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# Retrial queues with limited number of retrials : Numerical Investigations

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**Abstract** Retrial queues are frequently observed in a real world system likewise call center or internet service industries. In this paper, a retrial queue system in which the number of retrials of each customer is limited by a finite number, say m is considered. That is, if a customer fails to enter the service facility at *m*th retrial, then the customer leaves the system without service. The effects of restricting the number of retrials are investigated numerically by using the algorithmic method and simulation experiments.

**Keywords** Retrial queue, number of retrials, generalized truncated system, simulation.

## 1 Introduction

Retrial queues are characterized by the feature of retrial phenomena that a customer who finds all the servers busy upon arrival may join the virtual group of blocked customers, called orbit and retry for service after some random amount of time. Retrial queues have been widely used to model the many practical situations arising from telephone systems, telecommunication networks and call center model. For a detailed overviews of main results and the bibliographical information about retrial queues, see [1, 3].

In this paper, we consider the retrial queue where the number of retrials of each customer is limited by a finite number, say $m$. That is, each customer is allowed to attempt the service request at most $m+1$ times including primary attempt by an external arrival and if a customer fails to enter the service facility at $m+1$st attempt, then the customer leaves the system without service.

The model considered in this paper has at least two important features in the literature of retrial queues. One is that our model can be considered as a retrial queue with impatient customers. Retrial queue with impatient customers can be described in terms of persistence function $\{H_j, j=1,2,\cdots\}$, where $H_j$ is the probability that after the $j$th attempt fails, a customer will make the $(j+1)$st one [3]. Our model is the case of $H_j = 1$ for $j \leq m$ and $H_j = 0$ for $j > m$. Most of papers about the retrial queue with impatient customers deal with the case of $H_2 = H_3 = \cdots = H$, that is, the probability of a customer

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reattempting after failure of a retrial does not depend on the number of previous retrials. Closed form solutions for the system have not been obtained except for a few special cases, for example, $M/M/1$ retrial queue with $H < 1$, see [3, Section 3.3]. For more about the literature retrial queues with impatient customers, we refer the bibliographical remarks of [3, Chapter 5] and the references in [8].

Another feature is that our model can be considered as a control scheme in retrial queue. It can be expected intuitively that the smaller the number of retrials permitted to a customer is, the more lost customers are and hence the system congestion is reduced. One way to reduce the system congestion from this intuition is to restrict the number of retrials. This type of control scheme can be used in wireless LAN protocol e.g. [6]. However, it is very difficult to find the literature of the mathematical analysis or even numerical investigation of the system.

The objectives of this paper are to investigate numerically the effects of restricting the number of retrials for various parameters and to give an insight to the behavior of the system. A generalized truncation method in [5] and simulation are used for numerical results, and an analytical result is also given for a special case.

The paper is organized as follows. The mathematical model is described in Section 2. In Section 3 the balance equation for system behavior and an analytical result are given. The computing procedures are described in Section 4. Numerical results are are presented in Sections 5. Finally, conclusions are given in Section 6.

# 2 The Model

We consider an $M/M/s/s$ retrial queue which consists of an orbit with infinite capacity and a service facility with $s$ servers and no waiting space. Service times of customers are independent of each other and have a common exponential distribution with parameter $\mu$. Customers arrive from outside according to a Poisson process with rate $\lambda$. When an arriving customer finds that all the servers are busy, the customer joins orbit and repeats its request after an exponential amount of time. The time interval between retrials of a customer from orbit is called the inter-retrial time. We assume that the number of retrials of a customer from orbit is limited by $m$ and denote by $\Sigma^{(m)}$ the system. That is, each customer in $\Sigma^{(m)}$ is allowed to attempt the service request at most $m+1$ times including primary attempt by an external arrival and if a customer fails to enter the service facility at the $(m+1)$st attempt (this is the $m$th retrial), then the customer leaves the system without service.

The system $\Sigma^{(m)}$ can be viewed as a queueing network with $m+1$ nodes as shown in Figure 1 for $m = 3$. Service facility is denoted by node 0 and the customers who have been blocked to enter the service facility $k$ times are in node $k$, $k = 1,2,\cdots,m$. Thus if a customer at node $k$, $(1 \leq k < m)$ fails again to get into the service facility, then the customer enters node $k+1$ while the customer at node $m$ leaves the system forever and is lost upon failure of attempt to get service. The customers at node $k$ behave independently of each other, and the time a customer spends at node $k$ is assumed to be exponential with rate $\gamma_k > 0$, $k = 1,2,\cdots,m$. Thus when there are $n$ customers at node $k$, the retrial rate is $n\gamma_k$.

