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Returns to the origin for a randomized random walk

RETURNS TO THE ORIGIN
FOR A RANDOMIZED RANDOM WALK

by A. E. Gibson [1] and B. W. Conolly [2]

Introduction.

The expression « randomized random walk » (RRW) seems to have been introduced by Feller (1966) to describe an unrestricted linear random walk performed by a particle which moves unit positive or negative amounts X at randomly distributed time intervals. In the simplest case the intervals between successive positive steps are i. i. d. with d. f. $1 - e^{-\lambda t}$: similarly, those between negative steps have d. f. $1 - e^{-\mu t}$. It will immediately be seen that this is a generalization of the mechanism which generates the fundamental process $S(t)$ for the $M/M/1$ queue, in which connexion $S(t)$ has the physical interpretation of the number present in the system, waiting and in service, at time t. But for the queue, $S(t)$ cannot become negative, and a modification of the underlying mechanism takes place when $S(t) = 0$, which can be thought of as the erection of a barrier at the origin.

It is of practical interest to the operators of queueing systems and their relatives (e. g. inventory, population models) to understand the statistics of returns to the origin, i.e. the occasions when $S(t) = 0$, because they may imply an idle server, or an empty stock room, for example. In addition the return to the origin problem is of interest in its own right for R. R. W. because asymptotically the statistics display a similar surprising behaviour to that obtaining when there is no time dependence between steps in the walk, and the fundamental problem is that of describing S_n, the position of the particle at the nth step.

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This paper deals with both practical and theoretical aspects and the method used provides a unified treatment from which both the $M/M/1$ and R. R. W. results can be deduced immediately. Some of the results have been quoted without proof in a review paper by one of us (Conolly 1971).

2. The Problem.

The integer valued stochastic process $S(t)$ represents the position of the particle describing R. R. W. at time t. The initial condition will always be $S(0) = 0$, so that the walk begins at the origin. A return to the origin is said to take place at epoch t when $S(t-) \neq 0$, $S(t) = 0$. Because of the lack of memory property of the negative exponential distributions involved, a subsequent return to the origin recaptures the initial condition. Our main interest resides in the interrelated random variables T_k, the epoch of the kth return to zero, and $N(t)$, the number of returns to zero during $(0, t)$.

Unification of treatment for $M/M/1$ and R. R. W. consists in this. Whenever $S(t) \neq 0$ the d. f. s. of intervals between successive positive-negative steps are $1 - e^{-\lambda t} / 1 - e^{-\mu t}$. When $S(t) = 0$, the time interval to the next positive-negative step has d. f. $1 - e^{-\lambda_0 t} / 1 - e^{-\mu_0 t}$. For R. R. W., $\lambda_0 = \lambda$, $\mu_0 = \mu$: for $M/M/1$, $\lambda_0 = \lambda$, $\mu_0 = 0$.

Let

(2.1) $$F_k(t) = P_r[T_k \leq t \mid S(0) = 0]$$

with corresponding (assumed existing) p. d. f. $f_k(t)$. These may be defective. Also let

(2.2) $$k_n(t) = P_r[N(t) = n \mid S(0) = 0].$$

It is obvious and well known from renewal theory, that for $n = 0, 1, 2, \ldots$,

(2.3) $$k_n(t) = F_n(t) - F_{n+1}(t),$$

with the definition $F_0(t) = 1$.

Since the walk has to begin with a positive or negative step away from zero it follows that

(2.4) $$f_1(t) = [\lambda_0 e^{-(\lambda_0 + \mu_0) t}]^* f_{10}(t) + [\mu_0 e^{-(\lambda_0 + \mu_0) t}]^* f_{-10}(t)$$

where $*$ denotes convolution of the functions it separates and $f_{mn}(t)$ is the (assumed existing) first passage p. d. f. from m to n at epoch t. Since paths from ± 1 allowing a first return to the origin do not, by definition, include the origin except at the last step it follows that $f_{10}(t)$ and $f_{-10}(t)$ have the

known R. R. W. expressions (cf. Conolly (loc. cit.) for example) :

$$(2.5) \qquad f_{10}(t) = \varrho^{-\frac{1}{2}} e^{-(\lambda+\mu)t} I_1 \ (2t \sqrt{\lambda\mu})/t,$$

$$f_{-1,0}(t) = \varrho f_{10}(t),$$

with Laplace transforms (LTs) [1]

$$(2.6) \qquad \Phi_{10}(z) = 2\mu/(Z+R),$$

$$\Phi_{-1,0}(z) = \varrho \ \Phi_{+1,0}(z),$$

where

$$(2.7) \qquad Z = z + \lambda + \mu, \qquad R^2 = Z^2 - 4\lambda\mu.$$

Application of the Laplace transformation to (2.4) and use of (2.6) gives the *LT* $\Phi_1(z)$ of $f_1(t)$, vlz.

