

RESULTS ON THE TOPOLOGY OF GENERALIZED REAL BOTT MANIFOLDS

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Abstract

Generalized Bott manifolds (over \mathbb{C} and \mathbb{R}) have been defined by Choi, Masuda and Suh in [4]. In this article we extend the results of [7] on the topology of real Bott manifolds to generalized real Bott manifolds. We give a presentation of the fundamental group, prove that it is solvable and give a characterization for it to be abelian. We further prove that these manifolds are aspherical only in the case of real Bott manifolds and compute the higher homotopy groups. Furthermore, using the presentation of the cohomology ring with \mathbb{Z}_2 -coefficients, we derive a combinatorial characterization for orientability and spin structure.

1. Introduction

A generalized Bott tower has been defined by Choi, Masuda and Suh in [5, 4] as a sequence of projective bundles

$$(1.1) \quad B_k \xrightarrow{p_k} B_{k-1} \xrightarrow{p_{k-1}} \cdots \xrightarrow{p_2} B_1 \xrightarrow{p_1} B_0 = \{ \text{a point} \}$$

where B_i for $i = 1, \dots, k$ is the projectivization of the Whitney sum of $n_i + 1$ \mathbb{F} -line bundles over B_{i-1} , ($\mathbb{F} = \mathbb{C}$ or \mathbb{R}). This generalizes the construction of Bott towers by Grossberg and Karshon [10] in the case when $\mathbb{F} = \mathbb{C}$ and $n_i = 1$ for every i . The i th stage B_i of the tower can be alternately realized as a quasitoric manifold (when $\mathbb{F} = \mathbb{C}$) or a small cover (when $\mathbb{F} = \mathbb{R}$) over $\prod_{i=1}^k \Delta^{n_i}$ where Δ^{n_i} is the n_i -simplex. We call each B_i a generalized Bott manifold (over \mathbb{F}). In the case when $n_i = 1$ for every i we call each B_i a Bott manifold.

In [5], Choi, Masuda and Suh study these manifolds from the viewpoint of the *cohomological rigidity problem for toric manifolds*. Also in [4, Theorem 6.4], they characterize generalized Bott manifolds among the quasitoric manifolds over a product of simplices, as those which admit an invariant almost complex structure under the action of $(S^1)^n$ where $n = \sum_{i=1}^k n_i$. Moreover, in [4, Section 7] they describe the cohomology ring of any quasitoric manifold over a product of simplices and in [4, Section 8] they give a sufficient condition in terms of the cohomology ring for quasitoric manifolds over a product of simplices to be diffeomorphic to a generalized Bott manifold.

In this article our main objects of study are B_k (when $\mathbb{F} = \mathbb{R}$) which we call the generalized *real* Bott manifolds. We begin Section 2 with the definition of a *small cover* [6] over a product of simplices paralleling that of a quasitoric manifold as in [4]. It follows from [14, Section 4.1 and 4.2] that every generalized real Bott manifold is a small cover over a product of simplices. In Proposition 2.7 we prove the converse, that any small cover over a product

of simplices is a generalized real Bott manifold which is the small cover analogue of [4, Proposition 6.2].

In Section 3 we study the fundamental group of B_k . More precisely, in Theorem 3.2, we give a presentation of $\pi_1(B_k)$ which generalizes the description of the fundamental group of a real Bott manifold as given in [13, Lemma 3.2], [16, Lemma 3.2] and [7, Theorem 2.1]. We further derive a characterization of the generalized real Bott manifolds which have abelian fundamental group. Furthermore, we show that B_k is aspherical if and only if it is a real Bott manifold. We study further group theoretic properties of $\pi_1(B_k)$ in Propositions 3.8 and 3.10.

We begin Section 4 by recalling the description of the cohomology ring with \mathbb{Z}_2 -coefficients of the generalized real Bott manifolds. More precisely, in Proposition 4.1, we give a presentation with generators and relations of $H^*(B_k; \mathbb{Z}_2)$. Further, we give a combinatorial criterion for B_k to be orientable in Theorem 4.5 and for B_k to be spin in Theorem 4.7, by deriving closed formulae for the first and second Stiefel-Whitney classes of B_k (see Theorem 4.2, Corollary 4.3 and Corollary 4.4). These generalize the orientability criterion given by Kamishima and Masuda for real Bott manifolds in [13, Lemma 2.2] and the criterion for spin structure for real Bott manifolds given by Gaşior in [9] and the authors of the present paper in [8].

2. Small cover over a product of simplices

In this section we define small cover over a product of simplices and recall some preliminary results.

Let Δ^{n_i} be the standard n_i -simplex for $1 \leq i \leq k$ and let $P = \prod_{i=1}^k \Delta^{n_i}$. Then P is a simple polytope of dimension $n = \sum_{i=1}^k n_i$.

We know that an m -simplex has $m + 1$ facets. Let $f_{l_i}^i$ for $0 \leq l_i \leq n_i$ denote the facets of Δ^{n_i} for $1 \leq i \leq k$. Then the facets of P are as follows :

$$(2.1) \quad F_{l_i}^i = \Delta^{n_1} \times \dots \times \Delta^{n_{i-1}} \times f_{l_i}^i \times \Delta^{n_{i+1}} \times \dots \times \Delta^{n_k}$$

for $0 \leq l_i \leq n_i$ and $1 \leq i \leq k$. Let $\mathcal{F} = \{F_{l_i}^i : 1 \leq i \leq k, 0 \leq l_i \leq n_i\}$

REMARK 2.1. Note that $F_{l_i}^i \cap F_{l'_i}^i = \Delta^{n_1} \times \dots \times (f_{l_i}^i \cap f_{l'_i}^i) \times \dots \times \Delta^{n_k}$. If $n_i \geq 2$ then $f_{l_i}^i \cap f_{l'_i}^i \neq \emptyset$ for $0 \leq l_i, l'_i \leq n_i$. If $n_i = 1$, $f_{l_i}^i \cap f_{l'_i}^i \neq \emptyset$ if and only if $l_i = l'_i$. Also for $i \neq j$, we have $F_{l_i}^i \cap F_{l'_j}^j = \Delta^{n_1} \times \dots \times f_{l_i}^i \times \dots \times f_{l'_j}^j \times \dots \times \Delta^{n_k}$.

Thus, $F_{l_i}^i \cap F_{l'_j}^j = \emptyset$ if and only if $i = j, n_i = 1$ and $l_i \neq l'_j$.

Similarly, we know that an m -simplex has $m + 1$ vertices. Let $v_{l_i}^i$ for $0 \leq l_i \leq n_i$ denote the vertices of Δ^{n_i} for $1 \leq i \leq k$. Then the vertex set of P is as follows :

$$(2.2) \quad \mathcal{V} = \{v_{l_1 \dots l_k} = v_{l_1}^1 \times \dots \times v_{l_k}^k : 0 \leq l_i \leq n_i, 1 \leq i \leq k\}$$

We shall assume that $v_{l_i}^i$ is the vertex of Δ^{n_i} opposite the facet $f_{l_i}^i$. Then it is easy to see that $v_{l_1 \dots l_k}$ is the vertex at which the facets in $\mathcal{F} - \{F_{l_1}^1, \dots, F_{l_k}^k\}$ intersect.

DEFINITION 2.2. Let $\lambda : \mathcal{F} \rightarrow \mathbb{Z}_2^n$ be defined as follows,

$$(2.3) \quad \begin{aligned} \lambda(F_1^1) &= \mathbf{e}_1, \dots, \lambda(F_{n_1}^1) = \mathbf{e}_{n_1}, \dots, \lambda(F_{n_k}^k) = \mathbf{e}_n. \\ \lambda(F_0^1) &= \mathbf{a}_1, \lambda(F_0^2) = \mathbf{a}_2, \dots, \lambda(F_0^k) = \mathbf{a}_k. \end{aligned}$$

where $\mathbf{e}_i = (0, \dots, 1, \dots, 0)$ (with 1 in the i th position) for $1 \leq i \leq n$ and $\mathbf{a}_i = (\mathbf{a}_i^1, \dots, \mathbf{a}_i^k) \in \mathbb{Z}_2^n$, for $1 \leq i \leq k$ and each $\mathbf{a}_i^j = (a_{i,1}^j, a_{i,2}^j, \dots, a_{i,n_j}^j) \in \mathbb{Z}_2^{n_j}$. We assume that whenever a set of facets of P intersect along a face their images under λ form a part of a basis of \mathbb{Z}_2^n . The small cover over P associated to the characteristic function λ is defined as $M(\lambda) = \mathbb{Z}_2^n \times P / \sim$ where $(t, p) \sim (t', p')$ if and only if $p = p'$ and $t \cdot (t')^{-1} \in G_{F(p)}$. Here $F(p)$ is the unique face of P which contains p in its relative interior and $G_{F(p)}$ is the rank- l subgroup of \mathbb{Z}_2^n spanned by the images under λ of the l facets of P that intersect at $F(p)$ (see [6, Section 1.5]).