Let $X_k^{(m)}(t)$ be the number of customers at node $k$, $k = 0,1,2,\cdots,m$ in $\Sigma^{(m)}$ at time $t$. Then the $(m+1)$-dimensional stochastic process $\boldsymbol{X}^{(m)} = \{\boldsymbol{X}^{(m)}(t), t \geq 0\}$ with $\boldsymbol{X}^{(m)}(t) =$

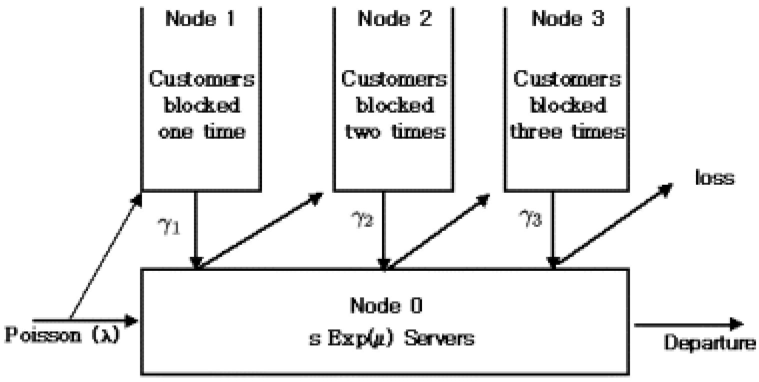


Figure 1: Schematic diagram of $\Sigma^{(3)}$ system

$(X_0^{(m)}(t),\cdots,X_m^{(m)}(t))$ is a Markov chain on the state space $\mathscr{S}=\{0,1,2,\cdots,s\}\times\mathbb{Z}_+^m$, where $\mathbb{Z}_+=\{0,1,2,\cdots\}$ is the set of nonnegative integers. The capacity of node 0 (the service facility) is finite and the customers at node $k$, $1\le k\le m$ behave like those in the queue with infinite servers. Thus $\Sigma^{(m)}$ is always stable.

# 3 Stationary distribution

Let $X_k^{(m)}$ be the stationary version of $\{X_k^{(m)}(t),\ t\ge 0\}$, $k=0,1,2,\cdots,m$. For simplicity, we write $X_k$ instead of $X_k^{(m)}$ by omitting the index $m$ in the following if it is not confused in the context.

Let

$$P(\boldsymbol{n})=\mathbb{P}(X_k=n_k,k=0,1,2,\cdots,m),$$

where $\boldsymbol{n}=(n_0,n_1,\cdots,n_m)\in\mathscr{S}$. Then the balance equations are given as follows: for $\boldsymbol{n}\in\mathscr{S}$,

$$\left(\lambda+n_0\mu+\sum_{k=1}^m n_k\gamma_k\right)P(\boldsymbol{n})=\lambda P(\boldsymbol{n}-\boldsymbol{e}_0)+(n_0+1)\mu P(\boldsymbol{n}+\boldsymbol{e}_0)$$
$$+\sum_{k=1}^m(n_k+1)\gamma_k P(\boldsymbol{n}-\boldsymbol{e}_0+\boldsymbol{e}_k),\ 0\le n_0<s, \tag{1}$$
$$\left(\lambda+s\mu+\sum_{k=1}^m n_k\gamma_k\right)P(\boldsymbol{n})=\lambda P(\boldsymbol{n}-\boldsymbol{e}_0)+\sum_{k=1}^m(n_k+1)\gamma_k P(\boldsymbol{n}-\boldsymbol{e}_0+\boldsymbol{e}_k)$$
$$+\lambda P(\boldsymbol{n}-\boldsymbol{e}_1)+\sum_{k=1}^m(n_k+1)\gamma_k P(\boldsymbol{n}+\boldsymbol{e}_k-\boldsymbol{e}_{k+1}),\ n_0=s, \tag{2}$$

where $\boldsymbol{e}_j$, $(j=0,1,\cdots,m)$ is the $(m+1)$-dimensional vector whose $j$th component (beginning from 0th component) is 1 and others are all 0 and $\boldsymbol{e}_{m+1}=(0,\cdots,0)\in\mathscr{S}$.