$$(2.8) \qquad \Phi_1(z) = 2 \ (\lambda_0 \mu + \lambda\mu_0)/Z_0 \ (Z + R)$$

where

$$Z_0 = \lambda_0 + \mu_0 + z.$$

Since T_k is the sum of k independent random variables each having the d. f. $F_1(t)$ it follows that

$$(2.9) \qquad f_k(t) = f_1^{(k)}(t),$$

the bracketed superscript denoting k-fold convolution, and hence that

$$(2.10) \qquad \Phi_k(z) = [\Phi_1(z)]^k .$$

A return to the time domain can be achieved by using Erdélyi et al. (1954). Thus

$$(2.11) \qquad f_k(t) = \frac{k \ (\lambda_0 \mu + \lambda\mu_0)^k}{(\lambda\mu)^{\frac{1}{2}k} (k-1)!} \int_0^t e^{-(\lambda_0+\mu_0)(t-s)} (t-s)^{k-1} e^{-(\lambda+\mu)s} I_k \ (2s \sqrt{\lambda\mu}) \ ds/s$$

for $k = 1, 2, 3, \ldots I_k(z)$ is the modified Bessel function of the first kind, order k.

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[1] As an example, $\Phi_{10}(z) = \int_0^\infty e^{-zt} f_{10}(t) \, dt$.

Integration of (2.11) gives the d. f. $F_k(t)$ in a form containing an incomplete gamma function under the sign of integration.

3. R. R. W.

In this case we put $\lambda_0 = \lambda$ and $\mu_0 = \mu$.

The most interesting situation accurs when $\lambda = \mu$:
otherwise there is a non-zero probability that no return to zero occurs. From the preceding analysis with $\lambda = \mu$ we obtain

$$k_n(t) = 2\mu e^{-2\mu t} \int_0^t I_n(2\mu s) \left[\frac{\{2\mu(t-s)\}^{n-1}}{(n-1)!} + \frac{\{2\mu(t-s)\}^n}{n!} \right] ds$$

(3.1)   for $n = 1, 2, 3, \ldots$, and

$$k_0(t) = e^{-2\mu t} \left[I_0(2\mu t) + 2\mu \int_0^t I_0(2\mu s)\, ds \right].$$

These formulae are easily obtained from application of the Laplace transformation to (2.3), manipulation and final return to the time domain via Erdélyi et al. (loc. cit.), p. 240, entry (30). It is not immediately obvious, but

(3.2) $$k_0(t) = k_1(t) + e^{-2\mu t}(1 + 2\mu t),$$

and $k_n(t)$ decreases steadily as n increases, for fixed t.

In particular, for quite moderate μt the second term in (3.2) becomes negligible, after which time it is virtually as likely that there is no return to zero as it is that there is a single return to zero.

Numerical values of $k_n(t)$ have been given by one of us (Gibson (1968)), but it is more instructive to examine the mean number of returns to zero over a long time interval. From (2.3) and (2.10) the L. T. of $k_n(t)$ is $\Phi_1^n(z)\{1 - \Phi_1(z)\}/z$ which leads, after manipulation, to $4\lambda\mu/zR(Z+R)$ for the L. T. of $E[N(t)]$. Returning to the time domain, we have

(3.3) $$E[N(t)] = 2\sqrt{\lambda\mu} \int_0^t e^{-(\lambda+\mu)s} I_1(2s\sqrt{\lambda\mu})\, ds.$$

This formula provides the means of exact calculation but tells little as it stands. The asymptotic behaviour is more interesting and may be obtained

by utilizing the asymptotic expression of the Bessel function. Thus, as $t \to \infty$,

$$E\left[N\left(t\right)\right] = \frac{1+\varrho}{|1-\varrho|} - 1 - \frac{2\varrho^{\frac{1}{4}}}{|1-\sqrt{\varrho}|}\left[1 - Erf\left(|1-\sqrt{\varrho}|\sqrt{2\mu t}\right)\right] +$$

$$+ O\left[(\mu t)^{-3/2}\right], \quad \varrho \neq 1;$$

$$= 2\sqrt{\frac{\mu t}{\pi}} + O\left[(\mu t)^{-1/2}\right], \quad \varrho = 1;$$

where $\varrho = \lambda/\mu$ and

$$Erf\left(x\right) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{x} e^{-\frac{1}{2}\mu^2}\,d\mu.$$

The case $\varrho = 1$ is of greater interest and shows, as is the case where time dependence between steps is ignored (see Feller (1957)), that the mean number of returns to zero behaves like $t^{\frac{1}{2}}$ for large t instead of like t, as intuition might suggest. This means that if $S(t)$ were plotted against t for $\varrho = 1$ one would be surprised *not* to observe that $S(t)$ held either positive or negative values for longer and longer periods.

