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Return times and conjugates of an antiperiodic transformation

STEVE ALPERN†
From the Department of Mathematics, London School of Economics, England

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Abstract. Denote by G the group of all μ-preserving bijections of a Lebesgue probability space (X, Σ, μ) and by C the conjugacy class of an antiperiodic transformation σ in G. We present several new results concerning the denseness of C in G with respect to various topologies. One of these asserts that given any weakly mixing transformation τ in G and any F with $\mu(F) < 1$, there is a transformation in C which agrees with τ a.e. on F.

1. *Introduction and statement of results*

In this paper we consider two related questions concerning an arbitrary antiperiodic automorphism σ of a Lebesgue probability space (X, Σ, μ). The first question concerns when we can find a sweep-out set for σ with a specified distribution of return times. The second asks in which ways automorphisms of X can be approximated by conjugates of σ.

Denote by $G = G(X, \Sigma, \mu)$ the group of all automorphisms (μ-preserving bijections) of (X, Σ, μ), and by $C(\sigma)$ the class of all conjugates, $\theta^{-1}\sigma\theta$, $\theta \in G$, of the antiperiodic automorphism σ. The first question seeks a sweep-out set B such that the relative distribution of return times to B is a given probability distribution $\pi = (\pi_1, \pi_2, \ldots)$. Suppose $d > 1$ divides all the k for which $\pi_k > 0$. Then for any such set B, and any m which is not a multiple of d, we would have

$$\mu(B \cap \sigma^m B) = 0.$$

So such a set B cannot exist in general, for example when σ is mixing. However, if no such d exists for π, then the required set B can always be found (by taking $B = \bigcup_{k=1}^{\infty} P_{k,1}$ in the following).

THEOREM 1. *Let $\sigma \in G$ be antiperiodic and let $\pi = (\pi_1, \pi_2, \ldots)$ be any denumerable probability distribution such that the ks with $\pi_k > 0$ are relatively prime. Then there is a partition $\{P_{k,i}\}$, $k = 1, \ldots, \infty$, $i = 1, \ldots, k$ of X satisfying*

(i) $P_{k,i} = \sigma^{i-1}(P_{k,1})$; *and*

(ii) $\mu\left(\bigcup_{i=1}^{k} P_{k,i}\right) = \pi_k$.

† Address for correspondence: Dr Steve Alpern, Department of Mathematics, London School of Economics, Houghton Street, London, England.

Theorem 1 generalizes corollary 2 of [**1**] which is restricted to finite dimensional distributions $\pi=(\pi_1,\ldots,\pi_n)$, and also Rohlin's lemma which is covered by $\pi_1=\varepsilon$ and $\pi_n=1-\varepsilon$. Basically, theorem 1 says that an antiperiodic automorphism can be represented by stacks $(\bigcup_{i=1}^k P_{k,i})$ of given heights (k) and given measures (π_k) as long as the heights are relatively prime. The proof of theorem 1 incorporates a substantial simplification for which I wish to thank the referee.

We turn now to the second question: the approximation of an automorphism $\tau\in G$ by a conjugate $\hat\sigma$ of σ $(\hat\sigma\in C(\sigma))$. We list below four types of approximation, which we shall then discuss in turn.

Approximation problems. Let $\tau,\sigma\in G$ with σ antiperiodic, $\varepsilon>0$ and $F, A_m\in\Sigma$, $m=1,\ldots,M$ be given. We seek a $\hat\sigma\in C(\sigma)$ satisfying:

(P1) $\mu(\hat\sigma(A_m)\triangle\tau(A_m))\le\varepsilon,\quad m=1,\ldots,M;$

(P2) $\mu(\hat\sigma(A_m)\triangle\tau(A_m))=0,\quad m=1,\ldots,M;$

(P3) $\mu\{x\in X:\hat\sigma(x)\neq\tau(x)\}\le\varepsilon;$

(P4) $\mu\{x\in F:\hat\sigma(x)\neq\tau(x)\}=0.$

The problem P1 was first studied by Halmos [**6**] and [**7**], who showed that P1 can be solved without any restrictions on τ or the A_m. That is, $C(\sigma)$ is dense in G in the coarse, or weak topology [**8**, p. 77].