The equations (1) and (2) are too complex to obtain the analytic solution for general case. Now we present the solution of the equation for $s = 1$ and $m = 1$. In this special case, the balance equations (1) and (2) become

$$\begin{aligned}(\lambda + j\gamma)p(0, j) &= \mu p(1, j),\ j \geq 0, &(3)\\ (\lambda + \mu + j\gamma)p(1, j) &= \lambda p(1, j-1) + \lambda p(0, j)\\ &+ (j+1)\gamma(p(0, j+1) + p(1, j+1)),\ j \geq 0. &(4)\end{aligned}$$

Combining (3) with (4), we have that

$$\begin{aligned}&[(\lambda(\lambda + j\gamma) + j\gamma(\lambda + \mu + j\gamma)]p(0, j) = (\lambda(\lambda + (j-1)\gamma)p(0, j-1)\\ +\ &(j+1)\gamma(\lambda + \mu + (j+1)\gamma)p(0, j+1),\ j \geq 0. \qquad (5)\end{aligned}$$

Thus

$$\begin{aligned}p(0, j) &= p(0,0)\prod_{i=1}^{j}\left(\frac{\lambda(\lambda + (i-1)\gamma)}{i\gamma(\lambda + \mu + i\gamma)}\right)\\ &= p(0,0)\left(\frac{\lambda + \mu}{\lambda + j\gamma}\right)\frac{1}{j!}\left(\frac{\lambda}{\gamma}\right)^{j}\prod_{i=0}^{j}\left(\frac{\frac{\lambda}{\gamma} + i}{\frac{\lambda+\mu}{\gamma} + i}\right),\ j \geq 0\end{aligned}$$

and hence from (3) that

$$p(1, j) = p(0,0)\left(\frac{\lambda + \mu}{\mu}\right)\frac{1}{j!}\left(\frac{\lambda}{\gamma}\right)^{j}\prod_{i=0}^{j}\left(\frac{\frac{\lambda}{\gamma} + i}{\frac{\lambda+\mu}{\gamma} + i}\right),\ j \geq 0.$$

Thus

$$p(0,0)^{-1} = \frac{\lambda + \mu}{\mu}\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{\lambda}{\gamma}\right)^{j}\prod_{i=0}^{j-1}\left(\frac{\frac{\lambda}{\gamma} + i}{\frac{\lambda+\mu}{\gamma} + i}\right).$$

# 4 Computing Procedure

## 4.1 Matrix Analytic method

Consider a continuous time Markov chain $\boldsymbol{X} = \{(X_0(t), X_1(t)), t \geq 0\}$, called level dependent quasi-birth-and-death (LDQBD) process on the state space $\{(i, j) : i \geq 0,\ 0 \leq j \leq K_i\}$ with generator of the form

$$Q = \begin{pmatrix} B_0 & A_0 & & & \\ C_1 & B_1 & A_1 & & \\ & C_2 & B_2 & A_2 & \\ & & C_3 & B_3 & \cdots \\ & & & \vdots & \end{pmatrix}, \qquad (6)$$

where $A_i$, $B_i$ and $C_i$ are the matrices of $K_i \times K_{i+1}$, $K_i \times K_i$ and $K_i \times K_{i-1}$, respectively and $K_i = K$, $i \geq i_0$ for some $i_0 > 0$. Note that $A_i$, $B_i$ and $C_i$ are square matrices of size $K$ for $i > i_0$.

We assume that $\boldsymbol{X}$ is positive recurrent and write the stationary distribution $\boldsymbol{x}$ of Q in the block partitioned form $\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2, \cdots)$ with $\boldsymbol{x}_i = (x_{i0}, x_{i1}, \cdots, x_{iK_i})$, $i \geq 0$. For computing $\boldsymbol{x}$, we use the generalized truncation method proposed in Neuts and Rao [5] with some modifications. The first step of the approximation is to modify the infinitesimal generator $Q$ to $Q_N$ by letting $A_i = A_N$, $B_i = B_N$, $C_i = C_N$ for $i \geq N$, where $N > i_0$ is a fixed positive integer. The second step is to find the stationary distribution $\boldsymbol{y} = (\boldsymbol{y}_0, \boldsymbol{y}_1, \cdots)$ of $Q_N$ and to increase $N$ until the individual elements of $\boldsymbol{y}$ do not change significantly. Finally, approximate $\boldsymbol{x}$ by $\boldsymbol{y}$.

