamma) = k\right\} \\ &= \#\left\{S_{qr}\right\} = N - 2^r. \end{aligned} \tag{47}$$

Otherwise, ${}_1^{\#}C_2^{(k)}(S_{qr}) = 0$.

(b) For $k \ge 0$, note that

$$ \begin{aligned} {}_2^{\#}C_2^{(k)}(S_{qr}) = {} & (k+1) \\ & \times \left[\#\left\{\gamma : \gamma \in S_{qr}, B_2(S_{qr}, \gamma) = k+1\right\}\right. \\ & \left. + \#\left\{\gamma : \gamma \in H_r, B_2(S_{qr}, \gamma) = k+1\right\}\right]. \end{aligned} \tag{48}$$

If $k = N/2 - 2^r - 1$, thus by (44)

$$ {}_2^{\#}C_2^{(k)}(S_{qr}) = \left(\frac{N}{2} - 2^r\right)\#\left\{S_{qr}\right\} = \left(\frac{N}{2} - 2^r\right)(N - 2^r). \tag{49}$$

Similarly, for $k = N/2 - 2^{r-1} - 1$, we have

$$ {}_2^{\#}C_2^{(k)}(S_{qr}) = \left(\frac{N}{2} - 2^{r-1}\right)\#\left\{H_r\right\} = \left(\frac{N}{2} - 2^{r-1}\right)(2^r - 1). \tag{50}$$

□

For GMC design S_{qr} $(r < q)$, the values of ${}_1^{\#}C_2$ and ${}_2^{\#}C_2$ only rely on two numbers N and r. In particular, if $r = q-1$, then $S_{q(q-1)} = T_q$. By Theorem 9, one has

$$ {}_1^{\#}C_2^{(k)}(T_q) = \begin{cases} \dfrac{N}{2}, & k = 0, \\ 0, & k \ne 0, \end{cases}$$

$$
{}_{2}^{\#}C_{2}^{(k)}\left(T_{q}\right)=\begin{cases}\dfrac{N}{4\,(N/2-1)}, & k=\dfrac{N}{4}-1,\\ 0, & k\neq\dfrac{N}{4}-1.\end{cases} \tag{51}
$$

The next example is used to illustrate this above result.

Example 10. Consider GMC 2^{16-11} design S_{54}. Since $r = 4$ and $N = 32$, one directly gets

$$
\begin{gathered}
{}_{1}^{\#}C_{2}^{(k)}(D)=\begin{cases}16, & k=0,\\ 0, & k\neq 0,\end{cases}\\
{}_{2}^{\#}C_{2}^{(k)}(D)=\begin{cases}120, & k=7,\\ 0, & k\neq 7.\end{cases}
\end{gathered} \tag{52}
$$

On the other hand, every GMC 2^{n-m} design D with $n \geq N/2$ can be constructed by the form $(D \setminus S_{qr}) \cup S_{qr}$, where $D \setminus S_{qr}$ consists of the last $n-(N-2^{r})$ columns of T_r. Then, $D_0 = D \setminus S_{qr}$. Based on Lemma 3 of Li et al. [10], we obtain the relationship of D and D_0 as follows:

$$
B_{2}(D,\gamma)=\begin{cases}n-\dfrac{N}{2}, & \gamma\in H_{q}\setminus H_{r},\\ B_{2}(D_{0},\gamma)+\dfrac{N}{2}-2^{r-1}, & \gamma\in H_{r}.\end{cases} \tag{53}
$$

Therefore, we can get the following result.

Theorem 11. *Consider GMC 2^{n-m} design $D = D_0 \cup S_{qr}$ with $r < q$, where $D_0 = a_{i_1}(a_{i_2}(\cdots(a_{i_{m-1}}(a_{i_m}D_m \cup D_{m-1})\cdots)\cup D_2)\cup D_1)$. Then*

(a)

$$
B_{2}(D,\gamma)=\begin{cases}\dfrac{n}{2}, & \gamma\in H_{j_m},\\ n-\dfrac{\left(a(t)+2^{i_t}\right)}{2}, & \gamma\in D_{t},\ t=1,\dots,m,\\ \dfrac{a(t)}{2}, & \gamma\in H_{j_{t-1}}\setminus H_{i_t},\ t=2,\dots,m,\\ \dfrac{N}{2}-2^{r-1}, & \gamma\in H_{r}\setminus H_{i_1},\\ n-\dfrac{N}{2}, & \gamma\in H_{q}\setminus H_{r},\end{cases} \tag{54}
$$