The function λ determines the following $k \times n$ matrix A , which can be viewed as a $k \times k$ vector matrix

$$(2.4) \quad A = \begin{pmatrix} a_{1,1}^1 & \cdots & a_{1,n_1}^1 & \cdots & a_{1,1}^k & \cdots & a_{1,n_k}^k \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{k,1}^1 & \cdots & a_{k,n_1}^1 & \cdots & a_{k,1}^k & \cdots & a_{k,n_k}^k \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1^1 & \cdots & \mathbf{a}_1^k \\ \vdots & \ddots & \vdots \\ \mathbf{a}_k^1 & \cdots & \mathbf{a}_k^k \end{pmatrix}$$

Thus $M(\lambda)$ can also be thought of a small cover associated to the matrix A and we sometimes denote it by $M(A)$.

REMARK 2.3. *Quotient construction.* A small cover over a simple convex polytope P of dimension n with m facets can be realized as the quotient of the moment-angle manifold of the polytope by certain free action of the compact real torus of dimension $m - n$. It can be seen that for the polytope $P = \prod_{i=1}^k \Delta^{n_i}$ the moment-angle manifold is $X = \prod_{i=1}^k S^{n_i}$ (see [6, Section 1.10 and 4.1]). In particular, it follows that $M(A)$ is the quotient of X by the action of $G = \mathbb{Z}_2^k$ given by

$$(2.5) \quad \begin{aligned} &(g_1, \dots, g_k) \cdot ((x_0^1, \dots, x_{n_1}^1), \dots, (x_0^k, \dots, x_{n_k}^k)) \\ &= ((g_1 \cdot x_0^1, (g_1^{a_{1,1}^1} \cdots g_k^{a_{k,1}^1}) \cdot x_1^1, \dots, (g_1^{a_{1,n_1}^1} \cdots g_k^{a_{k,n_1}^1}) \cdot x_{n_1}^1), \dots, \\ &\quad (g_k \cdot x_0^k, (g_1^{a_{1,1}^k} \cdots g_k^{a_{k,1}^k}) \cdot x_1^k, \dots, (g_1^{a_{1,n_k}^k} \cdots g_k^{a_{k,n_k}^k}) \cdot x_{n_k}^k)) \end{aligned}$$

where $(g_1, \dots, g_k) \in G$ and $(x_0^i, \dots, x_{n_i}^i) \in S^{n_i}$ for $i = 1, \dots, k$ (the proof is analogous to [4, Proposition 4.3]).

For $1 \leq j_1 \leq n_1, \dots, 1 \leq j_k \leq n_k$ let $A_{j_1 \dots j_k}$ be the following $k \times k$ submatrix of A :

$$(2.6) \quad A_{j_1 \dots j_k} = \begin{pmatrix} a_{1,j_1}^1 & \cdots & a_{1,j_k}^k \\ \vdots & & \vdots \\ a_{k,j_1}^1 & \cdots & a_{k,j_k}^k \end{pmatrix}$$

The following Proposition is the \mathbb{Z}_2 analogue of [4, Lemma 3.2].

Proposition 2.4. *Let A be a matrix of the form (2.4) that is associated to the characteristic function λ of a small cover over P . The following two properties are equivalent,*

- (1) The submatrices $A_{j_1 \dots j_k}$ for $1 \leq j_1 \leq n_1, \dots, 1 \leq j_k \leq n_k$ have the property that all their principal minors are 1.
- (2) If any n facets of P intersect at a vertex then the image of the corresponding facets under λ forms a basis of \mathbb{Z}_2^n .

Proof. Let $\delta = \begin{vmatrix} a_{i_1, j_{i_1}}^{i_1} & \dots & a_{i_1, j_{i_r}}^{i_r} \\ \vdots & & \vdots \\ a_{i_r, j_{i_1}}^{i_1} & \dots & a_{i_r, j_{i_r}}^{i_r} \end{vmatrix}$ be a principal minor of $A_{j_1 \dots j_k}$. Let I be the $n \times n$ identity

matrix. Let us identify the columns of I with $\lambda(F_1^1), \dots, \lambda(F_{n_1}^1), \dots, \lambda(F_1^k), \dots, \lambda(F_{n_k}^k)$ in that order. Let A' be the matrix obtained from I by replacing the columns corresponding to $\lambda(F_{j_{i_1}}^{i_1}), \dots, \lambda(F_{j_{i_r}}^{i_r})$ by the $i_1^{\text{th}}, \dots, i_r^{\text{th}}$ rows of the matrix A in (2.4). It is easy to see that $\delta = \det A'$. Now each column of A' is the image of some facet of P under λ and these facets intersect at the vertex v_J where $J = 0 \dots 0 j_{i_1} 0 \dots 0 j_{i_2} 0 \dots 0 j_{i_r} 0 \dots 0$. Further, we know that $\det A' \equiv 1 \pmod 2$ if and only if the columns of A' form a basis of \mathbb{Z}_2^n . Hence the proposition. □

Proposition 2.5. *Let A be a $k \times k$ vector matrix as in (2.4) associated to the characteristic function of a small cover over a product of simplices. Then A is conjugate by a permutation matrix to a unipotent upper triangular vector matrix of the following form :*

$$(2.7) \quad \begin{pmatrix} \mathbf{1} & \mathbf{b}_1^2 & \mathbf{b}_1^3 & \dots & \mathbf{b}_1^k \\ \mathbf{0} & \mathbf{1} & \mathbf{b}_2^3 & \dots & \mathbf{b}_2^k \\ \vdots & & & & \vdots \\ \mathbf{0} & \dots & \dots & \mathbf{1} & \mathbf{b}_{k-1}^k \\ \mathbf{0} & \dots & \dots & \mathbf{0} & \mathbf{1} \end{pmatrix}$$

where $\mathbf{0} = (0, \dots, 0)$, $\mathbf{1} = (1, \dots, 1)$ of appropriate sizes and $\mathbf{b}_i^j = (b_{i,1}^j, \dots, b_{i,n_j}^j) \in \mathbb{Z}_2^{n_j}$. Here multiplication of matrix entries is scalar multiplication and addition is vector addition.

Proof. Since the entries of A are all from \mathbb{Z}_2 it follows from Proposition 2.4 that all the principal minors of the submatrices $A_{j_1 \dots j_k}$ for $1 \leq j_1 \leq n_1, \dots, 1 \leq j_k \leq n_k$ are 1. The result then follows from [4, Lemma 5.1]. □

REMARK 2.6. Note that we can view P as $\prod_{i=1}^k \Delta^{n_{\sigma(i)}}$ where $\sigma \in S_k$. Then we obtain a matrix $\tilde{A} = E_\sigma A E_\sigma^{-1}$ where E_σ is the permutation matrix corresponding to σ . It is easy to see that the small covers associated to A and \tilde{A} are equivariantly diffeomorphic (the proof is similar to that of [3, Proposition 5.1] also see [4, Section 5]). In view of this fact and Proposition 2.5 we will henceforth assume that the matrix A has the form (2.7).

Proposition 2.7. *Every small cover over a product of simplices is a generalized real Bott manifold.*

Proof. The proof is the small cover analogue of the proof of [4, Proposition 6.2]. Let $M(A)$ be the small cover over $\prod_{i=1}^k \Delta^{n_i}$ associated to the matrix A . We may assume by Proposition 2.5 that A is of the form (2.7). Let $X_j = \prod_{i=1}^j S^{n_i}$, which is the moment-angle manifold of $\prod_{i=1}^j \Delta^{n_i}$ for $j = 1, \dots, k$. The group $G_k = \mathbb{Z}_2^k$ acts on X_k as in (2.5) and $X_k/G_k = M(A)$. Let

$B_j = X_j/G_k$ where G_k acts on X_j as follows

$$\begin{aligned} & (g_1, \dots, g_k) \cdot ((x_0^1, \dots, x_{n_1}^1), \dots, (x_0^j, \dots, x_{n_j}^j)) \\ &= ((g_1 \cdot x_0^1, (g_1^{a_{1,1}^1} \dots g_k^{a_{k,1}^1}) \cdot x_1^1, \dots, (g_1^{a_{1,n_1}^1} \dots g_k^{a_{k,n_1}^1}) \cdot x_{n_1}^1), \dots, \\ & \quad (g_j \cdot x_0^j, (g_1^{a_{1,1}^j} \dots g_k^{a_{k,1}^j}) \cdot x_1^j, \dots, (g_1^{a_{1,n_j}^j} \dots g_k^{a_{k,n_j}^j}) \cdot x_{n_j}^j)) \end{aligned}$$

We get a sequence $M(A) = B_k \xrightarrow{p_k} B_{k-1} \rightarrow \dots \rightarrow B_1 \xrightarrow{p_1} B_0 = \{\text{pt}\}$ induced by the natural projections of X_j on X_{j-1} for $j = 1, \dots, k$.