For description of $\boldsymbol{y}$, we need some matrices. For a fixed $N$, let $R$ be the $K \times K$ matrix that is the minimal nonnegative solution of the equation

$$A_N + RB_N + R^2 C_N = 0 \tag{7}$$

and $R^{(i)}$, $0 \leq i \leq N-1$ be the sequence of $K_i \times K_{i+1}$ matrices which are the nonnegative solutions to the system of equations

$$A_i + R^{(i)} B_{i+1} + R^{(i)} R^{(i+1)} C_{i+1} = 0 \tag{8}$$

with $R^{(N)} = R$. The following proposition can be obtained by combining the results in Bright and Taylor [2] and Neuts [4].

**Proposition 1.** *The stationary distribution* $\mathbf{y}$ *of* $Q_N$ *is given by*

$$\boldsymbol{y}_k = \begin{cases} \boldsymbol{y}_0 R[0,k-1], & 1 \leq k \leq N, \\ \boldsymbol{y}_0 R[0,N-1] R^{k-N}, & k \geq N+1, \end{cases} \tag{9}$$

*where* $R[0,k] = R^{(0)} R^{(1)} \cdots R^{(k)}$. *The vector* $\mathbf{y}_0$ *is the unique solution of the equation*

$$\boldsymbol{y}_0 (B_0 + R^{(0)} C_1) = 0$$

*with the normalizing condition*

$$\boldsymbol{y}_0 \left( I + \sum_{k=1}^{N} R[1,k-1] + R[1,N-1](I-R)^{-1} \right) \mathbf{e} = 1,$$

*where* $\mathbf{e}$ *is the column vector whose components are all 1.*

There are many algorithms for computing the matrices $R$ and $R^{(i)}$, e.g. see [7] and we omit the computing procedure.

Algorithm for $m = 1$. In this case, the stochastic process $\boldsymbol{X}^{(1)} = \{(X_1^{(1)}(t), X_0^{(1)}(t)), t \geq 0\}$ is an LDQBD process with generator $Q^{(1)}$ that is the same form as $Q$ in (6). Thus the procedure described in the previous subsection can be applied to this special case. Let $A^{(1)}$, $B_k^{(1)}$ and $C_k^{(1)}$ be the matrix components of $Q^{(1)}$ corresponding to $A_k$, $B_k$ and $C_k$ in $Q$, respectively. Then $A^{(1)}$, $B_n^{(1)}$ and $C_n^{(1)}$ are square matrices of order $s+1$ and the $(i,j)$-components $(0 \leq i, j \leq s)$ are given by

$$[A^{(1)}]_{ij} = \begin{cases} \lambda, & i = j = s \\ 0, & \text{otherwise} \end{cases}, \quad [C_n^{(1)}]_{ij} = \begin{cases} n\gamma, & j = i+1, 0 \leq i \leq s-1 \\ n\gamma, & i = j = s, \\ 0, & \text{otherwise} \end{cases}$$

and the matrix $B_n^{(1)}$ is the tridiagonal matrix whose diagonal vector $\mathbf{d}_n$, upper diagonal vector $\mathbf{u}_n$ and lower diagonal vector $\mathbf{l}_n$ are given as follows

$$\mathbf{d}_n = [\Lambda_{n,0}, \Lambda_{n,1}, \cdots, \Lambda_{n,s}], \mathbf{u}_n = [\lambda, \lambda, \cdots, \lambda], \mathbf{l}_n = [\mu, 2\mu, \cdots, s\mu],$$

where $\Lambda_{n,k} = -(\lambda + \gamma n + k\mu)$, $n \geq 0$.

Algorithm for $m = 2$. For a computation of the stationary distribution in $\Sigma^{(2)}$, we combine the direct truncation method and generalized truncation method. First, consider a truncated system in which the capacity of node 2 is $J$ and denote it by $\Sigma_J^{(2)}$. If a customer at node 1 is blocked to enter node 0 when there are $J$ customers at node 2, then the customer is lost. Let $X_i^J(t)$, $i = 0,1,2$ be the number of customers at node $i$ in $\Sigma_J^{(2)}$ at time $t$ and define $Y_0^J(t) = X_0^J(t)$, $Y_1^J(t) = X_1^J(t) + X_2^J(t)$, $Y_2^J(t) = X_2^J(t)$, $(t \geq 0)$. Then the stochastic process $\boldsymbol{Y}^J = \{(Y_1^J(t), Y_0^J(t), Y_2^J(t)), t \geq 0\}$ is an LDQBD process on the state space $\mathscr{S}_J = \{(n,i,j) : n \geq 0,\ 0 \leq i \leq s,\ 0 \leq j \leq J_n\}$, where $J_n = \min(J,n)$. Writing the state in lexicographic order, the generator $Q^{(2)}$ of $\boldsymbol{Y}^J$ is the same form as $Q$ in (6). Let $A^{(2)}$, $B_k^{(2)}$ and $C_k^{(2)}$ be the matrix components of $Q^{(2)}$ corresponding to $A_k$, $B_k$ and $C_k$ in $Q$, respectively. Then $B_n^{(2)}$ is the square matrix of order $(s+1)(J_n+1)$, $(n \geq 0)$ and the sizes of $A_n^{(2)}$ and $C_n^{(2)}$ are $((s+1)(J_n+1)) \times ((s+1)(J_{n+1}+1))$ and $((s+1)(J_n+1)) \times ((s+1)(J_{n-1}+1))$, respectively.