(b)

$$
{}_{1}^{\#}C_{2}^{(k)}(D)=\begin{cases}N-2^{r}, & k=n-\dfrac{N}{2},\\ n-N+2^{r}, & k=\dfrac{N}{2}-2^{r-1},\\ 0, & \textit{otherwise},\end{cases} \tag{55}
$$

(c)

$$
{}_{2}^{\#}C_{2}^{(k)}(D)
=\begin{cases}\dfrac{n\left(2^{j_m}-1\right)}{2}, & k=\dfrac{n}{2}-1,\\ \left(n-\dfrac{\left(a(t)+2^{i_t}\right)}{2}\right)\left(2^{i_t}-2^{j_t}\right), & k=n-\dfrac{\left(a(t)+2^{i_t}\right)}{2}-1,\\ & t=1,\dots,m,\\ \dfrac{a(t)\left(2^{j_{t-1}}-2^{i_t}\right)}{2}, & k=\dfrac{a(t)}{2}-1,\\ & t=2,\dots,m,\\ \left(\dfrac{N}{2}-2^{r-1}\right)\left(2^{r}-2^{i_1}\right), & k=\dfrac{N}{2}-2^{r-1}-1,\\ \left(n-\dfrac{N}{2}\right)\left(N-2^{r}\right), & k=n-\dfrac{N}{2}-1,\\ 0, & \textit{otherwise},\end{cases} \tag{56}
$$

where $a(t) = N - 2^{r} + c(t)$ and $c(t)$ is defined in (14).

Proof. (a) By (53) and Lemma 3, note that

$$
c(m+1)=\#\{D_{0}\}=\sum_{l=1}^{m}\left(2^{i_l}-2^{j_l}\right)=n-N+2^{r} \tag{57}
$$

yields (a).

(b) For $\gamma \in S_{qr}$, by (a), $B_2(D,\gamma) = n - N/2$. If $k = n - N/2$, then

$$
{}_{1}^{\#}C_{2}^{(k)}(D)=\#\left\{S_{qr}\right\}=N-2^{r}. \tag{58}
$$

Since $D_0 \subset H_r \setminus H_{i_1}$, for $k = N/2 - 2^{r-1}$, we have

$$
\begin{aligned}
{}_{1}^{\#}C_{2}^{(k)}(D)&=\#\{D_{0}\}\\
&=\#\left\{\bigcup_{t=1}^{m}D_{t}\right\}=\sum_{t=1}^{m}\left(2^{i_t}-2^{j_t}\right)=n-N+2^{r}.
\end{aligned} \tag{59}
$$

(c) Since

$$
\begin{aligned}
{}_{2}^{\#}C_{2}^{(k)}(D)=&\ (k+1)\\
&\times\Big[\#\left\{\gamma:\gamma\in H_{j_m},B_{2}(D,\gamma)=k+1\right\}\\
&+\sum_{t=1}^{m}\#\left\{\gamma:\gamma\in D_{l},B_{2}(D,\gamma)=k+1\right\}\\
&+\sum_{t=2}^{m}\#\left\{\gamma:\gamma\in H_{j_{t-1}}\setminus H_{i_k},B_{2}(D,\gamma)=k+1\right\}\\
&+\#\left\{\gamma:\gamma\in H_{r}\setminus H_{i_1},B_{2}(D,\gamma)=k+1\right\}\\
&+\#\left\{\gamma:\gamma\in H_{q}\setminus H_{r},B_{2}(D,\gamma)=k+1\right\}\Big],
\end{aligned} \tag{60}
$$

by (a), the result follows. □

When the factor number n of a GMC design satisfying $N/2 \le n \le N-1$ is even, by Theorem 11, we obtain values of the corresponding ${}_1^{\#}C_2$ and ${}_1^{\#}C_2$. The next example illustrates this point.

Example 12. Let $q = 8$, $r = 7$; consider GMC $2^{154-146}$ design $D = D_0 \cup S_{87}$. Since $n_0 = \#\{D_0\} = 26$ and

$$n_0 = 2^5 - 6 = \left(2^5 - 2^3\right) + \left(2^2 - 2\right), \tag{61}$$

we have $i_1 = 5$, $j_1 = 3$, $i_2 = 2$, and $j_2 = 1$. Thus, $2^{i_1} - 2^{j_1} = 24$, $2^{i_2} - 2^{j_2} = 2$ and $a(1) = N - 2^r + c(1) = 128$, $a(2) = N - 2^r + c(2) = 152$. By (b) and (c) of Theorem 11, one obtains

$$\begin{gathered} {}_1^{\#}C_2^{(k)}(D) = \begin{cases} 128, & k = 26, \\ 26, & k = 64, \\ 0, & \text{otherwise}, \end{cases} \\ {}_2^{\#}C_2^{(k)}(D) = \begin{cases} 3328, & k = 25, \\ 6144, & k = 63, \\ 1776, & k = 73, \\ 456, & k = 75, \\ 77 & k = 76, \\ 0, & \text{otherwise}. \end{cases} \end{gathered} \tag{62}$$