We next consider problem P2. First observe that, if $\hat\sigma$ satisfies P2, then $\hat\sigma(A)=\tau(A)$ (equality is always up to sets of measure 0) for all A belonging to the subalgebra $\mathcal{A}$ of Σ generated by $A_1,\ldots,A_M$. Now suppose that the set map $\tau/\mathcal{A}$ has a non-trivial periodic point, that is, an $A\in\mathcal{A}-\{\emptyset,x\}$ with $\tau^i(A)\in\mathcal{A}$ for $i=1,\ldots,k$, and $\tau^k(A)=A$. Then for any $\hat\sigma$ satisfying P2 we have $\hat\sigma^k(A)=\tau^k(A)=A$, so that $\hat\sigma$, and hence σ, could not be mixing. Consequently, P2 cannot be solved in general. However, if we exclude the above situation by hypothesis, then P2 can always be solved.

THEOREM 2. *Let $\tau,\sigma\in G$ with σ antiperiodic. Let $\mathcal{A}$ be a finite subalgebra of Σ such that $\tau/\mathcal{A}$ has no non-trivial periodic point. Then there is a $\hat\sigma\in C(\sigma)$ such that $\hat\sigma(A)=\tau(A)$ for all $A\in\mathcal{A}$.*

THEOREM 3. *Furthermore (continuing from above) let ρ be any totally bounded metric on X such that μ is positive on open sets. Let D denote the union of all atoms of $\mathcal{A}$ whose image under τ is connected. Then, given any $\varepsilon>0$, there is a $\hat\sigma\in C(\sigma)$ such that $\hat\sigma(A)=\tau(A)$ for all $A\in\mathcal{A}$ and $\rho(\hat\sigma(x),\tau(x))<\varepsilon$ for* (a.e.) *x in D.*

We note that theorem 3 is used in [**3**] to prove the existence of an ergodic non-stable Lebesgue measure preserving homeomorphism of $\mathbb{R}^4$, conditional on the existence of a non-ergodic one.

We briefly discuss problem P3 by observing that it can be solved when τ is antiperiodic, by applying Rohlin's lemma to both τ and σ. That is, $C(\sigma)$ is dense in the antiperiodic transformations with respect to the uniform topology [**5**, p. 112].

Finally, we discuss the strongest type of approximation, P4. First, it is obvious that we must require $\mu(F)<1$, for otherwise P4 implies τ is conjugate to σ. Next, the same argument used in the discussion of P2 shows that we must rule out the possibility that $\tau/\mathscr{A}$ has a non-trivial periodic point, where $\mathscr{A}$ is here the σ-algebra generated by measurable subsets of F. This possibility is excluded by either assumption of the following result, which is proved by using theorem 1.

THEOREM 4. *Let $\tau, \sigma \in G$, with σ antiperiodic, and $F \in \Sigma$ with $\mu(F) < 1$ be given. Assume either*

(1) *τ is ergodic and $\mu(F \cup \tau F) < 1$, or*

(2) *τ is weakly mixing.*

Then there is a $\hat{\sigma} \in C(\sigma)$ such that $\hat{\sigma}(x) = \tau(x)$ for (a.e.) *x in F.*