Writing the matrices $A_n^{(2)} = (A_{n:i,j}^{(2)})_{0 \leq i,j \leq s}$ in the block matrix form, where $A_{n:i,j}^{(2)}$, $(0 \leq i,j \leq s)$ is the $(J_n+1) \times (J_{n+1}+1)$ matrix, we can easily see that $A_{n:i,j}^{(2)} = O$ for $i \neq s$ or $j \neq s$ and

$$A_{n:s,s}^{(2)} = \begin{cases} (\lambda I_{n+1} : O), & n \leq J-1, \\ \lambda I_{J+1}, & n \geq J \end{cases}$$

and $I_k$ is the identity matrix of order $k$.

Analogously, write $B_n^{(2)} = (B_{n:i,j}^{(2)})_{0 \leq i,j \leq s}$ in the block partitioned form with $(J_n+1) \times (J_n+1)$ matrix $B_{n:i,j}^{(2)}$, $0 \leq i,j \leq s$. Then it can be seen that $B_n^{(2)}$ is a block tridiagonal matrix whose lower and upper diagonal blocks are, respectively given by

$$B_{n:i,i-1}^{(2)} = \mu_i I_{J_n+1},\ 0 \leq i \leq s,$$
$$B_{n:i,i+1}^{(2)} = \lambda I_{J_n+1},\ 0 \leq i \leq s-1$$

and the diagonal blocks $B_{n:i,i}^{(2)}$, $0 \leq i \leq s-1$ are diagonal matrices

$$B_{n:i,i}^{(2)} = \mathrm{diag}(b_{n;i,0}^{(2)}, b_{n;i,1}^{(2)}, \cdots, b_{n;i,J_n}^{(2)}),\ 0 \leq i \leq s-1$$

with

$$b_{n;i,k}^{(2)} = -(\lambda + \mu_i + \gamma_1(n-k) + \gamma_2 k),\ 0 \leq k \leq J_n$$

and

$$[B_{n:s,s}^{(2)}]_{ij} = \begin{cases} -(\lambda + \mu_s + \gamma_1(n-i) + \gamma_2 i), & 0 \leq i = j \leq J_n, \\ (n-i)\gamma_1, & j = i+1, 0 \leq i \leq J_n - 1, \\ 0, & \text{otehrwise.} \end{cases}$$

Writing the matrix $C_n^{(2)} = (C_{n:i,j}^{(2)})_{0\le i,j\le s}$ in the block partitioned form, we can see that $C_{n:i,j}^{(2)}$ is the $(J_n+1)\times(J_{n-1}+1)$ matrices with $C_{n:i,j}^{(2)} = 0,\ j \ne i+1$ and

$$[C_{n:i,i+1}^{(2)}]_{jj'} = \begin{cases} (n-j)\gamma_1, & j' = j,\ 0\le j\le J_n, \\ j\gamma_2, & j' = j-1,\ 1\le j\le J_n, \quad \text{for } 0\le i\le s-1, \\ 0, & \text{otherwise,} \end{cases}$$

$$[C_{n:s,s}^{(2)}]_{jj'} = \begin{cases} j\gamma_2, & j' = j-1,\ 1\le j\le J_n, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for } n\le J,$$

$$[C_{n:s,s}^{(2)}]_{jj'} = \begin{cases} j\gamma_2, & j' = j-1,\ 1\le j\le J_n, \\ (n-J)\gamma_1, & j' = j = J, \\ 0, & \text{otherwise.} \end{cases} \quad \text{for } n\ge J+1.$$

