Revenue Management Games

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Abstract: A well-studied problem in the literature on airline revenue (or yield) management is the optimal allocation of seat inventory among fare classes given a demand distribution for each class. In the literature thus far, passenger demand is an exogenous parameter. However, the seat allocation decisions of one airline affect the passenger demands for seats on other airlines. In this paper we examine the seat inventory control problem with two fare classes and two airlines under competition. Each airline chooses an optimal booking limit for the lower-fare class while taking into account the overflow of passengers from its competitor. We show that under certain conditions this 'revenue management game' has a pure-strategy Nash equilibrium, and for special cases we show that the equilibrium is unique. We also compare the total number of seats available in each fare class with, and without, competition. Analytical results for one special case as well as numerical examples demonstrate that, all else equal, under competition more seats are protected for higher-fare passengers than when a single airline acts as a monopoly or when airlines form an alliance to maximize overall profits.

1. Introduction

Consider the airline flight schedules displayed in Table 1. We see that TWA and Delta schedule flights between the same origin and destination, at exactly the same times, using the same equipment, and charging nearly the same price for advance-purchase tickets. This paper examines how such direct competition for customers affects a fundamental revenue management decision, the allocation of seat inventory among fare classes.

Airline	Flight	Aircraft	Departure	Price (advance purchase)
TWA	3832	Jetstream 41	8:35am	$213.50
Delta	6122	Jetstream 41	8:35am	$208.50
TWA	3834	Jetstream 41	2:24pm	$213.50
Delta	6204	Jetstream 41	2:24pm	$208.50
TWA	3836	Jetstream 41	5:50pm	$213.00
Delta	6206	Jetstream 41	5:50pm	$208.00

Table 1. Flight schedule for TWA and Delta ROC-JFK, April 23, 2000.

There is a substantial literature analyzing airline economics under competition as well as a recent stream of operations research literature on the problem of optimal seat allocation. However, there are no published papers that place the seat allocation problem in a competitive framework. As the example above illustrates, airlines face intense competition, and the impact of competition on seat inventory decisions and airline revenues is of interest to airline planning and marketing managers, as well as government regulators.

In this paper we model competition between two airlines offering two flights that serve as substitutes for customers. Each airline is faced with an initial demand from passengers who wish to purchase tickets, but each airline may also sell tickets to passengers that were denied a reservation on the competing airline. Hence, the optimal capacity limits for each class (the booking limits) on each airline are interdependent. We compare the optimal revenue management policies of two competing airlines with the policy of a monopolist who operates both flights or, equivalently, the policies of two airlines who form an alliance to maximize total profits. We show that under certain conditions a pure-strategy Nash equilibrium exists for the competitive case, and we identify special cases under which the equilibrium is unique and stable. Given the assumption that low and high-fare ticket prices remain constant, we find that under competition more seats are allocated for higher-fare customers, and fewer seats are allocated for the lower-fare customers, than under centralized control.

Readers familiar with the airline industry may find this juxtaposition of competitive analysis and yield management unusual. In general, competitive decisions and seat allocation decisions are made by different functional units within airlines, at different times in the planning process, and over extremely different time horizons. The decision to enter markets, the assignment of aircraft to particular markets, and the creation of a schedule takes place over a time horizon of years and months. The allocation of seat inventory among customer classes is an operational decision with a time horizon of weeks, days, or even minutes (see Jacobs, Ratliff and Smith [2000] for a general description of this planning process).

However, our simple model will show that competition on a particular route at a particular time can have a profound effect on yield management decisions. In general, airline planners have recently shown an increased interest in the integration of airline functions. For example, numerical experiments in Jacobs, Ratliff and Smith [2000] demonstrate the value of simultaneous optimization of yield management and scheduling decisions. Yuen and Irrgang [1998] emphasize the benefits of integrating sales, yield management, pricing and scheduling decisions.

Publications that consider the interactions among economic forces, strategic airline market entry decisions, and airline schedules include the network design models of Lederer and Nambimamdom [1999], Dobson and Lederer [1994], and the empirical work by Borenstein and Rose [1994]. Another body of research focuses on the airlines' scheduling decisions under competition using variants of the spatial model developed by Hotelling [1929]. See, for example, the recent empirical papers by Borenstein and Netz [1999] and Richard [1999]. These papers focus on broad competitive problems and ignore the specifics of seat inventory allocation. In this paper we will not be concerned with the reasons airlines schedule their flights at the same time or with the pricing decision for each flight. Rather, we will concentrate on the implications of competitive scheduling on seat inventory control.

There are numerous papers in the area of revenue management that focus on airline seat inventory control, although to our knowledge only one addresses the issues described here. For fundamental results on the general subject of seat inventory control see Belobaba [1989], Brumelle et.al. [1990], and a useful literature review by McGill and van Ryzin [1999]. Our paper is related to the work by Li and Oum [1998], which first introduced the seat allocation problem for two airlines in competition. The model developed by Li and Oum has a relatively restrictive assumption about how demand is allocated among airlines: total demand is split according to the proportion of seats available on each aircraft in each fare class, and the overflow process is not explicitly modeled. In addition, the paper by Li and Oum identifies one, symmetric equilibrium but does not determine whether the equilibrium is unique. Our approach is more general and the results more advanced; we will place no restrictions on how initial demand is distributed and will show that for special cases of the problem the equilibrium is unique.

The literature on inventory management has seen a stream of closely related papers devoted to competition among firms in which the firms determine inventory levels and customers may switch among firms until a suitable product is found (this has been described as a 'newsboy game'). Parlar [1988] examines the competition between two retailers facing independent demands. He establishes that a unique Nash equilibrium exists. Karjalainen [1992] formulates the problem for an arbitrary number of products with independent demands. Lippman and McCardle [1997] examine both the two-firm game and a game with an arbitrary number of players. In their models, initial industry demand is allocated among the players according to a pre-specified 'splitting rule.' This initial allocation may be either deterministic (e.g., 40% of demand to player 1) or stochastic (the rule itself depends on the outcome of a random experiment). For the two-firm game they establish the existence of a pure-strategy Nash equilibrium and show that the equilibrium is unique when the initial allocation is deterministic and strictly increasing in the total industry demand for each player. Recent extensions of these models include Mahajan and van Ryzin [1999] who model demand as a stochastic sequence of utility-maximizing customers. For an arbitrary number of firms, they demonstrate that an equilibrium exists and show that it is unique for a symmetric game. Rudi and Netessine [2000] analyze a problem similar to Parlar [1988] but for an arbitrary number of products. Given mild parametric assumptions they establish the existence of, and characterize, a unique, globally stable Nash equilibrium.

Many of these papers compare total inventory levels under firm competition with inventory levels when firms cooperate. Lippman and McCardle [1997] show that competition can lead to higher inventories, and Mahajan and van Ryzin [1999] derive similar results given their dynamic model of customer purchasing. On the other hand, with the substitution structure of the model of Rudi and Netessine [2000], under competition some firms may stock less than under centralization. In this paper we find that competition leads to an increase in high-fare seat 'inventory,' a result similar to earlier findings. However, our model differs in many respects from the newsboy competition described by Lippman and McCardle. As in Mahajan and van Ryzin, the method of allocating arriving customers to firms is more natural than the stylized splitting rules proposed by Lippman and McCardle. In our model, demand for each fare class on each airline can follow an arbitrary distribution, and we allow an arbitrary correlation structure among demands. Numerical experiments will demonstrate that the degree of correlation can have a significant impact on seat allocation decisions, and can even determine whether a pure-strategy equilibrium exists.

There is also a fundamental difference between the problem considered here and the problem considered in the inventory literature. Here we consider the allocation of a fixed inventory pool between two products, while the inventory literature assumes that the inventory of each product is a decision variable. In our problem, the effect of a change in one airline's booking limit is quite complex. As the booking limit rises, demand by low-fare passengers to a competitor declines while high-fare demand to the competitor rises. In addition, we will see that the booking limit of an airline can affect the volume of *its own* high-fare demand.