Since A is of the form (2.7), the last $k - j$ factors of G_k act trivially on X_j so that the G_k action reduces to a G_j action on X_j with $X_j/G_k \simeq X_j/G_j$. Further, the last factor of G_j acts on the last factor of X_j as scalar multiplication and trivially on other factors. It follows that $p_j : B_j = X_j/G_j \rightarrow B_{j-1} = X_{j-1}/G_{j-1}$ is the projectivization of a real vector bundle ξ_j over B_{j-1} associated to the principal G_{j-1} -bundle $X_{j-1} \rightarrow X_{j-1}/G_{j-1} = B_{j-1}$. In fact, $\xi_j = (X_{j-1} \times V_j)/G_{j-1}$ where $V_j = \mathbb{R}^{n_j+1}$ with the action of G_{j-1} given by

$$\begin{aligned} & (g_1, \dots, g_{j-1}) \cdot (x_0^j, \dots, x_{n_j}^j) \\ &= (x_0^j, (g_1^{a_{1,1}^j} \dots g_{j-1}^{a_{j-1,1}^j}) \cdot x_1^j, \dots, (g_1^{a_{1,n_j}^j} \dots g_{j-1}^{a_{j-1,n_j}^j}) \cdot x_{n_j}^j). \end{aligned}$$

□

REMARK 2.8. The statement of Proposition 2.7 above is also mentioned in [4, Remark 6.5].

Let $N = \mathbb{Z}^n$ be the lattice with basis $\{u_1^1, \dots, u_{n_1}^1, \dots, u_1^k, \dots, u_{n_k}^k\}$. Define $u_0^i = -(b_{i,1}^1 \cdot u_1^1 + \dots + b_{i,n_1}^1 \cdot u_{n_1}^1 + \dots + b_{i,1}^k \cdot u_1^k + \dots + b_{i,n_k}^k \cdot u_{n_k}^k)$ for $1 \leq i \leq k$ where $b_{i,l}^j$ are entries from the matrix (2.7). Let Σ_k be the fan in N consisting of cones generated by any sub-collection of $\{u_0^1, \dots, u_{n_1}^1, \dots, u_0^k, \dots, u_{n_k}^k\}$ that doesn't contain $\{u_0^i, \dots, u_{n_i}^i\}$ for $1 \leq i \leq k$.

Proposition 2.9. *The generalized real Bott manifold B_k is the real toric variety associated to the smooth projective fan Σ_k .*

Proof. We proceed by induction on k . When $k = 1$ we have $B_1 = \mathbb{RP}_1^n$ which is the real toric variety associated to the fan Σ_1 . Let $N' = \mathbb{Z}^{n-n_k}$ be the lattice with basis $\{(u_1^1)', \dots, (u_{n_1}^1)', \dots, (u_1^{k-1})', \dots, (u_{n_{k-1}}^{k-1})'\}$. For $1 \leq i \leq k - 1$ let

$$(u_0^i)' = -(b_{i,1}^1 \cdot (u_1^1)' + \dots + b_{i,n_{k-1}}^{k-1} \cdot (u_{n_{k-1}}^{k-1})').$$

Consider the fan Σ' in N' consisting of cones generated by any sub-collection of $\{(u_0^1)', \dots, (u_{n_1}^1)', \dots, (u_0^{k-1})', \dots, (u_{n_{k-1}}^{k-1})'\}$ that doesn't contain $\{(u_0^i)', \dots, (u_{n_i}^i)'\}$ for all $1 \leq i \leq k - 1$. Then Σ' is isomorphic to the fan Σ_{k-1} . By the induction hypothesis B_{k-1} is the real toric variety associated to the smooth projective fan $\Sigma_{k-1} \simeq \Sigma'$. Now $B_k = \mathbb{P}(\mathbf{1} \oplus L_1 \oplus \dots \oplus L_{n_k})$ where L_1, \dots, L_{n_k} are real line bundles over B_{k-1} . By Lemma 2.10 each L_i for $1 \leq i \leq n_k$, is isomorphic to a T -equivariant real algebraic line bundle L'_i over B_{k-1} where $T = (\mathbb{R}^*)^{n-n_k}$. Hence B_k is homeomorphic to $\mathbb{P}(\mathbf{1} \oplus L'_1 \oplus \dots \oplus L'_{n_k})$. Thus without loss of generality we can assume that L_i for $1 \leq i \leq n_k$ is a T -equivariant real algebraic line bundle over B_{k-1} . Let h_1, \dots, h_{n_k} be the support functions corresponding to L_1, \dots, L_{n_k} respectively. In view of

Remark 2.11,

$$(2.8) \quad h_j((u_{l_i}^i)') = 0 \quad \text{for } 1 \leq i \leq k - 1, 1 \leq l_i \leq n_i$$

and let

$$(2.9) \quad b_{i,j}^k := h_j((u_0^i)') \quad \text{for } 1 \leq i \leq k - 1.$$

Let $N'' = \mathbb{Z}^{n_k}$ with basis $\{(u_1^k)'', \dots, (u_{n_k}^k)''\}$. Let $(u_0^k)'' := -((u_1^k)'' + \dots + (u_{n_k}^k)'')$. Let Σ'' denote the fan in N'' consisting of cones generated by any proper subset of $\{(u_0^k)'', \dots, (u_{n_k}^k)''\}$. We get an exact sequence of fans

$$(2.10) \quad 0 \rightarrow (N'', \Sigma'') \xrightarrow{\phi''} (N, \Sigma_k) \xrightarrow{\phi'} (N', \Sigma') \rightarrow 0$$

where, $\phi''((u_{l_k}^k)'') = u_{l_k}^k$ for $1 \leq l_k \leq n_k$ and $\phi'(u_{l_i}^i) = (u_{l_i}^i)'$ for $1 \leq i \leq k - 1, 1 \leq l_i \leq n_i$ and $\phi'(u_{l_k}^k) = 0$ for $1 \leq l_k \leq n_k$.

Define $\psi : N' \simeq \mathbb{Z}_2^{n-n_k} \rightarrow \mathbb{Z}_2^n \simeq N$ by $\psi(y) = (y, -h_1(y), \dots, -h_{n_k}(y))$. Then by (2.8) ψ gives a splitting of (2.10). That is $N = N' \oplus N''$. Further, if we let $\tilde{\sigma}$ denote the image of any $\sigma \in \Sigma'$ under ψ then,

$$\tilde{\Sigma} = \{\tilde{\sigma} : \sigma \in \Sigma'\}$$

It can be seen that

$$\Sigma_k = \{\tilde{\sigma} + \sigma'' : \sigma \in \Sigma' \text{ and } \sigma'' \in \Sigma''\}$$

Then from the real analogue of [18, Proposition 1.33] we have that B_k is the real toric variety corresponding to the smooth projective fan Σ_k .

It also follows independently from [14, Section 4.1 and Section 4.2], that B_k is a small cover over the product of simplices $\prod_{i=1}^k \Delta^{n_i}$. □

Lemma 2.10. *Let X_Σ be the real toric variety with dense torus $(\mathbb{R}^*)^n$, associated to the smooth projective fan Σ in the lattice $N \simeq \mathbb{Z}^n$. Every real line bundle over X_Σ is isomorphic to an $(\mathbb{R}^*)^n$ -equivariant algebraic real line bundle.*

Proof. We know that $X_\Sigma = \bigcup_{\sigma \in \Sigma} U_\sigma$ where $U_\sigma = \text{Hom}_{sg}(M \cap \sigma^\vee, \mathbb{R})$, σ^\vee is the dual cone of σ in the dual lattice M . Let $\{\rho_j : 1 \leq j \leq d\}$ be the edge vectors in Σ and let v_j be the primitive vector generating the edge ρ_j for $1 \leq j \leq d$. Let \mathcal{L}_j be the algebraic real line bundle corresponding to the piecewise linear data $\{m_\sigma\}_{\sigma \in \Sigma}$ (see [18, Proposition 2.1]) where $m_\sigma \in M$ is defined as $m_\sigma = 0$ if v_j is not on an edge of σ and if v_j is on an edge of σ then $-m_\sigma$ is the element in the dual basis to the basis of N consisting of generators of σ such that $\langle -m_\sigma, v_j \rangle = 1$. Since $m_\sigma - m_{\sigma'} \in (\sigma \cap \sigma')^\perp$, we can define as in [18, p. 69], the real $(\mathbb{R}^*)^n$ -equivariant line bundle associated to $\{m_\sigma\}$ as follows :

$$\mathcal{L}_j = \bigcup_{\sigma \in \Sigma} U_\sigma \times \mathbb{R} / \sim \quad \text{where } (x, r) \sim (x, x(m_\sigma - m_{\sigma'}) \cdot r) \text{ for } x \in U_\sigma \cap U_{\sigma'}$$