Let $\boldsymbol{y}(J,N)$ be the stationary distribution of $Q_{J,N}^{(2)}$ which is obtained from $Q_J^{(2)}$ by setting $A_k^{(2)} = A_N^{(2)}$, $B_k^{(2)} = B_N^{(2)}$ and $C_k^{(2)} = C_N^{(2)}$ for $k\ge N$. Now we apply the computing procedure described in this section with slighted modification for $\boldsymbol{y}(J,N)$ that will be used for an approximation of the stationary distribution $\boldsymbol{x}_J^{(2)}$ of $Q_J^{(2)}$. Once $\boldsymbol{y} = \boldsymbol{y}(J,N)$ is obtained, the distributions of $X_k^{(2)}$, $k = 0,1,2$ can be obtained as follows. Note from the definition of $Y_k^J$, $k = 0,1,2$ that

$$y_{nij} = \mathbb{P}(X_0 = i, X_1+X_2 = n, X_2 = j),\ n\ge 0,\ 0\le i\le s,\ 0\le j\le J_n.$$

Thus

$$\mathbb{P}(X_0 = i) = \textstyle\sum_{n=0}^{\infty}\sum_{j=0}^{J_n} y_{nij}, \quad \mathbb{P}(X_1 = k) = \sum_{i=0}^{s}\sum_{n=k}^{k+J} y_{n,i,n-k},$$
$$\mathbb{P}(X_2 = k) = \textstyle\sum_{n=j}^{\infty}\sum_{i=0}^{s} y_{nij}, \quad \mathbb{P}(X_1+X_2 = n) = \sum_{i=0}^{s}\sum_{j=0}^{J_n} y_{nij}.$$

Algorithm for $m = \infty$. Consider the system with $m = \infty$ and $\gamma_k = \gamma$, $k = 1,2,\cdots$ and denote it by $\Sigma^*$. This system is the ordinary $M/M/s/s$ retrial queue. A very effective algorithm for computing the stationary distribution of $M/M/s/K$ retrial queue can be found in Shin and Moon [7] and details are omitted here.

## 4.2 Simulation

The matrix analytic method proposed in the case of $m = 2$ can be applied to the case of $m\ge 3$ by truncating the states $X_k^{(m)}(t)$ by $M_k$, $k = 2,3,\cdots,m$. Then the truncated process forms LDQBD process. However, the size of block matrix in the diagonal block of the generator can be $(s+1)(M_2+1)\cdots(M_m+1)$. So the size of the matrix that is required to be inverted increases rapidly as $m$ increases and the matrix analytic method for large $m$ does not seem to be appropriate for large $m$. Simulation method can be considered as an alternative for this general case.

# 5 Numerical Results

In this section, some numerical results are presented for investigating the effects of the parameters to the system behavior. We consider the following system characteristics :

**Table 1.** Comparisons of simulations with exact results for $s = 5$, $\mu = 1.0$, $\rho = 0.75$

| | | $P_B$ | | $L_0$ | | $L_{\text{Orbit}}$ | |
|---|---|---|---|---|---|---|---|
| $m$ | $\gamma$ | Exact | Sim. (C.I.) | Exact | Sim. (C.I.) | Exact | Sim. (C.I.) |
| 1 | 1.875 | 0.237 | 0.237 (0.003) | 3.308 | 3.309 (0.014) | 0.474 | 0.476 (0.013) |
| | 3.750 | 0.225 | 0.225 (0.002) | 3.247 | 3.262 (0.009) | 0.225 | 0.227 (0.003) |
| | 7.500 | 0.212 | 0.213 (0.002) | 3.193 | 3.199 (0.007) | 0.106 | 0.106 (0.001) |
| 2 | 1.875 | 0.277 | 0.277 (0.002) | 3.442 | 3.441 (0.008) | 0.853 | 0.853 (0.013) |
| | 3.750 | 0.261 | 0.264 (0.003) | 3.358 | 3.371 (0.010) | 0.425 | 0.430 (0.007) |
| | 7.500 | 0.241 | 0.241 (0.003) | 3.276 | 3.275 (0.010) | 0.207 | 0.206 (0.004) |
| $\infty^*$ | 1.875 | 0.388 | 0.384 (0.007) | 3.750 | 3.742 (0.022) | 2.508 | 2.487 (0.080) |
| | 3.75 | 0.407 | 0.401 (0.008) | 3.750 | 3.737 (0.024) | 1.975 | 1.836 (0.077) |
| | 7.5 | 0.425 | 0.411 (0.009) | 3.750 | 3.719 (0.025) | 1.695 | 1.415 (0.057) |