Part (1) of theorem 4 is similar to a recent result of Choksi and Kakutani for $G(\bar{X}, \bar{\Sigma}, \bar{\mu})$, where $(\bar{X}, \bar{\Sigma}, \bar{\mu})$ is an infinite σ-finite Lebesgue measure space. They demonstrated [**4**, theorem 6] that when $\tau, \sigma \in G(\bar{X}, \bar{\Sigma}, \bar{\mu})$, with τ ergodic, σ antiperiodic, and $F \in \bar{\Sigma}$ with $\mu(F) < \infty$ are given, there is a $\hat{\sigma} \in C(\sigma)$ with $\sigma(x) = \tau(x)$ for a.e. x in F. This can be expressed by saying that $C(\sigma)$ is dense in the ergodic automorphisms with respect to the 'strong topology' defined (in [**2**]) by basic neighbourhoods consisting of all automorphisms (of $(\bar{X}, \bar{\Sigma}, \bar{\mu})$) agreeing with a given one on a given set of finite measure. Theorem 4 can be similarly expressed by saying that $C(\sigma)$ is dense in the weakly mixing automorphisms with respect to the 'compact-equal' topology on G defined as follows. Identify (X, Σ, μ) with Lebesgue measure on the open unit interval. A basis is then given by sets of all automorphisms agreeing with a given one on a given compact set. This topology is finer than the uniform topology. For applications to the study of measure preserving homeomorphisms, (X, Σ, μ) can be identified with other non-compact spaces.

2. *Proof of theorem* 1

Our proof of theorem 1 will involve some special notation, for which we fix the σ and π of the theorem. For $j \geq 1$, let

$$S^j \equiv \{k \leq j : \pi_k > 0\},$$

$$s_j = \sum_{k \in S^j} \pi_k,$$

and

$$\pi^j = (\pi_1^j, \pi_2^j, \ldots, \pi_j^j, 0, 0, \ldots),$$

where $\pi_k^j = \pi_k / s_j$, for $k \leq j$. According to assumption on π, there is a j_0 such that S^{j_0} is relatively prime. Consequently, all integers greater than or equal to some fixed integer L may be represented as positive integer linear combinations of S^{j_0}.

A π-partition of X is a measurable partition $R = \{R_{k,i}\}$, $k = 1, \ldots, \infty$, $i = 1, \ldots, k$ such that

$$R_{k,i} = \sigma^{i-1}(R_{k,1}), \quad \text{and} \quad \mu(R_{k,1}) = 0$$

whenever $\pi_k = 0$. Define

$$R_k = \bigcup_{i=1}^{k} R_{k,i}$$

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and

$$d(R)=(\mu(R_1),\mu(R_2),\ldots).$$

In this notation, theorem 1 asserts the existence of a π-partition P with $d(P)=\pi$.

Our proof will obtain P as the limit of π-partitions P^j, $j\geq j_0$, with respect to the partition metric

$$\|R-Q\|=\mu\{x:x \text{ has different } R \text{ and } Q \text{ labels}\}$$

on the (complete) space of π-partitions. Each P^j will be a π-partition 'of type j' by which we mean that $\mu(P^j_k)=0$ for $k>j$.

To ensure that $d(P)=\pi$ we use the 'sum' metric on l^∞ and observe that

$$|d(R)-d(Q)|=\sum_{k=1}^{\infty}|\mu(R_k)-\mu(Q_k)|\leq 2\|R-Q\|.$$

So we would like to choose P^j with $|d(P^j)-\pi|$ very small. But unfortunately, since P^j is of type j, $|d(P^j)-\pi|$ is bounded away from 0. So instead we choose P^j with $|d(P^j)-\pi^j|$ small, or equivalently, with $\Delta_j(P^j)$ small, where

$$\Delta_j(R)=\max_{k\in S_j}(1-\pi^j_k/\mu(R_k)).$$

The construction of the P^j will be based on the following two lemmas.

LEMMA 1. *For any positive integer n there is a sweep-out set $B=B_n$ whose return times are not less than n. That is, there is a measurable subset B of X satisfying*:

(i) *the sets $B, TB,\ldots,T^{n-1}B$ are disjoint*; *and*

(ii) $\bigcup_{l=1}^{\infty}T^lB=X$.