In the next Section we describe the revenue management game and provide examples of scenarios in which Nash equilibria do, and do not, exist. We examine one variation of the game for which we establish the existence of a pure-strategy Nash equilibrium. Section 3 focuses on competition with partial overflow, models in which only low-fare or only high-fare passengers spill to a competing airline. In Section 4 we compare analytically the behavior of a monopolist (or alliance between airlines) with the behavior of two airlines under competition. Section 5 describes numerical examples and compares the service levels (percentage of customers who are able to purchase tickets) under the competitive and cooperative cases. In Section 6 we discuss the implications of our results on the practice of yield management and describe areas for future research.

2. The Full Revenue Management Game

Suppose two airlines offer direct flights between the same origin and destination, with departures and arrivals at similar times. We assume that other flights on this route are scheduled sufficiently far away in time so that they can be ignored. For simplicity, we assume that both flights have the same seat capacity and that there are only two fare classes available for passengers: a 'low-fare' and a 'high-fare.' A ticket purchased at either fare gives access to the same product: a coach-class seat on one flight leg. As is traditional in the literature on airline seat inventory control, we assume that demand for low-fare tickets occurs before demand for high-fare tickets, as is the case when advance-purchase requirements are used to distinguish between customers with different valuations on price and purchase convenience. Customers who prefer a low fare and are willing to accept the purchase restrictions will be called 'low-fare customers'. Customers who prefer to purchase later, at the higher price, are called 'high-fare customers'. We also assume that there are no customer cancellations.

To maximize expected profits, both airlines establish booking limits for low-fare seats. Once this booking limit is reached, the low fare is closed. Sales of high-fare tickets are accepted until either the airplane is full or the flight departs. If either type of customer is denied a ticket at one airline, the customer will attempt to purchase a ticket from the other (we call these "overflow passengers"). Therefore, both airlines are faced with a random initial demand for each fare class as well as demand from customers who are denied tickets by the other airline. Passengers denied a reservation by both airlines are lost.

Figure 1 shows both overflow processes as well as the following notation:

L, H = passenger classes, L for low-fare passengers and H for high-fare passengers.

C = capacity of the aircraft.

B_i = booking limit for low fare established by airline $i=1,2$.

D_{ki} = a random variable, demand for class k tickets at airline i, $k = L, H$ and $i = 1,2$.

p_k = revenue from class $k=L,H$ passengers less variable cost.

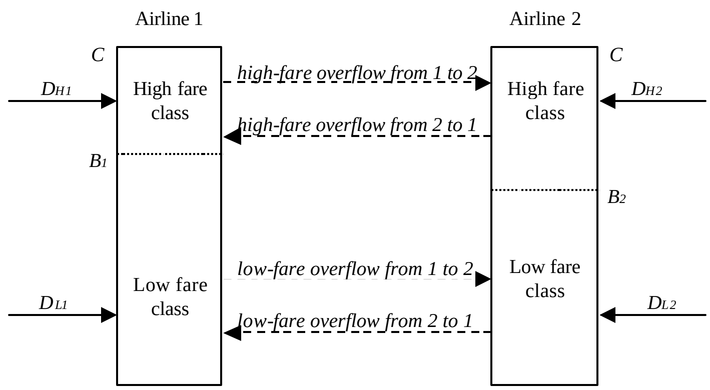

Figure 1. The Basic Competitive Model

We assume that both airline's prices are the same and that $p_L < p_H$. We also assume that the random variables D_{ki} have nonnegative support. However, to derive the following results establishing the existence of a pure-strategy equilibrium we do not assume that the cumulative distributions are continuous (we may have discrete or continuous probability distributions), and there may be an arbitrary correlation structure among demands. In Section 3, however, to establish the existence of a unique equilibrium we will assume that finite probability densities exist and that low and high-fare demands are independent.

In this paper we study competing airlines engaging in a noncooperative game with complete information. Each airline attempts to maximize its profits by adjusting its booking levels. In other words, the booking level $B_i \in [0,C]$ is the strategy space of airline *i* (for simplicity, we assume that the booking level may be any real number in this range). Each airline knows the strategy spaces and demand distributions of its own flight as well as those of the competing airline.

An important assumption of the model is that the initial demands D_{ki} are exogenous; they are not affected by the booking limits chosen by each airline. This assumption is consistent with the newsboy game models of Parlar, Karjalainen, and Lippman and McCardle. However, one might argue that the booking limits determine seat availability, and that in the long run this aspect of service quality affects initial demand. A more complete model would incorporate this relationship between booking limits and demand, and the solution would supply equilibrium demands as well as equilibrium booking limits. For our application, however, the relationship between booking limits and demand is weakened by marketing efforts such as advertising and frequent-flyer programs. In addition, the use of travel agents and on-line reservation tools reduces the marginal search cost associated with making each booking. Given low search costs, the decision as to which airline to query first may depend on factors that dominate the likelihood that the query will result in a booking.

Our model simplifies other aspects of the actual environment. For example, the model assumes that passengers denied a ticket in one class do not attempt to upgrade or downgrade to another class. The model also assumes that a passenger, when first denied a ticket, will not shift to a later or earlier flight operated by the same airline. However, all results presented in this paper also apply to a model in which some fraction (less than one) of passengers denied a ticket on one airline attempt to purchase a ticket from the other airline, while some fraction (greater than zero) are lost to both airlines. To simplify the model and minimize the number of parameters, we assume that all passengers denied a ticket from their first choice overflow to their second-choice airline.

The model contains only two fare classes, when in reality there may be many more (see Belobaba [1998] for an introduction to the complexities of real-world yield management systems). We also assume that the airlines' booking limits are static. That is, the booking limit is set before demand is realized and no adjustments are made as low-fare demand is observed. As we will see, even this relatively simple decision can be difficult to analyze in a competitive game, and this simple model allows us to focus on a few important questions. How will an optimal booking limit under competition differ from a booking limit under a centralized solution, with a single airline or when two airlines cooperate to maximize total profits? How does the existence of 'spill' demand affect the allocation of seat inventory? What is the effect of competition on profits, even when prices are held constant?

2.1 Low-fare then High-fare Spill

Thus far we have not described the order of events in the game. We begin with what may be the most natural order:

1. Airlines establish booking limits B_1 and B_2.
2. Low-fare passengers arrive to their first-choice airlines and are accommodated up to the booking limits.

3. Low-fare passengers not accommodated on their first-choice airlines 'spill' to the alternate airlines and are accommodated up to the booking limits.

4. High-fare passengers arrive to their first-choice airlines and are accommodated with any remaining seats, up to capacity C in each aircraft.

5. High-fare passengers not accommodated on their first-choice airlines 'spill' to the alternate airlines and are accommodated in any remaining seats, up to capacity C in each aircraft.