$^*$ The simulation results are for $m = 30$, C.I. : half length of 95% confidence interval

- $P_B = \mathbb{P}(X_0 = s)$ : blocking probability that an arriving customer finds that all the servers are busy
- $P_L = \frac{\gamma}{\lambda}\mathbb{E}(X_m; X_0 = s)$ : loss probability that the customer leaves the system without service
- $L_0 = \mathbb{E}(X_0)$ : mean number of customers at service facility
- $L_{\text{Orbit}} = \mathbb{E}\left(\sum_{k=1}^{m} X_k\right)$ : mean number of customers in orbit
- $\mathbb{E}(W) = \frac{1}{\lambda}(L_0 + L_{\text{Orbit}})$ : mean sojourn time

In Tables 1-3, the numerical results for $m = 1,2$ and $m = \infty$ are obtained by using the matrix analytic method and for $3 \le m < \infty$, simulation is used. Simulation models are developed with ARENA. Simulation run time is set to 11,000 minutes including 1,000 minutes of warm-up period, where the expected value of service time is one minutes. Ten replications are conducted for each case and the average value and the half length of 95% confidence interval are obtained. In Table 1, simulation results are compared with the exact results for correctness of simulation for $s = 5$, $\mu = 1.0$, $\rho = \frac{\lambda}{s\mu} = 0.75$ and three cases of retrial rate $\gamma = 0.5\lambda = 1.875$, $\lambda = 3.75$ and $2\lambda = 7.5$.

The parameters used in Tables 2-3 are as follows: $s = 5$, $\mu = 1.0$, the arrival rate $\lambda$ is given by $\lambda = s\rho$, $\rho = 0.5, 0.95$ and 1.5 and for each $\lambda$, four cases of retrial rates $\gamma = 0.1\lambda$, $\lambda$, $2\lambda$ and $10\lambda$ are considered.

Numerical results show that as $m$ increases, $P_B$, $L_0$ and $L_{\text{Orbit}}$ are increasing but the blocking probability $P_L$ is decreasing. Furthermore, as the retrial rate $\gamma$ increases, $L_{\text{Orbit}}$ is increasing but $P_L$ is decreasing.

**Table 2.** System characteristics for $s = 5$, $\mu = 1.0$

| $\rho$ | $m$ | $\gamma$ | $P_B$ | $P_L$ | $L_0$ | $L_{\text{Orbit}}$ | $\mathbb{E}(W)$ |
|---|---|---|---|---|---|---|---|
| 0.5 | 1 | 0.25 | 0.087 | 0.015 | 2.464 | 0.874 | 1.335 |
| | | 2.5 | 0.085 | 0.040 | 2.400 | 0.085 | 0.994 |
| | | 5. | 0.082 | 0.049 | 2.377 | 0.041 | 0.967 |
| | | 25. | 0.075 | 0.063 | 2.340 | 0.007 | 0.940 |
| | 5* | 0.25 | 0.093 | 0.000 | 2.507 | 1.135 | 1.457 |
| | | 2.5 | 0.104 | 0.005 | 2.483 | 0.211 | 1.081 |
| | | 5.0 | 0.105 | 0.013 | 2.464 | 0.133 | 1.041 |
| | | 25. | 0.098 | 0.031 | 2.424 | 0.064 | 0.997 |
| | 10* | 0.25 | 0.093 | 0.000 | 2.507 | 1.135 | 1.457 |
| | | 2.5 | 0.107 | 0.001 | 2.497 | 0.234 | 1.096 |
| | | 5.0 | 0.112 | 0.003 | 2.494 | 0.171 | 1.067 |
| | | 25. | 0.110 | 0.015 | 2.467 | 0.102 | 1.028 |
| | $\infty$ | 0.25 | 0.093 | 0.000 | 2.500 | 1.131 | 1.452 |
| | | 2.5 | 0.109 | 0.000 | 2.500 | 0.244 | 1.098 |
| | | 5. | 0.115 | 0.000 | 2.500 | 0.190 | 1.076 |
| | | 25. | 0.125 | 0.000 | 2.500 | 0.144 | 1.057 |

* simulation results.