To conclude this section we note that $f_r(t)$ can be expressed in terms of the generalised hypergeometric function $_2F_3\left[a_1, a_2; b_1, b_2, b_3; z\right]$ as follows:

$$(3.5) \quad f_r(t) = \frac{(2\lambda\mu)^r\,t^{2r-1}}{\Gamma(2r)}\,e^{-(\lambda+\mu)t}\,{}_2F_3\left[\frac{1}{2}\,r, \frac{1}{2}\,(r+1); r+1, r, r+\frac{1}{2}; \lambda\mu t^2\right].$$

From the asymptotic properties of $_2F_3$ (see, e.g., Luke (1962)) it follows that as $t \to \infty$

$$(3.6) \qquad f_r(t) = \frac{r\varrho^{-\frac{1}{4}}e^{-\mu(1-\sqrt{\varrho})^2 t}}{2\sqrt{\pi\mu}\,t^{3/2}} + O\left(rt^{-5/2}\right),$$

whence, by integration,

$$(3.7) \qquad P_r\left[T_r > t\right] = 1 - \xi^r + \frac{r\varrho^{-\frac{1}{4}}e^{-\mu t(1-\sqrt{\varrho})^2}}{\sqrt{\pi\mu t}}$$

$$- 2r\varrho^{-\frac{1}{4}}|1-\sqrt{\varrho}|\left[1 - Erf\left\{|1-\sqrt{\varrho}|\sqrt{2\mu t}\right\}\right] + O\left(rt^{-3/2}\right) \quad (\varrho \neq 1);$$

$$= \frac{r}{\sqrt{\pi\mu t}} + O\left(rt^{-3/2}\right), \quad (\varrho = 1);$$

where

$$\xi = 1 - \frac{|\, 1 - \varrho\, |}{1 + \varrho} \, .$$

It should be pointed out that the theory of stable distributions is applicable when $\varrho = 1$ and is consonant with the second part of (3.7).

Gibson (loc. cit.) gives extensive numerical tabulation of $f_r\,(t)$ both from the exact and asymptotic formulae to indicate the usefulness of the latter for easy practical calculation.

4. M/M/1.

The analysis of the preceding section can now be repeated for the queueing process M/M/1 using the substitution $\lambda_0 = \lambda$, $\mu_0 = 0$ in the general formulae. The details are left to the interested reader. Here we confine our remarks to the asymptotic forms. Note that T_r is defective unless $\varrho = \lambda/\mu \leq 1$, and that the theory of stable distributions can be shown to predict the form of the result when $\varrho = 1$. By the same methods as for R. R. W. we find that as $t \to \infty$,

$$(4.1) \qquad Pr\,[T_1 > t] = \frac{\varrho^{-3/4}\, e^{-\mu\,(1 - \sqrt{\varrho})^2\, t}}{\sqrt{\pi \mu t}} - $$

$$- 2\varrho^{-3/4}\,(1 - \sqrt{\varrho})\,[1 - Erf\,\{(1 - \sqrt{\varrho})\,\sqrt{2\mu t}\}] + O\,[(\mu t)^{-3/2}] \quad (\varrho < 1);$$

$$= \frac{1}{\sqrt{\pi \mu t}} + O\,[(\mu t)^{-3/2}], \qquad (\varrho = 1).$$

Similarly we can show that when $\lambda = \mu$

$$(4.2) \qquad E\,[N\,(t)] = \mu \int_0^t e^{-2\mu s}\,[I_1\,(2\mu s) + I_2\,(2\mu s)]\; ds,$$

which gives the asymptotic result that as $t \to \infty$

$$(4.3) \qquad E\,[N\,(t)] \doteq 2\,\sqrt{\frac{\mu t}{\pi}} + O\,[(\mu t)^{-1/2}].$$

Thus, even when traffic is heavy (mean arrival rate λ equal to mean service rate μ), and when, as can be seen from $\Phi_1(z)$, a return to zero is a persistent but null recurrent event, the mean number of times that the server becomes idle in a long period of time is proportional to $t^{1/2}$, as was the case for R. R. W.

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