To describe the problem in terms of customer demand and booking limits, define:

$D_{Li}^T = D_{Li} + (D_{Lj} - B_j)^+$, total demand for low-fare tickets on airline *i*, *i=1, j=2* and *i=2, j=1.*

$R_i = C - \min(D_{Li}^T, B_i)$, the number of seats available for high-fare passengers on airline *i = 1,2.*

$D_{Hi}^T = D_{Hi} + (D_{Hj} - R_j)^+$, total demand for high-fare tickets, *i=1, j=2* and *i=2, j=1.*

The total revenue for airline *i* is

$$p_i = E\left[p_L \min(D_{Li}^T, B_i) + p_H \min(D_{Hi}^T, R_i)\right]. \tag{1}$$

Each airline will maximize this expression, given the booking limit of its competitor. It will be instructive to examine the first derivative of this objective function. It is tedious to find the derivative by the traditional methods (e.g., applying Leibnitz's rule). Instead, by applying the techniques described in the Appendix of Rudi and Netessine [2000], we find for *i=1, j=2* and *i=2, j=1,*

$$\begin{aligned}\frac{\partial p_i}{\partial B_i} &= p_L \Pr(D_{Li}^T > B_i) - p_H \Pr(D_{Hi}^T > C - B_i, D_{Li}^T > B_i) \\ &\quad - p_H \Pr(D_{Li} > B_i, D_{Lj}^T < B_j, D_{Hj} > R_j, D_{Hi}^T < C - B_i).\end{aligned} \tag{2}$$

Although this is a complex expression, there is a straightforward interpretation for each term. An incremental increase in the booking limit B_i by airline *i* has three effects on that airline's total revenue. First, revenue from low-fare customers increases with probability $\Pr(D_{Li}^T > B_i)$. Second, revenue from the high-fare customers decreases with probability $\Pr(D_{Hi}^T > C - B_i, D_{Li}^T > B_i)$. While these two effects are direct consequences of the change in B_i, there is a third, indirect effect. Revenue from high-fare customers may decrease because (i) an increase is B_i may reduce the overflow of low-fare customers from i to j, (ii) a reduction in the number of low-fare customers at j may increase the number of seats available for high-fare customers at j, (iii) this may reduce the overflow of high-fare customers from j to i and (iv) a decline in the overflow from j may reduce the number of high-fare customers accommodated at i. The probability of this sequence of events is the third term on the right-hand side of equation (2), which implies that an increase in the booking limit of airline i can result in a decrease in high-fare demand to airline i.

Because the strategy spaces of the airlines are compact and the payoff functions are continuous (see Proposition 1, below), a Nash equilibrium in mixed strategies must exist. However, a pure-strategy Nash equilibrium may, or may not, exist for airlines playing this game. Figure 2 shows the best reply functions, or reaction functions $r_i(B_j)$, of two airlines, each with $C = 200$ and multivariate normal demands (the parameters for this example will be described in detail in Section 5). Figure 3, showing a game with multiple equilibria, was also generated with the multivariate normal distribution (again, details are given in Section 5). Figure 4 displays two reaction functions, each with two discontinuities, producing a game without any pure-strategy equilibrium. An extremely unlikely demand pattern was used to produce this outcome. Figure 4 was generated from:

- Bimodal demand distributions for each fare class and airline. The distributions were created by mixing two normal distributions, one representing low-volume demand (mean = 20 seats) and the other representing high-volume demand (mean = 150 seats).
- Strong negative correlations between low-fare and high-fare demands. When low-fare demand was chosen from the low-volume distribution, high-fare demand was chosen from the high-volume distribution, and vice-versa. As a result, $r(D_{Li}, D_{Hi}) = -0.9$ for i=1,2.[3]
- A large difference between high and low fares ($p_H / p_L = 4$).

[3] It is interesting to note that in practice the strong negative correlation would present an excellent opportunity for each airline to practice dynamic yield management, with an adjustable booking limit dependent on observed low-fare demand. Given such dynamic decision-making, there may well be a competitive equilibrium.

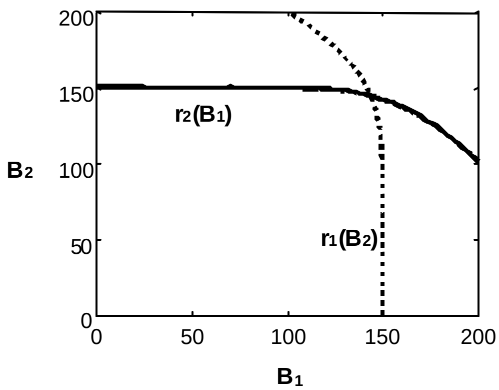

Figure 2: Unique Nash Equilibrium

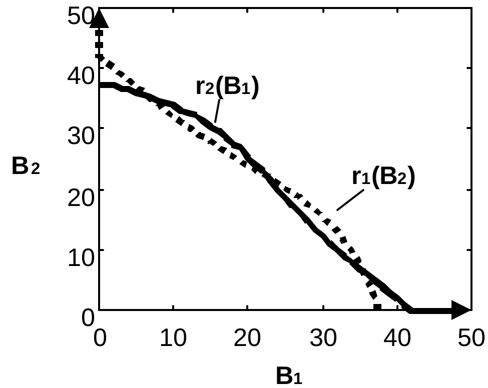

Figure 3: Multiple Equilibria

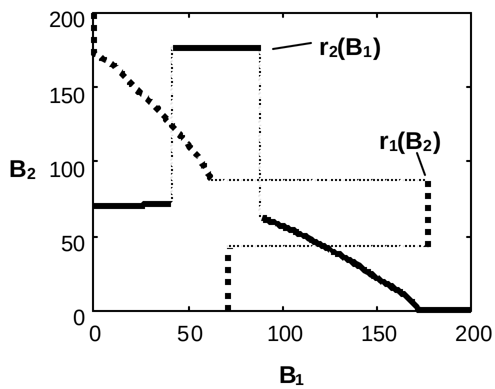

Figure 4: No Pure-Strategy Equilibrium

While we cannot specify analytically the general characteristics that would guarantee the existence of an equilibrium, expression (2) offers some insight. For most reasonable probability distributions and for most values of B_1 and B_2, the first two terms dominate the third term, so that

$$\frac{\partial p_i}{\partial B_i} \approx p_L \Pr(D_{Li}^T > B_i) - p_H \Pr(D_{Hi}^T > C - B_i, D_{Li}^T > B_i)\,. \tag{3}$$

This expression is similar to the first-order conditions for the standard two-fare seat allocation problem of a stand-alone airline, although here exogenous demands D_{ki} have been replaced by total demands D_{ki}^T. Brumelle et.al [1990] show that when the demands are monotonically associated, so that $\Pr(D_{Hi}^T > C - B_i \mid D_{Li}^T > B_i)$ is nondecreasing in B_i, then the objective function of airline *i* is quasi-concave in B_i. Given that the two players face objective functions that are continuous and quasi-concave in each booking limit, there exists a pure-strategy Nash equilibrium (Moulin, 1986).

This reasoning does not provide us with precise conditions for the existence of an equilibrium, but we have found this analysis to be helpful when examining the results of our numerical examples. When D_{Li} and D_{Hi} are strongly negatively correlated then the total demands D_{Li}^T and D_{Hi}^T are not monotonically associated. In this case, the objective functions for each airline are not unimodal, producing the discontinuities in the reaction functions shown in Figure 4. When D_{Li} and D_{Hi} are weakly negatively correlated, independent, or positively correlated, D_{Li}^T and D_{Hi}^T maintain the positive association property and a pure-strategy equilibrium exists. We will see in Section 5 that the latter case applies for most reasonable problem parameters.

Now we do identify two sufficient conditions for the existence of a pure-strategy equilibrium. First, if low-fare demand is extremely high so that $\Pr(D_{Li} > C) = 1$ *for* $i = 1,2$, then a pure-strategy equilibrium must exist and, under certain conditions, the equilibrium must be unique and stable. In this case, low-fare overflow is ignored by each airline because there is already a surplus of low-fare customers, and airlines only compete for high-fare customers. Because this is a special case of the model presented in Section 3.1, further discussion and a proof will be presented later (see Proposition 3 and Corollary 1). The second condition involves a revision of the timing of the game. This is presented in the next section.

2.2 High-fare then Low-fare Spill

We will now change the order of events and assume that low-fare customers that overflow are accepted only after all other passengers have been accommodated. The order of events is as follows:

1. Airlines establish booking limits B_1 and B_2.
2. Low-fare passengers arrive to their first-choice airlines and are accommodated up to the booking limits.
3. High-fare passengers arrive to their first-choice airlines and are accommodated with any remaining seats, up to capacity C in each aircraft.
4. High-fare passengers not accommodated on their first-choice airlines 'spill' to the alternate airlines and are accommodated in any remaining seats, up to capacity C in each aircraft.

5. Low-fare passengers not accommodated on their first-choice airlines 'spill' to the alternate airlines and are accommodated up to the booking limits.