Then $\{\mathcal{L}_j : 1 \leq j \leq d\}$ are the canonical real line bundles over X_Σ . Each \mathcal{L}_j has an $(\mathbb{R}^*)^n$ -equivariant sections s_j such that the zero locus of s_j is the $(\mathbb{R}^*)^n$ -invariant co-dimension 1 real subvariety $V(\rho_j)$ of X_Σ (see [18, Proposition 2.1] and [1, Example 2.10.5]). Here $V(\rho_j)$ is the closure of the orbit $O_{\rho_j} = \text{Hom}_{gp}(M \cap \rho_j^\perp, \mathbb{R}^*)$. Then, $w_1(\mathcal{L}_j) = [V(\rho_j)] \in H^1(X_\Sigma; \mathbb{Z}_2)$,

where $[V(\rho_j)]$ is the Poincaré dual to the homology class of $V(\rho_j)$. Now by [12, Theorem 4.3.1], as a \mathbb{Z}_2 -vector space, $H^1(X_\Sigma; \mathbb{Z}_2)$ is generated by $[V(\rho_j)]$ for $1 \leq j \leq d$. Let L be a real line bundle over X_Σ . Since $w_1(L) \in H^1(X_\Sigma; \mathbb{Z}_2)$ we have $w_1(L) = \sum_{j=1}^d c_j \cdot [V(\rho_j)] = w_1(\otimes \mathcal{L}_j^{c_j})$. Thus $L \simeq \otimes \mathcal{L}_j^{c_j}$ ([11, p. 250]). \square

REMARK 2.11. Since Σ is a smooth complete fan, we may assume without loss of generality that ρ_1, \dots, ρ_n span a cone in Σ . Then $\{v_1, \dots, v_n\}$ is a basis of N with dual basis $\{v_1^*, \dots, v_n^*\}$ of M . From [12, Theorem 4.3.1], for $1 \leq i \leq n$ we have the linear relations $\sum_{j=1}^d \langle v_i^*, v_j \rangle \cdot [V(\rho_j)]$ so that $H^1(X_\Sigma; \mathbb{Z}_2) = \langle [V(\rho_j)] : n + 1 \leq j \leq d \rangle \simeq \mathbb{Z}_2^{d-n}$. Therefore if L is any real line bundle over X_Σ then $L \simeq \bigotimes_{i=n+1}^d \mathcal{L}_i^{c_i} =: \mathcal{L}_h$. Here \mathcal{L}_h is the line bundle corresponding to the support function given by $h(v_i) = -c_i$ for $n + 1 \leq i \leq d$ and $h(v_i) = 0$ otherwise.

3. The fundamental group

We begin this section by giving a presentation of $\pi_1(B_k)$ with generators and relations. This is obtained by applying [19, Proposition 3.1]. Using the presentation we characterize those B_k with abelian fundamental group. Again, using [19, Theorem 6.1] we show that B_k is aspherical only when it is a real Bott manifold. In Section 3.2, we derive further group theoretic properties of $\pi_1(B_k)$. In particular we prove that $\pi_1(B_k)$ is solvable and that $\pi_1(B_k)$ is nilpotent if and only if it is abelian.

3.1. Presentation of $\pi_1(B_k)$. Recall that we have an exact sequence

$$(3.1) \quad 1 \rightarrow \pi_1(B_k) \rightarrow W(\Sigma_k) \rightarrow \mathbb{Z}_2^n \rightarrow 1$$

where $W(\Sigma_k)$ is the right angled Coxeter group associated to Σ_k with the following presentation :

$$(3.2) \quad \begin{aligned} W = W(\Sigma_k) = \langle s_{i,l_i} : 1 \leq i \leq k, 0 \leq l_i \leq n_i \mid s_{i,l_i}^2 = 1, (s_{i,l_i} \cdot s_{j,l'_j})^2 = 1 \text{ for } i \neq j \\ \text{and } (s_{i,l_i} \cdot s_{i,l'_i})^2 = 1 \forall i \text{ such that } n_i \geq 2 \rangle. \end{aligned}$$

The last arrow in the above exact sequence is obtained by composing the natural abelianization map from W to \mathbb{Z}_2^{n+k} with the characteristic map λ from \mathbb{Z}_2^{n+k} to \mathbb{Z}_2^n . (Here λ is extended by linearity from Definition 2.2)

For each $1 \leq j \leq k$ we define the reduced word

$$(3.3) \quad \alpha_j := s_{j,0} \cdot s_{1,1}^{a_{1,1}^1} \cdot s_{1,2}^{a_{1,2}^1} \cdots s_{1,n_1}^{a_{1,n_1}^1} \cdots s_{k,1}^{a_{k,1}^k} \cdots s_{k,n_k}^{a_{k,n_k}^k}$$

in W .

For every $\varepsilon = (\varepsilon_{1,1}, \dots, \varepsilon_{1,n_1}, \dots, \varepsilon_{k,1}, \dots, \varepsilon_{k,n_k}) \in \mathbb{Z}_2^n$ we define the following :

- (1) $t_\varepsilon = s_{1,1}^{\varepsilon_{1,1}} \cdots s_{1,n_1}^{\varepsilon_{1,n_1}} \cdots s_{k,1}^{\varepsilon_{k,1}} \cdots s_{k,n_k}^{\varepsilon_{k,n_k}}$.
- (2) $B^p = (B_{1,1}^p, \dots, B_{1,n_1}^p, \dots, B_{k,1}^p, \dots, B_{k,n_k}^p)$ where $B_{i,l_i}^p = \varepsilon_{i,l_i} + a_{p,l_i}^i$ for $1 \leq i, p \leq k$ and $1 \leq l_i \leq n_i$.
- (3) $C^{p,q} = (C_{1,1}^{p,q}, \dots, C_{1,n_1}^{p,q}, \dots, C_{k,1}^{p,q}, \dots, C_{k,n_k}^{p,q})$ where $C_{i,l_i}^{p,q} = \varepsilon_{i,l_i} + a_{p,l_i}^i + a_{q,l_i}^i$ for $1 \leq i, p, q \leq k$ and $1 \leq l_i \leq n_i$.

From the relations in W we get,

$$(3.4) \quad t_\varepsilon \cdot \alpha_j \cdot t_\varepsilon = \begin{cases} \alpha_j & , \quad \varepsilon_{j,1} = 0 \text{ or } n_j \geq 2 \\ \alpha_j^{-1} & , \quad \text{otherwise} \end{cases}$$

This can be seen as follows : if $n_j = 2$ then $s_{j,0}$ commutes with all s_{i,l_i} so that $t_\varepsilon \cdot \alpha_j \cdot t_\varepsilon = \alpha_j$. When $n_j = 1$ we get

$$t_\varepsilon \cdot \alpha_j \cdot t_\varepsilon = s_{j,1}^{\varepsilon_{j,1}} \cdot s_{j,0} \cdot s_{1,1}^{a_{j,1}^1} \cdots s_{1,n_1}^{a_{j,1}^1} \cdots s_{j,1}^{a_{j,1}^j} \cdots s_{k,n_k}^{a_{j,n_k}^k} \cdot s_{j,1}^{\varepsilon_{j,1}} = \begin{cases} \alpha_j & , \quad \varepsilon_{j,1} = 0 \\ \alpha_j^{-1} & , \quad \varepsilon_{j,1} = 1 \end{cases}$$

where $a_{j,1}^j = 1$. Note that $\alpha_j^2 = 1$ when $n_j \geq 2$.

REMARK 3.1. In view of Remark 2.6 we may assume that there exists an $l \leq k$ such that $n_1, \dots, n_l \geq 2$ and $n_{l+1} = \dots = n_k = 1$. Let M^l be the real part of the toric variety associated to the fan Σ_l . Note that if $l = 0$ then B_k is a real Bott manifold.