Proof. This result is, of course, a special case of the finite dimensional version of theorem 1 [**1**, corollary 2] which gives us (for example) a sweep-out set whose only return times are n and $n+1$. However, the lemma as formulated may be established directly by observing that any set which is maximal with respect to (i) must necessarily also satisfy (ii) [**8**, pp. 70–72]. □

LEMMA 2. *Let $j\geq j_0$ and $\varepsilon>0$ be given. Then to every π-partition R of type j there corresponds a π-partition Q of type j satisfying $|d(Q)-\pi^j|<\varepsilon$ and $\|Q-R\|\leq\Delta_j(R)$.*

Proof. Let $B=B_n$ be the set given by lemma 1 for some large n to be specified later. Partition B into sets B^l, $l=1,2,\ldots$ so that, if $x,y\in B^l$, then x and y have the same return time n^l to B, and $\sigma^m(x)$ and $\sigma^m(y)$ belong to the same element of R for $m=0,\ldots,n^l-1$. Next partition each B^l into sets B^l_0 and B^l_k, $k\in S^j$, so that

$$\mu(B^l_0/B^l)=\alpha_k \quad\text{and}\quad \mu(B^l_k/B^l)=\beta_k,$$

where $\alpha=1-\Delta_j(R)$ and $\beta_k=\pi^j_k-\alpha\mu(R_k)\geq 0$. Let C^l_k be the 'column' based on B^l_k, that is

$$C^l_k=\bigcup_{m=0}^{n^l-1}\sigma^m(B^l_k),$$

and let $D_k=\bigcup_l C^l_k$. Observe that

$$\mu(D_0)=\alpha=1-\Delta_j(R),\quad \mu(D_k)=\beta_k,$$

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and that

$$\mu(R_k \cap D_0) = \alpha\mu(R_k) = \pi^i_k - \beta_k.$$

We now define Q on D_0 to be the same as R. Regardless of how we subsequently define Q on the complement $\sim D_0$ of D_0 we shall have

$$\|Q - R\| \le \mu(\sim D_0) = 1 - \alpha = \Delta_j(R).$$

If we could define Q on $\sim D_0$ so that $\mu(Q_k/D_k) = 1$, we would have

$$\mu(Q_k) \ge \mu(R_k \cap D_0) + \mu(Q_k/D_k)\mu(D_k) = \alpha\mu(R_k) + \beta_k = \pi^i_k,$$

and consequently $d(Q) = \pi^i$. By defining Q on D_k so that $\mu(Q_k/D_k)$ is nearly 1, we shall ensure that

$$|d(Q) - \pi^i| < \varepsilon.$$

We define Q on D_k by specifying it on each column

$$C^l_k = \bigcup_{m=0}^{n^l-1} \sigma^m(B^l_k)$$

as follows. For simplicity take $B^l_k = E$ and $n^l = N$. Suppose $E \subset R_{(k_1,i_1)}$ and $\sigma^{N-1}(E) \in R_{(k_2,i_2)}$. We assign N Q-labels to the sets $E, \sigma E, \sigma^2 E, \ldots, \sigma^{N-1}E$ by beginning and ending with R-labels:

$$(k_1, i_1), (k_1, i_1+1), \ldots, (k_1, k_1), -, -, \ldots, -, -, (k_2, 1), (k_2, 2), \ldots, (k_2, i_2).$$

We then fill in successive blocks of the form

$$(k, 1), (k, 2), \ldots, (k, k),$$

beginning immediately after (k_1, k_1), until there are T blanks remaining between the final (k, k) and the label $(k_2, 1)$, where T satisfies $L \le T \le L + k$. Since $T \ge L$, the definition of L guarantees that these blanks may be filled in with blocks of the form

$$(k', 1), (k', 2), \ldots, (k', k'),$$

where $k' \in S^{i_0}$. This procedure ensures that Q is a π-partition. Furthermore, of the N labels in this sequence, all but at most

$$(k_1 - i_1 + 1) + T + k_2 \le k + (T + k) + k = T + 3k \le T + 3j$$

are in Q_k (first coordinate k). Thus

$$\mu(Q_k/C^l_k) \ge 1 - (T + 3j)/n^l \ge 1 - (T + 3j)/n,$$

and consequently

$$\mu(Q_k/D_k) \ge 1 - (T + 3j)/n \ge 1 - \tfrac{1}{2}\varepsilon,$$

if we take $n > \frac{1}{2}\varepsilon(T + 3j)$. Finally, we calculate

$$\pi^i_k - \mu(Q_k) \le \tfrac{1}{2}\varepsilon\beta_k$$

for $k \in S^i$ so that

$$|\pi^i - d(Q)| \le 2(\tfrac{1}{2}\varepsilon) \sum_{k \in S^i} \beta_k \le \varepsilon$$