To maintain the flavor of the timing described in Section 2.1, in Step 5 we only book low-fare passengers up to the booking limit, even if additional seats are available. Note that this game requires each airline to distinguish between low-fare passengers who choose that airline first from those that come to the airline as a second choice. While this may not always be possible, under this re-ordering, it is possible to establish the existence of a pure-strategy Nash equilibrium because an adjustment in B_i does not affect the high-fare demand faced by airline *i*.

First define:

$\min(D_{Li}, B_i)$, number of low-fare tickets sold in the first round

$D_{Hi}^T = D_{Hi} + (D_{Hj} - (C - \min(B_j, D_{Lj}))^+$, total demand for high-fare tickets at airline *i*

$(D_{Li} - B_i)^+$, overflow of low-fare passengers

$\left(\min(B_i, C - D_{Hi}^T) - D_{Li}\right)^+$, number of seats available to the overflow low-fare passengers.

The total revenue for airline *i* is:

$$p_i = E\left[\begin{array}{l} p_L \min(D_{Li}, B_i) \\ + p_L \min\left((D_{Lj} - B_j)^+, \left(\min(B_i, C - D_{Hi}^T) - D_{Li}\right)^+\right) + p_H \min\left(C - \min(B_i, D_{Li}), D_{Hi}^T\right) \end{array}\right] \qquad (4)$$

Proposition 1. *Given the game ordering defined by steps 1-5 above, a pure-strategy Nash equilibrium in booking limits (B_1, B_2) exists.*

Proof: We will show that the objective function for each player is continuous and submodular in *(B_1, B_2)*. Therefore, the objective function is continuous and supermodular in *(B_1, -B_2)*, which are sufficient conditions for the existence of a pure-strategy Nash equilibrium (Topkis, 1998).

To see that the objective function is continuous, note that the strategy space is finite so that for any given demand realization the objective function is bounded. In addition, the objective function is continuous in *(B_1, B_2)* for any given demand realization. Therefore, by the bounded convergence theorem, the expectation (4) is continuous (Billingsley, 1995).

To prove submodularity, note that the expectation of a submodular function is submodular, the sum of submodular functions is a submodular function, and a submodular function multiplied by a positive constant is a submodular function (Topkis, 1998, Lemma 2.6.1 and Corollary 2.6.2). Therefore, we will prove that for any given demand realization, each of the three terms in the sum (4) is submodular in *(B_i, B_j)*. The first term, $\min(D_{Li}, B_i)$, depends only on B_i, so it is submodular. For the last two terms we will employ the following two lemmas (for the sake of readability, in these lemmas and for the remainder of the proof the term 'increasing' implies nondecreasing and the term 'decreasing' implies nonincreasing):

<u>Lemma 1</u> (Adopted from Topkis, 1998, Example 2.6.2 (f).) If $g_i(B_i)$ is increasing and $g_j(B_j)$ is decreasing then $\min(g_i(B_i), g_j(B_j))$ is a submodular function in *(B_i, B_j)*.

<u>Lemma 2</u> (Topkis, 1978, Table 1) Suppose $g(B_i, B_j)$ is increasing in both B_i and B_j and is a submodular (supermodular) function in *(B_i, B_j)*. Also suppose that $f(z)$ is an increasing concave (convex) function. Then $f(g(B_i, B_j))$ is a submodular (supermodular) function in *(B_i, B_j)*.

We re-write the second term of the objective function as

$$\min\left((D_{Lj}-B_j)^+, \left(\min(B_i, C-D_{Hi}^T)-D_{Li}\right)^+\right)$$
$$= (D_{Lj}-B_j)^+ + \min\left(0, \left(\min(B_i, C-D_{Hi}^T)-D_{Li}\right)^+ - (D_{Lj}-B_j)^+\right). \quad (5)$$

The term $(D_{Lj}-B_j)^+$ depends only on B_j and hence is submodular. To prove that the second term in (5) is submodular, we will employ Lemmas 1 and 2. Since $f(z)=\min(0,z)$ is a concave increasing function of *z*, it remains to show that

$$g(B_i, B_j) = \left(\min(B_i, C-D_{Hi}^T)-D_{Li}\right)^+ - (D_{Lj}-B_j)^+ \quad (6)$$

is an increasing submodular function. We first show that it is an increasing function. It is obvious that this function is increasing in B_i. Further, from the definition of D_{Hi}^T above, $\left(\min(B_i, C-D_{Hi}^T)-D_{Li}\right)^+$ is either linearly decreasing in B_j (for some demand realizations) or does not change as B_j changes. In addition, $-(D_{Lj}-B_j)^+$ is also either linearly increasing or invariant in B_j. By examining the two terms in (6), we see that when the second term is linearly increasing in B_j then the first term is either linearly decreasing or does not change. When the first term is linearly decreasing in B_j then the second term must be increasing. Therefore, the second term dominates, and $g(B_i, B_j)$ is increasing in both B_j and B_i.

We now show that $g(B_i, B_j)$ is also submodular. First, $\min(B_i, C - D_{Hi}^T)$ is increasing in B_i, decreasing in B_j, and by Lemma 1 a submodular function in (B_i, B_j). Therefore, $\min(B_i, C - D_{Hi}^T) - D_{Li}$ is increasing and supermodular in $(B_i, -B_j)$. In addition, the function $f'(z) = (z)^+ = \max(0, z)$ is convex and increasing in z, so that by Lemma 2 $\left(\min(B_i, C - D_{Hi}^T) - D_{Li}\right)^+$ is a supermodular function in $(B_i, -B_j)$ and therefore a submodular function in (B_i, B_j). Hence, $g(B_i, B_j)$ is also a submodular function. This completes the proof for the second term of (4).

The third term of the objective function is $\min\left(C - \min(B_i, D_{Li}), D_{Hi}^T\right)$. Note that $C - \min(B_i, D_{Li})$ is decreasing in B_i, and D_{Hi}^T is increasing in B_j. By Lemma 1, $\min\left(C - \min(B_i, D_{Li}), D_{Hi}^T\right)$ is submodular. This completes the proof. ■

While we can be sure of a pure-strategy equilibrium in this case, we cannot be sure that the equilibrium is unique. Conditions for uniqueness will be described in the next section.

3. Competition with Partial Overflow

In this section we consider competing airlines with only high-fare passengers overflowing from one airline to another (Section 3.1) and with only low-fare passengers overflowing (Section 3.2). For each case we will find conditions under which a pure-strategy equilibrium exists and is unique.

It is, of course, reasonable to ask why we should be concerned with these special cases since both high and low-fare customers are likely to look for a seat on another airline if one cannot be found on the preferred airline. In fact, these special cases are good approximations of the general game described in Section 2.1, as long as the number of overflow customers from one of the two fare classes is small. In addition, the model to be presented in Section 3.1

includes the case when high-fare passengers switch airlines while demand for low-fare tickets is sufficiently large to sell all available low-fare tickets.

Moreover, analysis of these special cases sheds some light on the reasons why the full game presented in Section 2 may fail to have a pure-strategy equilibrium. We will see here that a game with only high-fare overflow always has a pure-strategy equilibrium, while a game with only low-fare overflow may not. If only high-fare customers spill to a competitor, then the airlines are involved in a supermodular game similar to the inventory game described by Parlar and by Lippman and McCardle. In terms of yield management, an increase in the booking limit by one airline increases demand by high-fare passengers to the competitor, thus lowering the competitor's booking limit. Each player's reaction function is monotonic in the other player's strategy, and an equilibrium must exist. However, when both types of overflow occur the response functions need not be monotonic, as in Figure 4. Additional conditions are needed to establish the existence of a pure-strategy equilibrium.

3.1 High-Fare Overflow Only

We now assume that there is no overflow of the low-fare passengers and only high-fare passengers approach the other airline when their first-choice airline is not available. Figure 5 illustrates the flow of passengers. Note that the following definitions differ slightly from the 'full-overflow' case presented in Section 2.1.

$R_i = C - \min(D_{Li}, B_i)$, the number of seats available for high-fare passengers on airline *i*.