Theorem 3.2. *The fundamental group of the generalized real Bott manifold B_k has a presentation $\pi_1(B_k) = \langle S \mid R \rangle$ with generators,*

$$(3.5) \quad S = \{\alpha_j : 1 \leq j \leq k\}$$

and relations,

$$(3.6) \quad R = \{(\alpha_p \alpha_q)^2 : 1 \leq p < q \leq l\} \cup \{\alpha_p^2 : 1 \leq p \leq l\} \\ \cup \{x_{p,q} : l+1 \leq p < q \leq k\} \cup \{x'_{p,q} : 1 \leq p \leq l < q \leq k\}$$

where,

$$x_{p,q} = \begin{cases} \alpha_p \alpha_q \alpha_p^{-1} \alpha_q^{-1} & , \quad a_{p,1}^q = 0 \\ \alpha_p \alpha_q^{-1} \alpha_p^{-1} \alpha_q^{-1} & , \quad a_{p,1}^q = 1 \end{cases}$$

and,

$$x'_{p,q} = \begin{cases} \alpha_p \alpha_q \alpha_p \alpha_q^{-1} & , \quad a_{p,1}^q = 0 \\ (\alpha_p \alpha_q)^2 & , \quad a_{p,1}^q = 1 \end{cases}$$

Proof. The proof uses [19, Theorem 3.2]. We can relate the above notations with those in the proof of [19, Theorem 3.2] as follows :

$$y_{j,\varepsilon} = t_\varepsilon \cdot \alpha_j \cdot t_\varepsilon$$

for $1 \leq j \leq k$ and $\varepsilon \in \mathbb{Z}_2^n$. Thus $y_{j,\varepsilon} = \alpha_j^{\pm 1}$ by (3.4). Then it can be seen that

$$y_{p,\varepsilon} \cdot y_{q,B^p} \cdot y_{p,C^{p,q}} \cdot y_{q,B^q} = t_\varepsilon \cdot \alpha_p \cdot t_{\mathbf{a}_p} \alpha_q t_{\mathbf{a}_p} \cdot t_{\mathbf{a}_q + \mathbf{a}_p} \alpha_p t_{\mathbf{a}_q + \mathbf{a}_p} \cdot t_{\mathbf{a}_q} \alpha_q t_{\mathbf{a}_q} \cdot t_\varepsilon \\ =: X_{p,q}^\varepsilon$$

whenever $1 \leq p, q \leq k$. By exchanging the rolls of p and q we get that $X_{p,q}^\varepsilon$ is conjugate to $X_{q,p}^\varepsilon$ or $(X_{q,p}^\varepsilon)^{-1}$ by an element from the free group generated by S . Thus we may assume that $p < q$. We now look at the following cases :

- (1) If $p, q \leq l$ then $X_{p,q}^\varepsilon = t_\varepsilon \cdot \alpha_p \alpha_q \alpha_p \alpha_q \cdot t_\varepsilon = (\alpha_p \alpha_q)^2$.
- (2) If $p \leq l$ and $q > l$ then

$$X_{p,q}^\varepsilon = \begin{cases} t_\varepsilon \cdot \alpha_p \alpha_q \alpha_p \alpha_q^{-1} \cdot t_\varepsilon & , \quad a_{p,1}^q = 0 \\ t_\varepsilon \cdot (\alpha_p \alpha_q^{-1})^2 \cdot t_\varepsilon & , \quad a_{p,1}^q = 1 \end{cases}$$

(3) If $p, q > l$ then

$$X_{p,q}^\varepsilon = \begin{cases} t_\varepsilon \cdot \alpha_p \alpha_q \alpha_p^{-1} \alpha_q^{-1} \cdot t_\varepsilon & , \quad a_{p,1}^q = 0 \\ t_\varepsilon \cdot \alpha_p \alpha_q^{-1} \alpha_p^{-1} \alpha_q^{-1} \cdot t_\varepsilon & , \quad a_{p,1}^q = 1 \end{cases}$$

From (3.4) it is easy to see that $X_{p,q}^\varepsilon$ in (2) is conjugate to either $x'_{p,q}$ or $(x'_{p,q})^{-1}$ and $X_{p,q}^\varepsilon$ in (3) is conjugate to either $x_{p,q}$ or $x_{p,q}^{-1}$ by an element from the free group generated by S . Also by definition of α_j it is clear that $\alpha_j^2 = 1$ for $1 \leq j \leq l$. Thus it follows that R gives a complete set of relations of $\pi_1(B_k)$. Hence the theorem. \square

REMARK 3.3. Alternately, the presentation for $\pi_1(B_k)$ can be derived from (3.13).

Corollary 3.4. *The fundamental group $\pi_1(B_k)$ is abelian if and only if $B_k \simeq M^l \times (S^1)^{k-l}$ for some $0 \leq l \leq k$.*

Proof. From Theorem 3.2 it is easy to see that $\pi_1(B_k)$ is abelian if and only if for all $1 \leq p \leq k$, $l + 1 \leq q \leq k$ and $p < q$, $a_{p,1}^q = 0$, where l is as in Remark 3.1. Hence the corollary follows. \square

Lemma 3.5. *B_k is an aspherical manifold if and only if it is a real Bott manifold.*

Proof. We know from [19, Theorem 6.1] that a real toric variety is aspherical if and only if the corresponding fan is flag-like. A fan is flag-like if and only if for every collection S of edge vectors of the fan if $\langle u, u' \rangle$ spans a cone for all $u, u' \in S$ then S spans a cone. Now if $n_i \geq 2$ for some i then the set of edge vectors $\{u_{i_j}^i\}_{j=0}^{n_i}$ of Σ_k is such that $\langle u_{i_0}^i, u_{i_1}^i \rangle$ spans a cone in Σ_k but $\langle u_{i_0}^i, \dots, u_{i_{n_i}}^i \rangle$ does not. Conversely, if $n_i = 1$ for all i then it can be easily seen that Σ_k is flag-like. The theorem follows. \square

REMARK 3.6. The Lemma 3.5 above also follows from the quotient construction (see Remark 2.3) of B_k and the fact that some finite cover of B_k is a product of spheres and it is a torus if and only if B_k is a real Bott manifold, in other words, a sphere of dimension greater than 1 appears in the factor of the product unless B_k is a real Bott manifold.

Corollary 3.7. *The group $\pi_1(B_k)$ is torsion free if and only if B_k is a real Bott manifold.*

Proof. If B_k is a real Bott manifold then by Lemma 3.5 it is aspherical and hence $\pi_1(B_k)$ is torsion free. Conversely, if $n_i \neq 1$ for some i , then from Theorem 3.2, $\alpha_i \neq 1$ but $\alpha_i^2 = 1$, so that α_i is a torsion element. \square

3.2. Further group theoretic properties of $\pi_1(B_k)$.

Proposition 3.8. *The commutator subgroup $[\pi_1(B_k), \pi_1(B_k)]$ is abelian. In particular $\pi_1(B_k)$ is a solvable group.*

Proof. By [19, Lemma 4.1] we know that $[W, W]$ is abelian if and only if for each edge u in the fan Σ_k , there is at most one edge u' such that $\langle u, u' \rangle$ does not span a cone in Σ_k . We

have $\langle u_i^j, u_p^j \rangle$ spans a cone in Σ_k whenever $i \neq j$ or $i = j, n_i \geq 2$. When $i = j$ and $n_i = 1$ there are only two edge vectors u_0^i and u_1^i and they do not span a cone in Σ_k . Hence $[W, W]$ is abelian. Since $\pi_1(B_k)$ is a subgroup of W it follows that $[\pi_1(B_k), \pi_1(B_k)]$ is abelian. Finally, $1 \leq [\pi_1, \pi_1] \leq \pi_1(B_k)$ gives an abelian tower for $\pi_1(B_k)$ so that $\pi_1(B_k)$ is solvable. \square

Let $\bar{\alpha}_j$ denote the image of α_j under the canonical abelianization homomorphism

$$(3.7) \quad \pi_1(B_k) \rightarrow H_1(B_k; \mathbb{Z}) \simeq \pi_1(B_k)/[\pi_1(B_k), \pi_1(B_k)].$$

We then have the following description of $H_1(B_k; \mathbb{Z})$.

Proposition 3.9. *The group $H_1(B_k; \mathbb{Z})$ has a presentation with generators*

$$(3.8) \quad \langle \bar{\alpha}_j : 1 \leq j \leq k \rangle$$

and relations as follows :

$$(3.9) \quad \bar{\alpha}_p \cdot \bar{\alpha}_q \cdot \bar{\alpha}_p^{-1} \cdot \bar{\alpha}_q^{-1}$$

for $1 \leq p, q \leq k$ and

$$(3.10) \quad \bar{\alpha}_q^2$$

for $1 \leq q \leq l$ and for those $l < q \leq k$ for which there exists a $p < q$ such that $a_{p,1}^q = 1$. Thus additively we have an isomorphism $H_1(B_k; \mathbb{Z}) \simeq \mathbb{Z}^{k-r-l} \oplus \mathbb{Z}_2^{l+r}$ where r is the number of $l < q \leq k$ as above.