as required. □

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Proof of theorem 1. For $j \geq j_0$ choose positive numbers ε_j going to 0 and sufficiently small so that

$$|d(Q)-\pi^j|<\varepsilon_j \text{ implies } \Delta_j(Q)<2^{-(j+1)}$$

for any π-partition Q. For $j \geq j_0$ we construct a π-partition P^j of the type j satisfying:

(i) $|d(P^j)-\pi^j|<\varepsilon_j$; and

(ii) $\|P^j - P^{j-1}\| \leq 2^{-j} + \pi_j/s_j \quad (j>j_0)$.

It then follows that $\|P^j - P\| \to 0$ for some π-partition P, which necessarily satisfies $d(P)=\pi$ (see remarks preceeding lemma 1).

The P^j are constructed as follows. The first one, P^{j_0}, satisfying property (i), may be obtained directly from the finite version of theorem 1 [**1**, corollary 2] – in fact with $d(P^{j_0})=\pi^{j_0}$. However, we may make this proof self-contained by observing that the algorithm of lemma 2, used with $\alpha=0$ and $\beta_k=\pi_k^{j_0}$, yields the required π-partition P^{j_0} directly. Now suppose

$$P^{j_0},\ldots,P^{j-1}$$

have been found satisfying (i) and (ii). Since P^{j-1} satisfies

$$|d(P^{j-1})-\pi^{j-1}|<\varepsilon_{j-1}$$

we know by choice of ε_{j-1} that

$$\Delta_{j-1}(P^{j-1})<2^{-j}.$$

Now observe that any π-partition R of type $j-1$ is also of type j and that

$$\Delta_j(R) \leq \Delta_{j-1}(R)+\pi_j/s_j.$$

If we apply this inequality to P^{j-1} we obtain

$$\Delta_j(P^{j-1}) \leq 2^{-j}+\pi_j/s_j.$$

Now apply lemma 2 taking $R=P^{j-1}$ and $\varepsilon=\varepsilon_j$ to obtain (as Q) the partition P^j satisfying (i) and (ii). □

3. *Proof of theorem* 4

Since τ is ergodic and $\mu(F)<\mu(X)$, it follows that the τ-orbit of a.e. point of F eventually leaves F. Consequently, we can partition the set

$$F \cup \tau F = \bigcup_{k=2}^{\infty} \bigcup_{i=1}^{k} F_{k,i},$$

where $F_{k,i}=\tau^{i-1}(F_{k,1})$, $F_{k,i} \subset F-\tau F$ for $i<k$, and $F_{k,k} \subset TF-F$. Let

$$F_{1,1}=X-(F \cup TF)$$

and define

$$\pi_k=\mu\Big(\bigcup_{i=1}^{k} F_{k,i}\Big).$$

We claim that the ks for which $\pi_k>0$ are relatively prime. The demonstration of this fact breaks up into two cases. In case (1), the hypothesis $\mu(F \cup TF)<\mu(X)$ ensures that $\pi_1>0$. Next suppose that we are in case (2), but not case (1), so that τ is weakly mixing, and $\mu(F \cup TF)=\mu(X)$. Suppose that p, the greatest common

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divisor of the ks for which $\pi_k > 0$, is greater than 1. Let $D = F_{k,1}$ for some k with $\pi_k > 0$. Then

$$\mu(X) > \mu(D) = \pi_k/k > 0$$

but

$$\mu(D \cap \tau^{np+1}D) = 0 \quad \text{for all } n.$$

But this contradicts the hypothesis that τ is weakly mixing, so we must have $p = 1$. Thus in either case, π satisfies the hypothesis of theorem 1.