$D_{Hi}^T = D_{Hi} + (D_{Hj} - R_j)^+$, total demand for high-fare tickets on airline *i*, *i=1, j=2* and *i=2, j=1*.

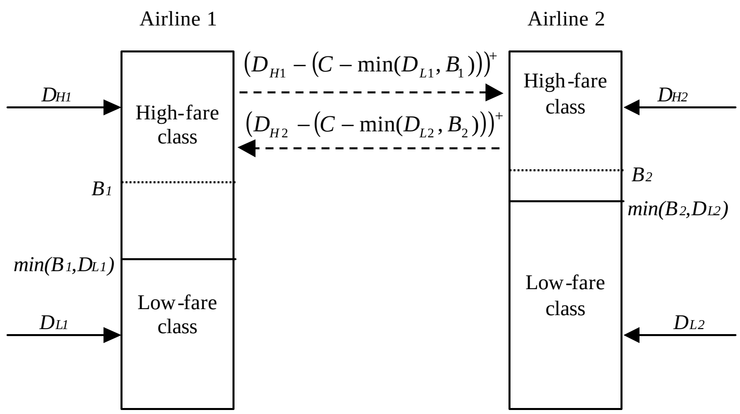

Figure 5: High-fare passengers overflow

The total revenue for airline *i* is

$$p_i = E\left[p_L \min(D_{Li}, B_i) + p_H \min(D_{Hi}^T, R_i)\right]. \tag{7}$$

The first derivative of the objective function will be useful in the following theorems. We find

$$\frac{\partial p_i}{\partial B_i} = p_L \Pr(D_{Li} > B_i) - p_H \Pr(D_{Hi}^T > C - B_i, D_{Li} > B_i). \tag{8}$$

The existence of a pure-strategy Nash equilibrium, established in the following proposition, follows from the supermodularity of the game. This result holds for any demand distribution, including distributions with correlation among airlines and among fare classes. As was the case with Proposition 1, the demand distribution may be continuous or discrete.

Proposition 2. *Given overflow by high-fare customers only, a pure-strategy Nash equilibrium in booking limits (B_1, B_2) exists.*

Proof: By the reasoning presented in the proof of Proposition 1, the objective function (7) is continuous. We will now show that both $\min(D_{Li}, B_i)$ and $\min(D_{Hi}^T, R_i)$ are submodular, so that the objective function is submodular for any given demand realization and therefore the expectation (7) is submodular. This is sufficient to establish the existence of a pure-strategy Nash equilibrium (Topkis, 1999).

Observe that min(D_{Li}, B_i) depends only on B_i and hence is submodular. By the definitions above, R_i is decreasing in B_i and D_{Hi}^T is increasing in B_j. By Lemma 1 in the proof of Proposition 1, min(D_{Hi}^T, R_i) is submodular. ■

To show there is a single, *unique* equilibrium, we make the following assumptions:

Assumption 1: There exists, for each random variable, a finite probability density function $f_{D_{ki}}(t) = d\,\Pr(D_{ki} < t)/dt$. In addition, the density functions $f_{D_{Hi}}(t) > 0$ for $0 \le t \le C$ and *i=1,2*.

Assumption 2: Demands for low-fare and high-fare tickets are independent. More formally, let $D_L = (D_{L1}, D_{L2})$ and $D_H = (D_{H1}, D_{H2})$. We assume that D_L and D_H are mutually independent.

Assumption 3: $\Pr(D_{Li} > C) > 0$ for *i=1,2*.

Proposition 3. *Given overflow by high-fare customers only and Assumptions 1-3, there is a unique, globally stable Nash equilibrium in (B_1, B_2).*

Proof: We will characterize the best reply functions (reaction functions) of the players in the game and then will show that the functions are a contraction on (B_1, B_2). Therefore, a single, unique equilibrium exists and is stable.

We will first show that each function p_i, with B_j held constant, reaches its maximum at a unique point $B_i \in [0, C)$. Given Assumption 2, the first derivatives of the objective functions may be written as

$$\frac{\partial p_i}{\partial B_i} = \Pr(D_{Li} > B_i)(p_L - p_H \Pr(D_{Hi}^T > C - B_i)) \qquad i = 1,2 . \tag{9}$$

From Assumption 3, the first derivative is always less than zero at the upper boundary C:

$$\left.\frac{\partial p_i}{\partial B_i}\right|_{B_i = C} = \Pr(D_{Li} > C)(p_L - p_H \Pr(D_{Hi}^T > 0)) = \Pr(D_{Li} > C)(p_L - p_H) < 0 . \tag{10}$$

Now consider two cases. If $\Pr(D_{Hi}^T > C) < p_L / p_H$ then (9) is positive when evaluated at the lower boundary:

$$\left.\frac{\partial p_i}{\partial B_i}\right|_{B_i=0} = \Pr(D_{Li} > 0)(p_L - p_H \Pr(D_{Hi}^T > C)) = p_L - p_H \Pr(D_{Hi}^T > C) > 0\,. \tag{11}$$

By assumption 1, $\Pr(D_{Hi}^T > C - B_i)$ is strictly increasing in B_i, and the objective function is strictly quasi-concave in the interval [*0*, *C*]. If there is an interior solution it is determined by the following first-order conditions (note that we have expanded the term D_{Hi}^T):

$$\Pr(D_{Hi} + (D_{Hj} - C + \min(D_{Lj}, B_j))^+ > C - B_i) = \frac{p_L}{p_H},\ i = 1,\ j = 2 \text{ and } i = 2,\ j = 1\,. \tag{12}$$

If $\Pr(D_{Hi}^T > C) \geq p_L / p_H$, then the objective function is not increasing at 0 and the slope does not change sign in the interval [*0*, *C*]. Therefore, the objective is maximized at $B_i = 0$.

Equation (12) and the boundary condition specify reaction functions $r_1(B_2)$ and $r_2(B_1)$ for the two players. When the value of the reaction function is in the interior (*0*, *C*) then implicit differentiation of (12) finds the magnitudes of the slopes of the reaction functions:[4]

$$\left|\frac{\partial r_i(B_j)}{\partial B_j}\right| = \left| - \frac{f_{D_{Hi}^T | D_{Hj} > C - B_j}(C - B_i)\Pr(D_{Hj} > C - B_j)}{f_{D_{Hi}^T}(C - B_i)}\right| < 1\,. \tag{13}$$

If the value of the reaction function is a boundary solution, $B_i = 0$, then $\left|\partial r_i(B_j) / \partial B_j\right| = 0 < 1$. Therefore, the reaction functions $\left(r_1(B_2), r_2(B_1)\right)$ are a contraction on (B_1, B_2).

From Proposition 2, we know that at least one equilibrium point exists. From the proof of Theorem 2.5 of Friedman [1986], if there is at least one equilibrium point and the reaction function is a contraction then the game has exactly one equilibrium point.

In addition, the expression for the derivative in (13) implies that

$$\left|\frac{\partial r_1(B_2)}{\partial B_2}\right| \left|\frac{\partial r_2(B_1)}{\partial B_1}\right| < 1 \tag{14}$$

so that the equilibrium is stable (Moulin, 1986). ■

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[4] In (13), the expression $f_{D_{Hi}^T | A}(t)$ represents the density function of D_{Hi}^T given event *A*.

This result allows us to say something stronger about the full-overflow case of Section 2.1 when low-fare demand is sufficient to fill both aircraft.

Corollary 1. *Assume overflow by both low-fare and high-fare customers. Given Assumptions 1 and 2, and given that low-fare demand is extremely large* ($\Pr(D_{Li} > C) = 1$ *for i*=1,2), *there is a unique, globally stable Nash equilibrium in (B_1, B_2).*

Proof: In this case, airlines only compete for high-fare customers and the overflow of low-fare customers can be ignored because there are no extra seats to accommodate them. In the full model objective function (1), we replace $\min(D_{Li}^T, B_i)$ with B_i. This is a special case of the model examined in Section 3.1. Therefore, the uniqueness and stability results hold here. ■

3.2 Low-Fare Overflow Only

We will now assume that high-fare passengers do not overflow and that only low-fare passengers switch airlines if their first choice is fully booked (see Figure 6).