Proof. The proof follows readily from (3.7) and (3.6). \square

Proposition 3.10. *The group $\pi_1(B_k)$ is nilpotent if and only if it is abelian.*

Proof. Observe by (3.2) that

$$(3.11) \quad \alpha_j^2 = (s_{j,0} \cdot s_{1,1}^{a_{j,1}^1} \cdot s_{1,2}^{a_{j,2}^1} \cdots s_{1,n_1}^{a_{j,n_1}^1} \cdots s_{k,1}^{a_{j,1}^k} \cdots s_{k,n_k}^{a_{j,n_k}^k})^2 = (s_{j,0} \cdot s_{j,1})^2 \neq 1$$

for $j > l$ and $\alpha_j^2 = 1$ for $j \leq l$. If $\pi_1(B_k)$ is not abelian then by Theorem 3.2, there exists a $q > l$ and a $p < q$ such that $a_{p,1}^q = 1$. Then it follows from Proposition 3.9 that $\alpha_q^2 \in [\pi_1(B_k), \pi_1(B_k)]$. Further, by (3.6), $\alpha_p \alpha_q^{-2} \alpha_p^{-1} \alpha_q^2 = (\alpha_p \alpha_q^{-1} \alpha_p^{-1})^2 \alpha_q^2 = \alpha_q^2 \alpha_q^2 = \alpha_q^4 \in [\pi_1(B_k), [\pi_1(B_k), \pi_1(B_k)]]$. Proceeding similarly by induction we get that

$$\alpha_p \alpha_q^{-2^m} \alpha_p^{-1} \alpha_q^{2^m} = \alpha_q^{2^{m+1}} \in \pi_1(B_k)^{(m)}.$$

Here $\pi_1(B_k)^{(1)} := [\pi_1(B_k), \pi_1(B_k)]$ and $\pi_1(B_k)^{(m)} := [\pi_1(B_k), \pi_1(B_k)^{(m-1)}]$. Finally, by (3.11) and the fact that $s_{q,0} \cdot s_{q,1}$ is of infinite order in W , we have that $\alpha_q^{2^m} = (s_{q,0} \cdot s_{q,1})^{2^m} \neq 1$ for any m . The proposition follows. \square

REMARK 3.11. Here we mention that the fundamental group of a real toric variety is not in general solvable. For example, the non-orientable surfaces of genus g are real toric varieties (see for example [12, Section 4.5, Remark 4.5.2] and [19, Remark 3.3]), whose fundamental groups contain free subgroups of rank $g - 1$ (see for example [2, p. 62, Section 4]). Thus whenever $g \geq 3$, these groups are not solvable.

REMARK 3.12. We wish to mention here that from [15, Corollary 3.7], if a small cover M is aspherical and its fundamental group is virtually solvable (i.e., it has the structure of an infra-solvmanifold), then M is homeomorphic to a real Bott manifold. On the other hand, Proposition 3.8 in the present paper says that the fundamental group of a generalized real Bott manifold B_k is always solvable and Lemma 3.5 says that if B_k is aspherical then it is a real Bott manifold.

REMARK 3.13. In view of Remark 3.12, it may be worthwhile to ask if a small cover M (not necessarily aspherical) having a virtually solvable fundamental group is in fact a *generalized real Bott manifold*.

3.3. Remark on higher homotopy groups.

Theorem 3.14. *The higher homotopy groups $\pi_j(B_k)$ for $j \geq 2$ are isomorphic to $\pi_j(\mathbb{P}^{n_1}) \times \cdots \times \pi_j(\mathbb{P}^{n_k})$.*

Proof. Note that the fibration $\mathbb{P}^{n_k} \xrightarrow{i} B_k \xrightarrow{p_k} B_{k-1}$ induces a long exact sequence of homotopy groups

$$(3.12) \quad \cdots \rightarrow \pi_j(\mathbb{P}^{n_k}) \xrightarrow{i_*} \pi_j(B_k) \xrightarrow{(p_k)_*} \pi_j(B_{k-1}) \xrightarrow{\partial_*} \pi_{j-1}(\mathbb{P}^{n_k}) \rightarrow \cdots$$

Since $B_k = \mathbb{P}(\mathbf{1} \oplus L_1 \oplus \cdots \oplus L_{n_k})$ the zero sections of each $L_i \rightarrow B_{k-1}$ for $1 \leq i \leq n_k$ define a section of $p_k : B_k \rightarrow B_{k-1}$, namely the map $s_k : B_{k-1} \rightarrow B_k$ that sends any point to $[(1, 0, \dots, 0)]$ in the fiber above that point. The induced map, $(s_k)_* : \pi_j(B_{k-1}) \rightarrow \pi_j(B_k)$ is a right inverse of $(p_k)_*$. This in particular, makes all the maps $(p_k)_*$ surjective so that the maps ∂_* are identically 0 and hence the maps i_* are injective. Thus we have the split exact sequence

$$(3.13) \quad 0 \rightarrow \pi_j(\mathbb{P}^{n_k}) \xrightarrow{i_*} \pi_j(B_k) \xrightarrow{(p_k)_*} \pi_j(B_{k-1}) \rightarrow 0.$$

Since $\pi_j(B_k)$ is abelian for $j \geq 2$, by induction on k we get that $\pi_j(B_k) \simeq \pi_j(\mathbb{P}^{n_1}) \times \cdots \times \pi_j(\mathbb{P}^{n_k})$. □

4. Orientability and Spin structure

The results in this section extend those in [8, Section 3]. We begin by recalling the presentation of the cohomology ring with \mathbb{Z}_2 coefficients of B_k . We then give a recursive formula for the total Stiefel-Whitney class and closed formulae for the first and second Stiefel-Whitney classes of B_k . We hence obtain necessary and sufficient conditions for orientability and spin structure on B_k in terms of certain identities on the entries a_{i,l_j}^j of the matrix A .

Proposition 4.1. *Let \mathcal{R} be the ring $\mathbb{Z}_2[x_{10}, \dots, x_{1n_1}, \dots, x_{k0}, \dots, x_{kn_k}]$ and let \mathcal{I} denote the ideal in \mathcal{R} generated by the following set of elements*

$$(4.1) \quad \{x_{i0} x_{i1} \cdots x_{in_i}, x_{il_i} + \sum_{j=1}^i a_{jl_i}^i x_{j0} \mid 1 \leq i \leq k, 1 \leq l_i \leq n_i\}.$$

As a graded \mathbb{Z}_2 -algebra $H^*(B_k; \mathbb{Z}_2)$ is isomorphic to \mathcal{R}/\mathcal{I} .

Proof. Since B_k is the toric variety associated to the smooth projective fan Σ_k the result follows from [12, Theorem 4.3.1]. Alternately, since B_k is the small cover over the polytope $\prod_{i=1}^k \Delta^{n_i}$ with characteristic function λ , the result follows from [6, Theorem 4.14]. \square

Let $w_p(B_k)$ denote the p th Stiefel-Whitney class of B_k for $0 \leq p \leq n$ with the understanding that $w_0(B_k) = 1$. Then $w(B_k) = 1 + w_1(B_k) + \dots + w_n(B_k)$ is the total Stiefel-Whitney class of B_k .

Theorem 4.2. *The following holds in the \mathbb{Z}_2 -algebra \mathcal{R}/\mathcal{I} :*

$$(4.2) \quad w(B_k) = w(B_{k-1}) \cdot \left(\prod_{l_k=1}^{n_k} (1 + \sum_{j=1}^k a_{j,l_k}^k x_{j0}) \right) \cdot (1 + x_{k0}).$$

Proof. From [6, Corollary 6.8] for B_k we have

$$(4.3) \quad w(B_k) = \prod_{i=1}^k \prod_{l_i=0}^{n_i} (1 + x_{il_i}).$$

Note that the defining matrix for B_{k-1} is the $(k-1) \times (n-n_k)$ submatrix of A obtained by deleting the k th row and the last n_k columns of A . Moreover, via pullback along p_k^* , where p_k is the map in (1.1), $H^*(B_{k-1}; \mathbb{Z}_2)$ can be identified with the subring $\mathcal{R}'/\mathcal{I}'$ of \mathcal{R}/\mathcal{I} where $\mathcal{R}' = \mathbb{Z}_2[x_{10}, \dots, x_{1n_1}, \dots, x_{k-1,0}, \dots, x_{k-1, n_{k-1}}]$ and \mathcal{I}' is the ideal generated by the relations

$$(4.4) \quad \{x_{i0} x_{i1} \cdots x_{in_i}, x_{il_i} + \sum_{j=1}^i a_{j,l_i}^i x_{j0} \text{ for } 1 \leq i \leq k-1, 1 \leq l_i \leq n_i\}.$$

Since B_k is a \mathbb{P}^{n_k} -bundle over B_{k-1} , we further have the following presentation of $H^*(B_k; \mathbb{Z}_2)$ as an algebra over $H^*(B_{k-1}; \mathbb{Z}_2)$:

$$(4.5) \quad H^*(B_k; \mathbb{Z}_2) \simeq H^*(B_{k-1}; \mathbb{Z}_2)[x_{k0}, \dots, x_{kn_k}]/J$$

where J is the ideal generated by the relations

$$(4.6) \quad x_{k0} x_{k1} \cdots x_{kn_k}, x_{kl_k} + \sum_{j=1}^k a_{j,l_k}^k x_{j0} \text{ for } 1 \leq l_k \leq n_k.$$