Let $P = \{P_{k,i}\}$ be the partition given by theorem 1 to the σ of this theorem and the distribution π we have just constructed. Define $\theta \in G(X, \Sigma, \mu)$ so that

$$\theta(P_{k,1}) = F_{k,1} \quad \text{and} \quad \theta(x) = \tau^{i-1}\sigma^{1-i}(x)$$

for $x \in P_{k,i}$, for all k with $\pi_k > 0$ and all i such that $1 \leq i \leq k$. It follows from this construction that

$$\tau(x) = \theta^{-1}\sigma\theta(x)$$

whenever $x \in F_{k,i}$ where $i < k$. But

$$F \subset \bigcup_k \bigcup_{i<k} F_{k,i},$$

so the theorem is proved. □

4. *Proofs of theorems* 2 *and* 3

The following discussion, through the statement of proposition 1, is taken from [**1**]. Let $T = \{t_{ij}\}$ be an $n \times n$ matrix consisting of 0s and 1s. We call T 'aperiodic' if for some integer N, T^N has all positive entries. T induces a map $\hat{T}: \Gamma \to \Gamma$, where Γ is the power set of $\{1, \ldots, n\}$, by $j \in \hat{T}(\gamma)$ if $t_{ij} = 1$ for some $i \in \gamma \in \Gamma$. Let Γ_1 denote the subalgebra of Γ given by: $\gamma \in \Gamma_1$ if $t_{ij} = 1$ and $j \in \hat{T}(\gamma)$ imply $i \in \gamma$. We say that a probability distribution $(p_1, \ldots, p_n)$ is consistent with T if it satisfies

(1) $\sum_{i \in \gamma} p_i = \sum_{j \in \hat{T}(\gamma)} p_j$ for all $\gamma \in \Gamma_1$; and

(2) $\sum_{i \in \gamma} p_i < \sum_{j \in \hat{T}(\gamma)} p_j$ for all $\gamma \in \Gamma - \Gamma_1$.

PROPOSITION 1 (immediate consequence of theorem 1, [**1**]). *Let* $\{P_i\}_{i=1}^n$ *be a measurable partition of* (X, Σ, μ) *whose distribution* $(\mu P_1, \mu P_2, \ldots, \mu P_n)$ *is consistent with an* $n \times n$ 0–1 *matrix T. Let* $\sigma \in G$ *be antiperiodic. Then there is a* $\hat{\sigma} \in C(\sigma)$ *such that* $\mu(\hat{\sigma}P_i \cap P_j) = 0$ *for all i, j with* $t_{i,j} = 0$.

Proof of theorem 2. Let A_l, $l = 1, \ldots, L$ denote the atoms of $\mathscr{A}$. Let $\{P_{ij}\}_{i,j=1}^n$ be a measurable partition of (X, Σ, μ) which refines the partitions given by the atoms of $\mathscr{A}$ and the atoms of $\tau(\mathscr{A})$. Define an $n \times n$ 0–1 matrix T by

$$(3) \quad t_{ij} = \begin{cases} 1 & \text{if } P_i \subset A_l \text{ and } P_j \subset \tau(A_l), \text{ some } l; \\ 0 & \text{otherwise.} \end{cases}$$

First observe that $\gamma \in \Gamma_1$ if and only if $\bigcup_{i \in \gamma} P_i \in \mathscr{A}$, and that consequently $(\mu P_1, \ldots, \mu P_n)$ is consistent with T. To show that T is aperiodic it is sufficient to prove that, for any $\gamma \in \Gamma - \{\varnothing\}$, the sequence

$$\mu(\gamma), \mu(\hat{T}\gamma), \mu(\hat{T}^2\gamma), \ldots$$

is eventually 1, where $\mu(\gamma) = \sum_{i\in\gamma} \mu P_i$. (If N is the longest number of steps it takes, then $T^N > 0$.) It follows from (3) that $\mu(\gamma) \leq \mu(\hat{T}\gamma)$ with equality only for $\gamma \in \Gamma_1$. Consequently, the only way such a sequence can fail to reach 1 is if $\hat{T}/\Gamma_1$ has a non-trivial period point γ_0. But then $A = \bigcup_{i\in\gamma_0} P_i$ would be a non-trivial periodic point of $T/\mathscr{A}$. Since this possibility is excluded by assumption, our argument shows that T is aperiodic. The automorphism $\hat{\sigma}$ given by proposition 1 now proves the theorem. □