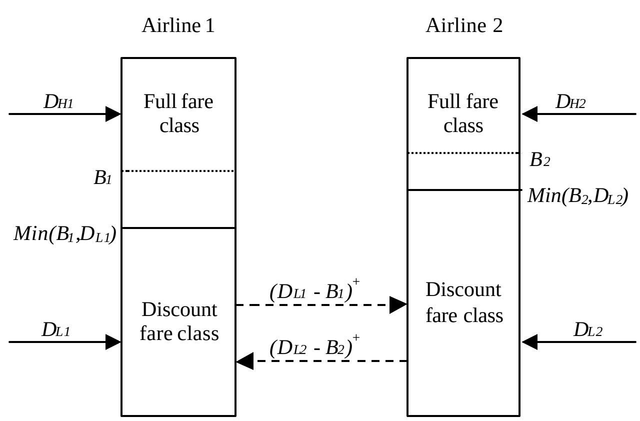

Figure 6. Low-fare passengers overflow

First define:

$D_{Li}^T = D_{Li} + (D_{Lj} - B_j)^+$, total demand for low-fare tickets on airline *i*, *i=1*, *j=2* and *i=2*, *j=1*.

$R_i = C - \min(D_{Li}^T, B_i)$, the number of seats available for high-fare passengers on airline *i*.

The number of low-fare tickets sold is equal to $\min(D_{Li}^T, B_i)$ and the total revenue for airline i is

$$\pi_i = E\left[p_L \min(D_{Li}^T, B_i) + p_H \min(D_{Hi}, R_i)\right]. \tag{15}$$

Surprisingly, a pure-strategy equilibrium need not exist for this simple game. The objective function is not necessarily submodular or quasi-concave. However, under Assumptions 1-3 the equilibrium is unique and stable.

Proposition 4. *Given overflow by low-fare customers only and Assumptions 1-3, there is a unique, globally stable Nash equilibrium in (B_1, B_2).*

Proof: Given independence between high and low-fare demands, the first derivative of the objective function (15) is

$$\frac{\partial \pi_i}{\partial B_i} = \Pr(D_{Li}^T > B_i)(p_L - p_H \Pr(D_{Hi} > C - B_i)). \tag{16}$$

The objective function is quasi-concave on [0,C] and it can be shown that the optimal solution is always in the interior, (0,C). The first-order condition

$$\Pr(D_{Hi} > C - B_i) = \frac{p_L}{p_H} \tag{17}$$

depends only on B_i and not on the competitor's action B_j. Therefore, (17) defines the unique optimal solution for each airline and each reaction function has a slope of zero. The reaction functions are a contraction on (B_1, B_2) and, following the reasoning of the proof of Proposition 3, this contraction leads to a unique, globally stable equilibrium. ■

This solution is identical to the solution for a stand-alone airline. When high-fare customers do not switch airlines and high-fare and low-fare demands are independent, the optimal booking limits for both stand-alone airlines and airlines in competition are not influenced by the demand distributions of low-fare customers.

4. Comparing the Competitors and a Monopolist

We will now compare the behavior of two airlines in competition with the behavior of a monopolist. Note that the term 'monopolist' does not necessarily imply that a single firm is the only carrier on a particular route. The 'monopolist' may be two airlines in an alliance to coordinate yield management decisions. In addition, two airlines may compete on a particular route at certain times of day, while each airline may hold a virtual monopoly at other times of day because its competitor has not scheduled a competing flight at a point close in time. For example, United Airlines has the only direct flight from Rochester, NY to the Washington DC area in the evening, while most of its flights during the morning and afternoon compete directly with flights by US Airways.

In general, we will find that the total booking limit for the monopolist is never less than the sum of the booking limits of two competing airlines. In this section we provide a proof of this result, given a model with high-fare overflow only (the model presented in Section 3.1). In the following section we will present numerical experiments utilizing the full model of Section 2.1. To simplify the comparison, we assume that the price ration p_L/p_H and the distributions of consumer demands D_{ki} are equal under the competitive and monopoly environments. In Section 6 we will discuss the implications of these assumptions.

Our results are consistent with the findings of Lippman and McCardle [1997], who analyze competing newsvendors. They find that competition never leads to a decrease in total inventory. The 'inventory' of each newsvendor is analogous to the stock of protected high-fare seats, $C - B_i$, and the demand for newspapers is analogous to demand by high-fare customers. However, our problem incorporates a significant complication, the stochastic demand by low-fare, as well as high-fare, customers.

First we review the case with no competition and only one aircraft with capacity C in the market (for further details, see Belobaba, 1989, and Brumelle et.al., 1990). Since there is just one aircraft, we will suppress the subscript $i=1,2$ which denotes the aircraft in the competitive case. After establishing a booking limit B, the airline will sell $\min(D_L, B)$ low-fare tickets and $\min\left(D_H, C - \min(D_L, B)\right)$ high-fare tickets. Therefore the total revenue is

$$p = E\left[p_D \min(D_L, B) + p_F \min\left(D_H, C - \min(D_L, B)\right)\right]. \tag{18}$$

The first derivative is

$$\frac{\partial p}{\partial B} = p_L \Pr(D_L > B) - p_H \Pr(D_H > C - B, D_L > B). \tag{19}$$

As mentioned in Section 2, the first-order conditions are sufficient for a solution when $\Pr(D_H > C - B \mid D_L > B)$ is nondecreasing in B [Brumelle et al. 1990]. Note that this condition is satisfied if D_H and D_L are independent. Given this property, a solution B^* within the interval (0,C) satisfies[5]

$$\Pr(D_H > C - B^* \mid D_L > B^*) = \frac{p_L}{p_H}. \tag{20}$$

Now consider an airline with a monopoly (or an alliance between two airlines) operating two flights. Passenger arrivals and overflows follow the order of events described by Steps 1-5 at the beginning of Section 2.1. While this case may seem to be more complex than the single-flight problem, it reduces to the simpler problem described above, since the passenger overflow from one aircraft is captured by the same firm in the other aircraft. We can write the objective function in this two-aircraft case as

$$p = E\left[p_L \min(D_{L1} + D_{L2}, B_1 + B_2) + p_H \min(D_{H1} + D_{H1}, 2C - \min(D_{L1} + D_{L2}, B_1 + B_2))\right] \tag{21}$$

and the first derivative is similar to (19) above, with $B = B_1 + B_2$:

$$\frac{\partial p}{\partial B} = p_L \Pr(D_{L1} + D_{L2} > B) - p_H \Pr(D_{H1} + D_{H2} > 2C - B, D_{L1} + D_{L2} > B). \tag{22}$$

Now we consider the situation introduced in Section 3.1. Assume that low-fare customers do *not* overflow to a second-choice flight while high-fare passengers do overflow. The objective function for the monopoly airline is

$$p = E\left[\begin{array}{l} p_L\left(\min(D_{L1}, B_1) + \min(D_{L2}, B_2)\right) \\ \qquad + p_H \min\left(D_{H1} + D_{H2}, 2C - \min(D_{L1}, B_1) - \min(D_{L2}, B_2)\right)\end{array}\right]. \tag{23}$$

An interior solution (B_1^*, B_2^*) satisfies the following first-order conditions for *i=1, j=2* and *i=2, j=1*:[6]

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[5] There is also a boundary condition. If $\Pr(D_H > C) \geq p_L / p_H$ then $B^* = 0$.