Furthermore, via p_k^* we can identify $w(B_{k-1})$ with the expression :

$$(4.7) \quad w(B_{k-1}) = \prod_{i=1}^{k-1} \prod_{l_i=0}^{n_i} (1 + x_{il_i}).$$

in \mathcal{R}' where x_{il_i} for $1 \leq i \leq k-1$ and $1 \leq l_i \leq n_i$ satisfy the relations (4.4). Now by (4.3) and (4.7), it follows that,

$$(4.8) \quad w(B_k) = w(B_{k-1}) \cdot \prod_{l_k=0}^{n_k} (1 + x_{kl_k}).$$

Using the relations in \mathcal{I} we get (4.2). Hence the theorem. \square

Corollary 4.3. *We have the following expression for $w_1(B_k)$ in \mathcal{R}/\mathcal{I} :*

$$(4.9) \quad w_1(B_k) = \sum_{j=1}^k (1 + \sum_{i=1}^k \sum_{l_i=1}^{n_i} a_{j,l_i}^i) \cdot x_{j0}.$$

Proof. Note that under the isomorphism of graded algebras $H^*(B_k; \mathbb{Z}_2)$ and \mathcal{R}/\mathcal{I} , $w_1(B_k) \in H^1(B_k; \mathbb{Z}_2)$ corresponds to a polynomial of degree 1 in x_{il_i} , $1 \leq i \leq k$, $1 \leq l_i \leq n_i$ modulo \mathcal{I} . Thus by comparing the degree 1-terms on either side of (4.2) and using induction on k the lemma follows. □

Corollary 4.4. *We have the following expression for $w_2(B_k)$ in \mathcal{R}/\mathcal{I} :*

$$(4.10) \quad w_2(B_k) = \sum_{s=1}^l T'_s \cdot x_{s0}^2 + \sum_{\substack{r,s=1 \\ r < s}}^l T'_{rs} \cdot x_{r0} \cdot x_{s0} + \sum_{\substack{1 \leq r \leq k \\ l < s \leq k \\ r < s}} (T'_{rs} + a_{r,1}^s \cdot T'_s) \cdot x_{r0} \cdot x_{s0}$$

Where

$$(4.11) \quad T'_s = \sum_{\substack{1 \leq i \leq k \\ 1 \leq l_i \leq n_i}} a_{s,l_i}^i + \sum_{\substack{1 \leq i \leq k \\ 1 \leq l_i < l'_i \leq n_i}} a_{s,l_i}^i a_{s,l'_i}^i + \sum_{\substack{1 \leq i < j \leq k \\ 1 \leq l_i \leq n_i \\ 1 \leq l_j \leq n_j}} a_{s,l_i}^i a_{s,l_j}^j$$

and

$$(4.12) \quad T'_{rs} = 1 + \sum_{\substack{1 \leq i \leq k \\ 1 \leq l_i \leq n_i}} (a_{r,l_i}^i + a_{s,l_i}^i) + \sum_{\substack{1 \leq i \leq k \\ 1 \leq l_i, l'_i \leq n_i \\ l_i \neq l'_i}} a_{r,l_i}^i a_{s,l'_i}^i + \sum_{\substack{1 \leq i < j \leq k \\ 1 \leq l_i \leq n_i \\ 1 \leq l_j \leq n_j}} a_{r,l_i}^i a_{s,l_j}^j$$

Proof. Using the relations (4.1) in (4.3), $w(B_k)$ can be identified with the class of the following term in \mathcal{R}/\mathcal{I} :

$$(4.13) \quad \prod_{i=1}^k (1 + x_{i0}) \cdot \prod_{i=1}^k \prod_{l_i=1}^{n_i} \left(1 + \sum_{j=1}^k a_{j,l_i}^i \cdot x_{j0} \right)$$

(Here we note that $a_{j,l_i}^i = 0$ for $j > i$.) Since Proposition 4.1 gives an isomorphism of graded \mathbb{Z}_2 -algebras, the degree 2 term of $w(B_k)$, namely $w_2(B_k)$ can be identified with the degree 2 term of the expression (4.13) which is the class of the following term in \mathcal{R}/\mathcal{I} :

$$(4.14) \quad \sum_{s=1}^k T'_s \cdot x_{s0}^2 + \sum_{\substack{r,s=1 \\ r < s}}^k T'_{rs} \cdot x_{r0} \cdot x_{s0}$$

For $l \leq s \leq k$, we can use the relations in (4.1) to obtain,

$$(4.15) \quad x_{s0}^2 = \sum_{j=1}^{s-1} a_{j,1}^s \cdot x_{j0} \cdot x_{s0}$$

Using (4.15) in (4.14) we get (4.10). Hence the corollary. □

Theorem 4.5. *The space B_k is orientable if and only if*

$$(4.16) \quad \sum_{i=1}^k \sum_{l_i=1}^{n_i} a_{j,l_i}^i \equiv 1 \pmod 2 \text{ for all } 1 \leq j \leq k.$$

Proof. Note that via the isomorphism in Theorem 4.1 of graded algebras \mathcal{R}/\mathcal{I} and $H^*(B_k; \mathbb{Z}_2)$, $\{x_{10}, \dots, x_{k0}\}$ corresponds to a basis over \mathbb{Z}_2 of $H^1(B_k; \mathbb{Z}_2)$. Thus by (4.9) it follows that $w_1(B_k) = 0$ if and only if

$$(4.17) \quad \sum_{i=1}^k \sum_{l_i=1}^{n_i} a_{j,l_i}^i \equiv 1 \pmod 2 \text{ for all } 1 \leq j \leq k.$$

Furthermore, since a necessary and sufficient condition for a compact connected differentiable manifold M to be orientable is $w_1(M) = 0$, the theorem follows. \square

REMARK 4.6. Note that Theorem 4.5 also follows as a direct consequence of [17, Theorem 1.7].

Theorem 4.7. *Let*

$$(4.18) \quad T_s := \sum_{\substack{1 \leq i \leq k \\ 1 \leq l_i < l'_i \leq n_i}} a_{s,l_i}^i a_{s,l'_i}^i + \sum_{\substack{1 \leq i < j \leq k \\ 1 \leq l_i \leq n_i \\ 1 \leq l_j \leq n_j}} a_{s,l_i}^i a_{s,l_j}^j$$

and

$$(4.19) \quad T_{rs} := \sum_{\substack{1 \leq i \leq k \\ 1 \leq l_i \leq n_i}} a_{r,l_i}^i a_{s,l_i}^i.$$

The orientable generalized real Bott manifold B_k admits a spin structure if and only if the following identities hold :

$$(4.20) \quad T_s \equiv 1 \pmod 2 \quad \text{for } 1 \leq s \leq l$$

$$(4.21) \quad T_{rs} \equiv 0 \pmod 2 \quad \text{for } 1 \leq r < s \leq l$$

$$(4.22) \quad T_{rs} + a_{r,1}^s \cdot (1 + T_s) \equiv 0 \pmod 2 \quad \text{for } 1 \leq r \leq k, l < s \leq k.$$

Proof. Using (4.16) we can simplify T'_s and T'_{rs} from Corollary 4.4 as follows,

$$(4.23) \quad T'_s = 1 + \sum_{\substack{1 \leq i \leq k \\ 1 \leq l_i < l'_i \leq n_i}} a_{s,l_i}^i a_{s,l'_i}^i + \sum_{\substack{1 \leq i < j \leq k \\ 1 \leq l_i \leq n_i \\ 1 \leq l_j \leq n_j}} a_{s,l_i}^i a_{s,l_j}^j = 1 + T_s$$

$$(4.24) \quad \begin{aligned} T'_{rs} &= 1 + \sum_{\substack{1 \leq i \leq k \\ 1 \leq l_i, l'_i \leq n_i \\ l_i \neq l'_i}} a_{r,l_i}^i a_{s,l'_i}^i + \sum_{\substack{1 \leq i < j \leq k \\ 1 \leq l_i \leq n_i \\ 1 \leq l_j \leq n_j}} a_{r,l_i}^i a_{s,l_j}^j \\ &= 1 + \sum_{\substack{1 \leq i \leq k \\ 1 \leq l_i \leq n_i}} a_{r,l_i}^i \cdot (1 + a_{s,l_i}^i) = 1 + 1 + T_{rs} = T_{rs} \end{aligned}$$

Using (4.23), (4.24) in (4.10), $w_2(B_k)$ can be identified with the class of the following term in \mathcal{R}/\mathcal{I} :

$$(4.25) \quad \sum_{s=1}^l (1 + T_s) \cdot x_{s0}^2 + \sum_{\substack{r,s=1 \\ r < s}}^l T_{rs} \cdot x_{r0} \cdot x_{s0} + \sum_{\substack{1 \leq r \leq k \\ l < s \leq k \\ r < s}} (T_{rs} + a_{r,1}^s \cdot (1 + T_s)) \cdot x_{r0} \cdot x_{s0}$$

By Proposition 4.1, as a \mathbb{Z}_2 -vector space, $H^2(B_k; \mathbb{Z}_2)$ is isomorphic to the subspace of \mathcal{R}/\mathcal{I} generated by the classes of x_{s0}^2 for $1 \leq s \leq k$ and $x_{r0} \cdot x_{s0}$ for $1 \leq r < s \leq k$. Further, since B_k is constructed as an iterated sequence of projective bundles, the mod 2 Poincaré polynomial of B_k has the following expression

$$(4.26) \quad P_t(B_k; \mathbb{Z}_2) = \prod_{i=1}^k P_t(\mathbb{P}^{n_i}; \mathbb{Z}_2) = \prod_{i=1}^k (1 + t + \dots + t^{n_i}).$$