Proof of theorem 3. This proof is very similar to the proof of theorem 2, so we only indicate the differences. Let $A_1, \ldots, A_{L_1}$ be the atoms of $\mathscr{A}$ whose τ images are connected. Let $\{P_{ij}\}$, $i, j = 1, \ldots, n$ additionally satisfy

$$\rho(P_i) < \tfrac{1}{2}\varepsilon \quad \text{and} \quad \rho(\tau(P_i)) < \tfrac{1}{2}\varepsilon,$$

which is possible because X is totally bounded ($\rho(\ )$ denotes diameter). Define T by

$$t_{ij} = \begin{cases} 1 & \text{if } P_i \subset A_l \text{ and } P_j \subset T(A_l), \text{ some } l > L_1; \\ 1 & \text{if } P_i \subset A_l \text{ and } P_j \subset T(A_l), \text{ some } l \leq L_1 \text{ and} \\ & \overline{\tau P_i} \cap \bar{P}_j \neq \varnothing, \text{ where bar denotes closure;} \\ 0 & \text{otherwise.} \end{cases}$$

The proof that T is aperiodic and that $(\mu P_1, \ldots, \mu P_n)$ is consistent with T is the same as that for theorem 2 except that the connectivity of $T(A_l)$, $l \leq L_1$, is used to identify Γ_1 with $\mathscr{A}$. Let $\hat{\sigma}$ be the automorphism given by proposition 1. To establish the final estimate of the theorem, assume $P_i \subset D$, or equivalently, $P_i \subset A_l$, some $l \leq L_1$. Then

$$\rho(\hat{\sigma}P_i \cup \tau P_i) \leq \max_{j:t_{ij}=1} \rho(P_j \cup \tau P_i) \leq \rho(P_j) + \rho(\tau P_i) < \varepsilon,$$

since $\bar{P}_j \cap \overline{\tau P_i} \neq \varnothing$. It follows that

$$\rho(\hat{\sigma}(x), \tau(x)) < \varepsilon \quad \text{for a.e. } x \text{ in } D.$$ □

The application (in [**3**]) of theorem 3 mentioned in the introduction uses only the following special case.

COROLLARY 1. *Let (X, Σ, μ, ρ) be as in theorem* 3. *Let $A \in \Sigma$ and let $\tilde{\tau}: A \to X$ be any μ-preserving injection such that $\mu(\tilde{\tau}(A) \triangle A) > 0$ and $\tilde{\tau}(A)$ is connected. Then given any $\varepsilon > 0$ there is a μ-preserving injection $\tilde{\sigma}: A \to \tilde{\tau}(A)$ such that $\rho(\tilde{\tau}(x), \tilde{\sigma}(x)) < \varepsilon$ for* a.e. *x in A, and $\tilde{\sigma}$ has no non-trivial invariant sets.*

Proof. We may assume without loss of generality that $\mu(A \cap \tilde{\tau}(A)) > 0$, for otherwise simply take $\tilde{\sigma} = \tilde{\tau}$. Let $\tau \in G(X, \Sigma, \mu)$ be any extension of $\tilde{\tau}$ and let $\mathscr{A} = \{\varnothing, X, A, \sim A\}$. The set D of theorem 3 is either A or X, but in any case $D \supseteq A$. Let $\sigma \in G$ be any ergodic automorphism and let $\hat{\sigma}$ be the conjugate of σ which approximates τ in the sense of theorem 3. Then the restriction $\tilde{\sigma}$ of $\hat{\sigma}$ to A has the required properties. □

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