[6] Again, there are boundary conditions. We present conditions for 'extreme' solutions here. If $\Pr(D_{H1} + D_{H2} > 2C) \geq p_L / p_H$ then $(B_1^*, B_2^*) = (0,0)$. If $\Pr(D_{H1} + D_{H2} > C - \min(D_{Li}, C) \leq p_L / p_H$ for i=1,2, then $(B_1^*, B_2^*) = (C, C)$.

$$\left.\frac{\partial p}{\partial B_i}\right|_{(B_i^*,B_j^*)} = p_L \Pr(D_{Li} > B_i^*) - p_H \Pr(D_{H1} + D_{H2} > 2C - B_i^* - \min(D_{Lj}, B_j^*), D_{Li} > B_i^*) = 0\,. \quad (24)$$

There may be multiple values of (B_1^*, B_2^*) that satisfy (24).

This first-order condition and the first-order conditions (12) that uniquely determine the competitive equilibrium allow us to compare, analytically, the centralized and competitive solutions.

Proposition 5. *Assume overflow by high-fare customers only and Assumptions 1-3. Also assume that the optimal solution for the monopolist as well as the equilibrium under competition are in the interior, e.g.,* $B_i \in (0, C)$. *Then the total number of protected seats,* B_1+B_2*, is lower under competition than under the centralized solution.*

Proof: Define B_i^a, i=1,2, as the optimal decisions for the monopolist ('a' for alliance) and define B_i^c, i=1,2, as the equilibrium decisions under competition. The alliance solution is determined by the first-order conditions, equations (24). Given Assumptions 1 and 2, these first-order conditions may be re-written as

$$\Pr\left(D_{H1} + D_{H2} > 2C - B_i^a - \min(D_{Lj}, B_j^a)\right) = \frac{p_L}{p_H} \quad \text{for } i=1, j=2 \text{ and } i=2, j=1. \quad (25)$$

The competitive optimality conditions (12) may be re-written as:

$$\begin{aligned}&\Pr\left(D_{H1} + D_{H2} > 2C - B_i^c - \min(D_{Lj}, B_j^c)\right) \\ &\qquad + \Pr\left(D_{H1} + D_{H2} < 2C - B_i^c - \min(D_{Lj}, B_j^c), D_{H1} > C - B_i^c\right) = \frac{p_L}{p_H}\end{aligned} \quad (26)$$

for $i=1, j=2$ and $i=2, j=1$. Note that (25) and (26) differ by a single probability term in the left-hand side of (26). Since this extra term in nonnegative and the right-hand sides are equal, the following inequalities hold simultaneously:

$$\Pr\left(D_{H1} + D_{H2} > 2C - B_1^c - \min(D_{L2}, B_2^c)\right) \le \Pr\left(D_{H1} + D_{H2} > 2C - B_1^a - \min(D_{L2}, B_2^a)\right), \quad (27)$$

$$\Pr\left(D_{H1} + D_{H2} > 2C - B_2^c - \min(D_{L1}, B_1^c)\right) \le \Pr\left(D_{H1} + D_{H2} > 2C - B_2^a - \min(D_{L1}, B_1^a)\right). \quad (28)$$

Since this must be true for any value of C, these inequalities define stochastic orders on two pairs of single-valued functions of random variables D_{Hi} and D_{Li}. To make this clear, after some algebraic manipulation, (27) and (28) may be re-written as

$$D_{H1} + D_{H2} + \min(B_1^c + D_{L2}, B_1^c + B_2^c) \leq_{st} D_{H1} + D_{H2} + \min(B_1^a + D_{L2}, B_1^a + B_2^a), \qquad (29)$$

$$D_{H1} + D_{H2} + \min(B_2^c + D_{L1}, B_1^c + B_2^c) \leq_{st} D_{H1} + D_{H2} + \min(B_2^a + D_{L1}, B_1^a + B_2^a), \qquad (30)$$

where $X \leq_{st} Y$ indicates that *X* is smaller than *Y* in the usual stochastic order. Because of the independence between low-fare and high-fare demands (Assumption 1) and the preservation of stochastic order under convolution (Shaked and Shanthikumar, 1994),

$$\min(B_1^c + D_{L2}, B_1^c + B_2^c) \leq_{st} \min(B_1^a + D_{L2}, B_1^a + B_2^a), \qquad (31)$$

$$\min(B_2^c + D_{L1}, B_1^c + B_2^c) \leq_{st} \min(B_2^a + D_{L1}, B_1^a + B_2^a). \qquad (32)$$

Finally, by contradiction, assume that $B_1^c + B_2^c > B_1^a + B_2^a$. Then for both inequalities (31) and (32) to hold we would need simultaneously $B_1^c < B_1^a$ and $B_2^c < B_2^a$, which is inconsistent with the assumption. Hence, $B_1^c + B_2^c < B_1^a + B_2^a$. ■

Proposition 5 implies that, under competition, at least as many seats are held for high-fare customers as is optimal under joint profit maximization. For the monopolist, every high-fare passenger who does not find a seat at airline *i* and turns to airline *j* is not 'lost' to the firm. Under competition, however, when airline *i* establishes a lower booking limit, airline *j* lowers its booking limit as well as the two airlines compete for high-fare passengers.

5. Numerical Experiments

To determine whether the previous section's results apply to the full-fledged game described in Section 2.1, we calculate numerically both the competitive equilibrium and the optimal monopoly solution under a wide variety of parameter values. Our goal is to see whether the booking limit set by the monopoly, $B_1^a + B_2^a$, is consistently greater than or equal to the total booking limit under competition, $B_1^c + B_2^c$.

For each scenario, demand is distributed according to a multivariate normal distribution and truncated at zero; any negative demand is added to a mass point at zero. Solutions are found by a simple gradient algorithm and the gradients themselves, expressions (2) and (22), were evaluated by Monte Carlo integration (a simple search procedure was also used if the objective function was not quasi-concave). The scenarios are created by combining the following parameters.

- Ratio of high fare to low fare: To cover a range that includes many actual price ratios, we use the following values: p_H / p_L = [1.5, 2, 3, 4].

- Proportion of demand due to low-fare passengers: Let m_{Li} (m_{Hi}) be the average low-fare (high-fare) demand for airline *i*, *i*=1,2. Because in practice low-fare demand is often greater than high-fare demand, we assume that $m_{Li} \geq m_{Hi}$, and we use proportions $m_{Li} /(m_{Li} + m_{Hi})$ = [0.5, 0.75, 0.9]. Below we will also discuss experiments in which $m_{Li} /(m_{Li} + m_{Hi}) < 0.5$.

- Proportion of demand due to airline *i*: Let m_{k1} (m_{k2}) be the average demand for airline 1 (2), for demand class *k=L,H.* Due to symmetry, we need only test scenarios where $m_{k1} < m_{k2}$. We use ratios $m_{k1} /(m_{k1} + m_{k2})$ = [0.1, 0.25, 0.5].

- Variability: To limit the number of parameters, we assume that all four customer demand distributions have the same coefficient of variation, *CV*. We use values *CV* = [0.25, 0.5, 1, 1.5, 2]. Note that *CV*'s higher than 1 rarely occur in practice (Jacobs, Ratliff and Smith [2000] describe 0.2 to 0.6 as a reasonable range for the *CV*). However, we felt that there is some value in examining environments with highly variable demands. When we present the results below, we present both the aggregate results and the results for low *CV*s (*CV* = 0.25 or 0.5).

- Correlation: Again, to limit the number of parameters, we assume that the correlations among all demands are equal. When four random variables are distributed according to the multivariate normal distribution, the lowest possible common correlation is $-1/(4-1) = -0.33$; when the common correlation is lower than this bound the covariance matrix is not positive definite (Tong, 1980). For correlation, we use values r = [-0.3, 0.0, 0.5, 0.9].

When combined, these parameters define $4*3*3*5*4 = 720$ scenarios.

Before we examine aggregate statistics from the 720 scenarios, let us focus on a single 'baseline' scenario. We choose $p_H / p_L = 2$, $CV = 0.5$, and $r = 0$, set the mean low-fare demand to each airline at 150 passengers, and set the mean high-fare demand at 50 passengers so that $m_{Li} /(m_{Li} + m_{Hi})$ =0.75 and $m_{k1} /(m_{k1} + m_{k2})$ = 0.5. While certain parameter values included in the ranges above are unlikely to occur in practice, this scenario is relatively plausible.