**Table 3.** System characteristics for $s = 5$, $\mu = 1.0$

| $\rho$ | $m$ | $\gamma$ | $P_B$ | $P_L$ | $L_0$ | $L_{\text{Orbit}}$ | $\mathbb{E}(W)$ |
|---|---|---|---|---|---|---|---|
| 0.95 | 1 | 0.475 | 0.391 | 0.179 | 3.909 | 3.901 | 1.646 |
| | | 4.75 | 0.335 | 0.221 | 3.685 | 0.334 | 0.849 |
| | | 9.5 | 0.317 | 0.239 | 3.631 | 0.159 | 0.797 |
| | | 47.5 | 0.280 | 0.256 | 3.530 | 0.028 | 0.751 |
| | 10* | 0.475 | 0.690 | 0.039 | 4.576 | 25.002 | 6.213 |
| | | 4.75 | 0.593 | 0.095 | 4.285 | 2.847 | 1.503 |
| | | 9.5 | 0.545 | 0.129 | 4.149 | 1.508 | 1.187 |
| | | 47.5 | 0.394 | 0.206 | 3.775 | 0.298 | 0.857 |
| | 30* | 0.475 | 0.779 | 0.006 | 4.718 | 45.01 | 10.46 |
| | | 4.75 | 0.726 | 0.041 | 4.545 | 6.566 | 2.340 |
| | | 9.5 | 0.684 | 0.066 | 4.431 | 3.715 | 1.715 |
| | | 47.5 | 0.525 | 0.146 | 4.050 | 0.880 | 1.038 |
| | $\infty$ | 0.475 | 0.792 | 0.000 | 4.736 | 50.71 | 11.67 |
| | | 4.75 | 0.820 | 0.000 | 4.704 | 16.05 | 4.369 |
| | | 9.5 | 0.820 | 0.000 | 4.658 | 13.05 | 3.729 |
| | | 47.5 | 0.749 | 0.000 | 4.252 | 7.534 | 2.481 |

* simulation results.

**Table 4.** System characteristics for $s = 5$, $\mu = 1.0$

| $\rho$ | $m$ | $\gamma$ | $P_B$ | $P_L$ | $L_0$ | $L_{\text{Orbit}}$ | $\mathbb{E}(W)$ |
|---|---|---|---|---|---|---|---|
| 1.5 | 1 | 0.75 | 0.620 | 0.405 | 4.464 | 6.196 | 1.421 |
| | | 7.5 | 0.552 | 0.427 | 4.298 | 0.552 | 0.647 |
| | | 15. | 0.522 | 0.436 | 4.232 | 0.261 | 0.599 |
| | | 75. | 0.474 | 0.448 | 4.139 | 0.047 | 0.558 |
| | 10* | 0.75 | 0.906 | 0.348 | 4.898 | 61.04 | 8.784 |
| | | 7.5 | 0.865 | 0.357 | 4.832 | 5.944 | 1.434 |
| | | 15. | 0.810 | 0.368 | 4.731 | 2.878 | 1.016 |
| | | 75. | 0.622 | 0.414 | 4.389 | 0.521 | 0.654 |
| | 30* | 0.75 | 0.965 | 0.339 | 4.964 | 182.6 | 24.99 |
| | | 7.5 | 0.959 | 0.340 | 4.957 | 18.05 | 3.065 |
| | | 15.0 | 0.943 | 0.341 | 4.932 | 8.846 | 1.842 |
| | | 75. | 0.790 | 0.378 | 4.669 | 1.676 | 0.845 |
| | 50* | 0.75 | 0.979 | 0.336 | 4.978 | 303.4 | 41.15 |
| | | 7.5 | 0.977 | 0.336 | 4.976 | 30.08 | 4.683 |
| | | 15.0 | 0.972 | 0.337 | 4.970 | 14.96 | 2.654 |
| | | 75. | 0.867 | 0.362 | 4.794 | 2.864 | 1.020 |

* simulation results.

# 6 Conclusions

In this work, the effects of restricting the number of retrials are investigated numerically and we show that some performance measures severely depend on the parameter $m$, the maximum number of retrial permitted to a customer. Our study is the first step for further research related with the model considered in this paper and is to give an insight to estimating even roughly the behavior of the system. There are a number of further research issues that remain to be addressed. One of them is to develop a simple approximation method which can be applied to the general system. Our results may be used for error analysis of the approximation. In practical situation such as wire less local area networks, it is important to determine the threshold value $m$ under the given loss probability $P_L$ or blocking probability $P_B$. It is necessary to develop an effective way for determining the value $m$ as a function of $P_L$ or $P_B$.

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