It follows from (4.26) and (4.15) that $H^2(B_k; \mathbb{Z}_2)$ is freely generated by the classes of x_{s0}^2 for $1 \leq s \leq l$ and $x_{r0} \cdot x_{s0}$ for $1 \leq r < s \leq k$. Moreover, the necessary and sufficient condition for an orientable manifold M to admit a spin structure is $w_2(M) = 0$. Thus the theorem follows from (4.25). □

4.1. The case of a real Bott manifold.

Corollary 4.8. *When $l = 0$, the orientable real Bott manifold B_k is Spin if and only if*

$$(4.27) \quad T_{rs} + a_{r,1}^s \cdot (1 + T_s) \equiv 0 \pmod 2 \quad \text{for all } 1 \leq r < s \leq k$$

where T_{rs} and T_s are as in (4.19) and (4.18) respectively. Further, if we let $C := A - I$ (where A be the $k \times k$ upper triangular matrix associated to B_k and I is the $k \times k$ identity matrix) then (4.27) is equivalent to

$$(4.28) \quad \sum_{p=1}^k c_{r,p} c_{s,p} + c_{r,s} \cdot \sum_{\substack{p,q=1 \\ p < q}}^k c_{r,p} c_{s,q} \equiv 0 \pmod 2 \quad \text{for all } 1 \leq r < s \leq k$$

where $c_{i,j}$ are the entries of C . In particular, [8, Theorem 3.2] and, [9, Theorem 1.2] follow.

Proof. The first statement follows immediately from Theorem 4.7 by putting $l = 0$. The left hand side of (4.27) is

$$(4.29) \quad \sum_{p=1}^k a_{r,1}^p a_{s,1}^p + a_{r,1}^s \cdot (1 + \sum_{\substack{p,q=1 \\ p < q}} a_{s,1}^p a_{s,1}^q)$$

Since $c_{i,j} = a_{i,1}^j$ when $i \neq j$ and $c_{i,i} = a_{i,1}^i - 1 = 0$, (4.29) reduces to,

$$\sum_{\substack{p=1 \\ p \neq r,s}}^k c_{r,p} c_{s,p} + a_{r,1}^r a_{s,1}^r + a_{r,1}^s a_{s,1}^s + c_{r,s} \cdot \left(1 + \sum_{\substack{p,q=1 \\ p < q, p,q \neq s}}^k c_{s,p} c_{s,q} + a_{s,1}^s \cdot \sum_{\substack{i=1 \\ i \neq s}}^k a_{s,1}^i \right).$$

Further, using (4.16) the above expression can be rewritten as

$$\sum_{\substack{p=1 \\ p \neq r,s}}^k c_{r,p} c_{s,p} + 1 \cdot 0 + c_{r,s} \cdot 1 + c_{r,s} \cdot \left(1 + \sum_{\substack{p,q=1 \\ p < q, p,q \neq s}}^k c_{s,p} c_{s,q} + 1 \cdot (1 + a_{s,1}^s) \right).$$

This further can be simplified to

$$\sum_{p=1}^k c_{r,p} c_{s,p} + c_{r,s} \cdot \sum_{\substack{p,q=1 \\ p < q}}^k c_{r,p} c_{s,q}.$$

Thus (4.27) is equivalent to (4.28). Hence [8, Theorem 3.2] follows. This also implies [9, Theorem 1.2] by [8, Theorem 3.10, Corollary 3.11]. \square

4.2. Digraph interpretation. Choi, Masuda and Suh in [3] associate an acyclic directed graph to a real Bott manifold $M(A)$ by viewing the associated matrix A as the adjacency matrix of the digraph. In [3, Lemma 4.1], they give necessary and sufficient conditions on the digraph for $M(A)$ to be orientable and to admit a symplectic form. In [8, Theorem 4.5] the authors give a necessary and sufficient condition on the digraph for $M(A)$ to admit a spin structure.

One can similarly associate a *labeled multidigraph* to a generalized real Bott manifold. Let B_k be the generalized real Bott manifold associated to the vector matrix A from (2.7). Define D_{B_k} as the graph with k vertices $\{w_1, \dots, w_k\}$ and edges as follows : there is an edge from w_i to w_j with label $a_{i,l_j}^j \in \{0, 1\}$. Also for every i there are n_i loops (edges beginning and ending at the same vertex), each with label 1, based at w_i . The *out-degree* of a vertex w_i is the number of edges with label 1 that begin at w_i . It can be seen that B_k is orientable if and only if every vertex in D_{B_k} has odd out-degree. One can similarly make interpretations on D_{B_k} for B_k to admit a spin structure. The authors are trying to find such an interpretation that generalizes the one for real Bott manifolds.

EXAMPLE 4.9.

- (1) Let $P = \Delta^2 \times \Delta^1$. The possible 2-step generalized Bott manifolds over P correspond to the characteristic matrices given by

$$A = \left(\begin{array}{cc|c} a_{1,1}^1 & a_{1,2}^1 & a_{1,1}^2 \\ a_{2,1}^1 & a_{2,2}^1 & a_{2,1}^2 \end{array} \right) = \left(\begin{array}{cc|c} 1 & 1 & a_{1,1}^2 \\ 0 & 0 & 1 \end{array} \right)$$

where $a_{1,1}^2 = 0$ or 1 . From Theorem 4.5 $B_2 = M(A)$ is orientable if and only if $a_{1,1}^1 + a_{1,2}^1 + a_{1,1}^2 = 1 + 1 + a_{1,1}^2 \equiv 1 \pmod 2$ and $a_{2,1}^1 + a_{2,2}^1 + a_{2,1}^2 = 0 + 0 + 1 \equiv 1 \pmod 2$. Thus B_2 is orientable if and only if $a_{1,1}^2 = 1$.

Further, $T_1 = a_{1,1}^1 a_{1,2}^1 + a_{1,1}^1 a_{1,1}^2 + a_{1,2}^1 a_{1,1}^2 = 1 + 1 + 1 \equiv 1 \pmod 2$ and $T_{12} + a_{1,1}^2 \cdot (1 + T_2) = 1 + 1(1 + 0) \equiv 0 \pmod 2$. So by Theorem 4.7 B_2 is also spin if and only if $a_{1,1}^2 = 1$.

- (2) Let $P = \Delta^2 \times \Delta^2$. The possible 2-step generalized Bott manifolds over P correspond to the characteristic matrices given by

$$A = \left(\begin{array}{cc|cc} a_{1,1}^1 & a_{1,2}^1 & a_{1,1}^2 & a_{1,2}^2 \\ a_{2,1}^1 & a_{2,2}^1 & a_{2,1}^2 & a_{2,2}^2 \end{array} \right) = \left(\begin{array}{cc|cc} 1 & 1 & a_{1,1}^2 & a_{1,2}^2 \\ 0 & 0 & 1 & 1 \end{array} \right)$$

where $a_{1,1}^2, a_{1,2}^2 \in \{0, 1\}$. $B_2 = M(A)$ is orientable if and only if $a_{1,1}^2 = 1$ and $a_{1,2}^2 = 0$ or $a_{1,1}^2 = 0$ and $a_{1,2}^2 = 1$. In both these cases $T_{12} = a_{1,1}^2 + a_{1,2}^2 \equiv 1 \pmod{2}$ so that B_2 is not spin.

- (3) Let $P = \Delta^2 \times \Delta^1 \times \Delta^1$. The possible 3-step generalized Bott manifolds over P correspond to the characteristic matrices given by

$$A = \left(\begin{array}{cc|cc} a_{1,1}^1 & a_{1,2}^1 & a_{1,1}^2 & a_{1,1}^3 \\ a_{2,1}^1 & a_{2,2}^1 & a_{2,1}^2 & a_{2,1}^3 \\ a_{3,1}^1 & a_{3,2}^1 & a_{3,1}^2 & a_{3,1}^3 \end{array} \right) = \left(\begin{array}{cc|cc} 1 & 1 & a_{1,1}^2 & a_{1,1}^3 \\ 0 & 0 & 1 & a_{2,1}^3 \\ 0 & 0 & 0 & 1 \end{array} \right)$$

where $a_{1,1}^2, a_{1,1}^3, a_{2,1}^3 \in \{0, 1\}$. Then $B_3 = M(A)$ is orientable if and only if $(a_{1,1}^2, a_{1,1}^3, a_{2,1}^3) = (1, 0, 0)$ or $(a_{1,1}^2, a_{1,1}^3, a_{2,1}^3) = (0, 1, 0)$. In both cases we see that $T_1 = 1 + a_{1,1}^2 + a_{1,1}^3 + a_{2,1}^3 \equiv 0 \pmod{2}$ so that B_3 is not spin.

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