Figure 2 displays the reaction functions of the airlines, given these parameter values. There is a unique equilibrium, resulting in $B_1^c = B_2^c = 144$. Therefore, the total booking limit is 288 and the airlines reserve a total of 112 seats for high-fare customers. A monopolist, on the other hand, has an optimal total booking limit of $B_1^a + B_2^a = 300$ seats, with 100 seats set aside for high-fare customers. If we define the "service level" as the probability that a customer is able to purchase a seat on either aircraft, the difference in booking limits produces significantly different service levels for each customer class. Under competition, 45% of low-fare customers found a seat on either flight, while under a monopoly the low-fare service level rises to 50%. On the other hand, high-fare passengers benefit from competition. Their service level is 77% under competition, 70% under the monopoly.

While this particular example produced a unique equilibrium, in Section 2 we saw that the full-overflow game may have multiple equilibria or may not have any equilibria at all. Such an outcome would complicate the comparison between competitive and monopoly booking limits. However, by examining the airline response functions for each of the 720 scenarios, we saw that in *every* case an equilibrium exists and was unique. All response functions were continuous, and most produced a stable equilibrium, as in Figure 2. As mentioned above, an extremely low (negative) correlation between high and low-fare demands can generate the outcome shown in Figure 4, in which no pure-strategy equilibrium exists. We have also found instances of multiple equilibria when the ratio $m_{Li} /(m_{Li} + m_{Hi})$ is low (e.g., 0.1) and correlation is negative or zero. We will discuss these cases at the end of this Section.

First we compare the total booking limits in the competitive and monopoly environments for the original 720 scenarios. In every scenario, the booking limit for the monopoly is equal to, or greater than, the sum of the booking limits for the airlines in competition. The mean difference $(B_1^a + B_2^a) - (B_1^c + B_2^c)$ across all scenarios is 15 seats, and the difference varies from 0 seats to 131 seats. When we examine only those scenarios with CV=0.25 or CV=0.5, the differences are smaller. Under these scenarios, the average difference is 9 seats with a range from 0 to 103 seats.

In general, the largest differences occur when correlation is low ($r = -0.3$) and expected demands are equally balanced among airlines and classes (when $m_{Li} /(m_{Li} + m_{Hi})$ =0.5 and $m_{k1} /(m_{k1} + m_{k2})$ =0.5). Table 2 displays the difference $(B_1^a + B_2^a) - (B_1^c + B_2^c)$ for each value of ρ. Each column of Table 2 represents an average over 180 scenarios. As the correlation increases, the difference between the monopoly and competitive cases decreases.

	ρ = –0.3	ρ = 0.0	ρ = 0.5	ρ = 0.9
Avg. monopoly total booking limit $B_1^a + B_2^a$	299	266	235	220
Avg. competitive total booking limit $B_1^c + B_2^c$	265	249	228	218
Average $(B_1^a + B_2^a) - (B_1^c + B_2^c)$	34	17	7	2
Low-fare service level (monopoly-competitive)	10.0%	4.1%	1.3%	0.4%
High-fare service level (monopoly-competitive)	-10.0%	-4.7%	-1.7%	-0.4%

Table 2. Demand correlation and the effects of competition.

The differences in booking limits have a significant effect on the service levels offered to each customer class. Over all cases, the service level offered to low-fare customers rose an average of 4% under the monopoly (39% to 43%), while the service level offered to high-fare customers declines an average of 4% under the monopoly (75% to 71%). For scenarios with low CVs the average differences were a bit smaller: 3.7% and 3.4%, respectively. In addition, the range of results was extremely large. In five scenarios out of 720, monopoly low-fare service levels were over *50%* greater than the low-fare service levels under competition. The difference in high-fare service levels was as high as 31%.

In general, the difference in total profits between the monopoly and competitive cases was small. Averaged over all 720 scenarios, profits to the monopoly are just 0.3% higher than the total profits under competition, with a range from 0% to 5%. When restricted to scenarios with CV=0.25 or CV=0.5, the average difference in profits is 0.2% with a range from 0% to 3.5%. The largest differences in profit were seen when correlation is low, p_H / p_L is high, and expected demands are equally balanced among airlines and classes. These small differences in profit are not unexpected since in most cases the objective function is relatively 'flat' near the optimum.

It is more difficult to make these comparisons when the proportion of demand due to low-fare passengers is small ($m_{Li}/(m_{Li}+m_{Hi}) < 0.5$) because scenarios with multiple competitive equilibria begin to appear. For example, with $m_{Li}/(m_{Li}+m_{Hi})=0.1$, we identified one scenario, shown in Figure 3, with three equilibria: $(B_1^c = 6, B_2^c = 36)$, $(B_1^c = 36, B_2^c = 6)$, and $(B_1^c = 22, B_2^c = 22)$. However, under this scenario the monopoly solution is $B_1^a + B_2^a = 93$. As was true for the original 720 scenarios, at each competitive equilibrium the total booking limit is smaller than or equal to the booking limit chosen by a monopoly. This was true for *all* examined scenarios with $m_{Li}/(m_{Li}+m_{Hi}) < 0.5$.

6. Observations and Future Research

In this paper we have examined how competition affects a fundamental decision in yield management, the allocation of seats among low and high-fare classes. Besides the technical results concerning the existence and uniqueness of competitive equilibria and the analytical expressions for the first-order conditions, our primary finding is that the sum of the airlines' booking limits under competition is no higher than the total booking limit produced when total profits from both flights are maximized (as in a monopoly or when airlines cooperate in setting booking limits). Under competition more high-fare tickets and fewer low-fare tickets may be sold than under a monopoly. This is not an obvious result, for in many standard economic models competition leads to a fall in prices (e.g., a simple Bertrand model of price competition). Here, we have held prices constant, but competition leads to a reallocation of inventory among customer segments, producing a rise in the average price paid for an airline seat.

Under the monopoly solution, low-fare customers are more likely to find a seat, and are more likely to find a seat on a first-choice airline, than under competition. With prices held constant, a monopolist would improve service for the low-price segment while diminishing service for the high-price segment. This may be particularly interesting in regulatory environments in which antitrust laws prohibit airlines from colluding on prices but allow them to coordinate yield management decisions.

As we mentioned in Section 2, our model ignores many real-world aspects of yield management, such as the spill-over of passengers between fare classes and the more general seat inventory control problem based on the origin and destination of each passenger rather than the individual flight leg (Belobaba, 1998). We have also ignored the use of seat inventory as part of a long-term strategy to gain market share on a particular route (see Yuen and Irrgang, 1998, for a description of this practice). Additional research is needed in these areas, and we are particularly interested in how our comparison between total booking limits under competitive and monopoly environments may be extended to a game with more than two fare classes.

Another significant concern with the analysis is that when comparing competitive and cooperative booking limits we assume that both prices and exogenous demand are constant. For some comparisons this assumption may be reasonable. For example, two competing airlines often charge the same prices throughout the day for travel on a particular route, and some hours in the day are 'competitive' while others are monopolized by a single airline (as in the example of the Rochester to Washington DC route described in Section 4). Prices are uniform over all flights, but competition throughout the day may significantly affect the yield management decisions of both airlines.

On a more strategic level, the existence of multiple airlines on a route increases the competition for passengers on most flights. Will our analysis change significantly if the entry into a market by a competing airline leads to lower fares, as economic theory predicts? First we note that for both the monopolist and competitive airlines the booking limits depend primarily on the *ratio* of high to low-fare prices, p_L / p_H, and not on the absolute prices. There is empirical evidence that competition on a particular route reduces the spread of fares (Morrison and Winston, 1995). This implies that competition may increase the ratio p_L / p_H and thus raise the average booking limit, an effect that may counteract the decline in booking limits under competition described in Sections 4 and 5. Which effect dominates may be the subject of empirical research that compares the actual yield management practices of airlines in markets with, and without, competition.

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