Theorem 13. *Consider GMC 2^{n-m} design $D = D_0 \cup S_{qr}$ with $r < q$, where $D_0 = a_{i_1}(a_{i_2}(\cdots(a_{i_{m-1}}(a_{i_m}H_{i_m} \cup D_{m-1})\cdots)\cup D_2)\cup D_1)$. Then*

(a)

$$B_2(D,\gamma) = \begin{cases} \dfrac{(n-1)}{2}, & \gamma \in H_{i_m}, \\ n - \dfrac{\left(a(t) + 2^{i_t}\right)}{2}, & \gamma \in D_t,\ t = 1,\ldots,m-1, \\ \dfrac{a(t)}{2}, & \gamma \in H_{j_{t-1}} \setminus H_{i_t},\ t = 2,\ldots,m, \\ \dfrac{N}{2} - 2^{r-1}, & \gamma \in H_r \setminus H_{i_1}, \\ n - \dfrac{N}{2}, & \gamma \in H_q \setminus H_r, \end{cases} \tag{63}$$

(b)

$${}_1^{\#}C_2^{(k)}(D) = \begin{cases} N - 2^r, & k = n - \dfrac{N}{2}, \\ n - N + 2^r, & k = \dfrac{N}{2} - 2^{r-1}, \\ 0, & \textit{otherwise}, \end{cases} \tag{64}$$

(c)

$${}_2^{\#}C_2^{(k)}(D) = \begin{cases} \dfrac{(n-1)\left(2^{i_m}-1\right)}{2}, & k = \dfrac{(n-1)}{2} - 1, \\ \left(n - \dfrac{\left(a(t)+2^{i_t}\right)}{2}\right)\left(2^{i_t} - 2^{j_t}\right), & k = n - \dfrac{\left(a(t)+2^{i_t}\right)}{2} - 1, \\ & t = 1,\ldots,m-1, \\ \dfrac{a(t)\left(2^{j_{t-1}} - 2^{i_t}\right)}{2}, & k = \dfrac{a(t)}{2} - 1, \\ & t = 2,\ldots,m, \\ \left(\dfrac{N}{2} - 2^{r-1}\right)\left(2^r - 2^{i_1}\right), & k = \dfrac{N}{2} - 2^{r-1} - 1, \\ \left(n - \dfrac{N}{2}\right)\left(N - 2^r\right), & k = n - \dfrac{N}{2} - 1, \\ 0, & \textit{otherwise}, \end{cases} \tag{65}$$

where $a(t) = N - 2^r + c(t)$ and $c(t)$ is defined in (14).

Proof. Only prove (a). Since

$$\#\{D_0\} = \sum_{l=1}^{m-1}\left(2^{i_l} - 2^{j_l}\right) + 2^{i_m} - 1 = n - N + 2^r, \tag{66}$$

one has $c(m) = n - (N - 2^r) - (2^{i_m} - 1)$. By (53) and Lemma 4, yields (a). □

The proof of (b) and (c) is similar to those of Theorem 11. The following example serves to show its application.

Example 14. Let $q = 8$, $r = 7$, and $N = 256$ and consider GMC $2^{135-127}$ design $D = D_0 \cup S_{87}$. Since $\#\{D_0\} = 2^3 - 1$, we have $i_1 = 3$. By (b) and (c) of Theorem 13, one gets

$$\begin{gathered} {}_1^{\#}C_2^{(k)}(D) = \begin{cases} 128, & k = 7, \\ 7, & k = 64, \\ 0, & \text{otherwise}, \end{cases} \\ {}_2^{\#}C_2^{(k)}(D) = \begin{cases} 896, & k = 6, \\ 7680, & k = 63, \\ 469, & k = 66, \\ 0, & \text{otherwise}. \end{cases} \end{gathered} \tag{67}$$

5. Concluding Remark

Based on construction of GMC 2^{n-m} designs with $5N/16 + 1 \le n \le N-1$, we obtain the mathematical formulation to calculate the values of ${}_1^{\#}C_2$ and ${}_2^{\#}C_2$ in the AENP. These results are very useful to analyze the confounding information among lower-order factors of two-level GMC designs. For GMC 2^{n-m} designs satisfying $n \notin [5N/16 + 1, N-1]$, some further studies in this direction are in progress.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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