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REVERSED HARDY-LITTEWOOD-SOBOLEV INEQUALITY

JINGBO DOU AND MEIJUN ZHU

Abstract In this paper, we obtain a reversed Hardy-Littlewood-Sobolev inequality: for $0 < p,\ t < 1$ and $\lambda = n - \alpha < 0$ with $1/p + 1/t + \lambda/n = 2$, there is a best constant $N(n, \lambda, p) > 0$, such that

$$|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)|x-y|^{-\lambda} g(y)dxdy| \geq N(n,\lambda,p)||f||_{L^p(\mathbb{R}^n)}||g||_{L^t(\mathbb{R}^n)}$$

holds for all nonnegative functions $f \in L^p(\mathbb{R}^n)$, $g \in L^t(\mathbb{R}^n)$. For $p = t$, we prove the existence of extremal functions, classify all extremal functions via the method of moving sphere, and compute the best constant.

1. Introduction

The classic sharp Hardy-Littlewood-Sobolev (HLS) inequality ([9, 10, 18, 15]) states that

$$|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)|x-y|^{-(n-\alpha)} g(y)dxdy| \leq N(n,\lambda,p)||f||_{L^p(\mathbb{R}^n)}||g||_{L^t(\mathbb{R}^n)} \tag{1.1}$$

holds for all $f \in L^p(\mathbb{R}^n)$, $g \in L^t(\mathbb{R}^n)$, $1 < p,\ t < \infty$, $0 < \lambda := n - \alpha < n$ and $1/p + 1/t + \lambda/n = 2$. Lieb [15] proved the existence of the extremal functions to the inequality with sharp constant and computed the best constant in the case of $p = t$ (or one of these two parameters is two). The sharp HLS inequality implies sharp Sobolev inequality, Moser-Trudinger-Onofri and Beckner inequalities [1], as well as Gross's logarithmic Sobolev inequality [6]. All these inequalities play significant role in solving global geometric problems, such as Yamabe problem, Ricci flow problem, etc. Besides recent extension of the sharp HLS on the Heisenberg group by Frank and Lieb [5], there are at least two other directions concerning the extension of the above sharp HLS inequality: (1) Extending the sharp inequality on general manifolds, see, for example, Dou and Zhu [3] for such an extension on the upper half space and related research; (2) Extending it for the negative exponent λ (that is for the case of $\alpha > n$). In this paper, we extend the sharp HLS inequality for the negative exponent λ.

More specifically, in this paper, we prove that the reversed Hardy-Littlewood-Sobolev inequality for $0 < p,\ t < 1$, $\lambda < 0$ holds for all nonnegative $f \in L^p(\mathbb{R}^n)$, $g \in L^t(\mathbb{R}^n)$. For $p = t$, the existence of extremal functions is proved, all extremal functions are classified via the method of moving sphere, and the best constant is computed.

Prior to our research, it seems that the only result concerning $\lambda < 0$ was discussed by Stein and Weiss [20] in 1960, where they showed that a HLS inequality (not in the sharp form) for $p \in ((n-1)/n, n/\alpha)$ holds (Theorem G in [20]). However, the range

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for p does not include the important conformal invariant case $p = t = 2n/(n+\alpha)$, thus it seems hard to find the sharp constant. On the other hand, recent results on sharp Sobolev type inequalities with negative exponents on $\mathbb{S}^n$, (see, e.g. Yang and Zhu [21], Hang and Yang [8] for the Paneitz operator on $\mathbb{S}^3$, and Ni and Zhu [17] for the Laplacian operator on $\mathbb{S}^1$), strongly indicate that certain HLS inequalities for $\lambda < 0$ shall hold.

The main purpose of this paper is to establish the following reversed HLS inequality and its sharp form.

Theorem 1.1. *For $n \geq 1, 0 < p,\, t < 1$ and $\lambda = n - \alpha < 0$ satisfying*

$$\frac{1}{p} + \frac{1}{t} + \frac{\lambda}{n} = 2, \tag{1.2}$$

there is a best constant $N^*(n,\alpha,p) > 0$, such that, for all nonnegative $f \in L^p(\mathbb{R}^n)$, $g \in L^t(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)|x-y|^{-\lambda}g(y)dxdy \geq N^*(n,\alpha,p)||f||_{L^p(\mathbb{R}^n)}||g||_{L^t(\mathbb{R}^n)}. \tag{1.3}$$

For $p < 1$, the convention notation for $f(x) \in L^p(\mathbb{R}^n)$ means $\int_{\mathbb{R}^n}|f(x)|^p dx < \infty$.

For $p = t = 2n/(n+\alpha)$, we are able to compute the sharp constant. In this case, inequality (1.3) is equivalent to the following reversed HLS on sphere $\mathbb{S}^n$: for all nonnegative $F \in L^p(\mathbb{S}^n)$, $G \in L^p(\mathbb{S}^n)$,

$$\int_{\mathbb{S}^n}\int_{\mathbb{S}^n} F(\xi)|\xi-\eta|^{\alpha-n}G(\eta)dS_\xi dS_\eta \geq N^*(n,\alpha)||F||_{L^p(\mathbb{S}^n)}||G||_{L^p(\mathbb{S}^n)}, \tag{1.4}$$

where and throughout the paper $|\xi - \eta|$ is denoted as the chordal distance from ξ to η in $\mathbb{R}^{n+1}$, and $N^*(n,\alpha)$ is the same as $N^*(n,\alpha,2n/(n+\alpha))$.

For $\alpha \in (0,\infty)$, define the classic singular integral operator on $\mathbb{S}^n$ by

$$\tilde{I}_\alpha F(\xi) = \int_{\mathbb{S}^n}\frac{F(\eta)}{|\xi-\eta|^{n-\alpha}}dS_\eta,\ \forall \xi \in \mathbb{S}^n. \tag{1.5}$$

We have

Theorem 1.2. *Let $1 \leq n < \alpha$. For all nonnegative $F \in L^{2n/(n+\alpha)}(\mathbb{S}^n)$,*

$$||\tilde{I}_\alpha F||_{L^{\frac{2n}{n-\alpha}}(\mathbb{S}^n)} \geq N^*(n,\alpha)||F||_{L^{\frac{2n}{n+\alpha}}(\mathbb{S}^n)}, \tag{1.6}$$

where the best constant

$$N^*(n,\alpha) = \pi^{(n-\alpha)/2}\frac{\Gamma(\alpha/2)}{\Gamma(n/2+\alpha/2)}\{\frac{\Gamma(n/2)}{\Gamma(n)}\}^{-\alpha/n}; \tag{1.7}$$

And equality holds if and only if

$$F(\xi) = a(1-\xi\cdot\eta)^{-\frac{n+\alpha}{2}}$$

for some $a > 0$ and $\eta \in \mathbb{R}^{n+1}$ with $|\eta| < 1$.

For $\alpha \in (0,\infty)$, define the classic singular integral operator on $\mathbb{R}^n$ by

$$I_\alpha f(x) = \int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}dy,\ \forall x \in \mathbb{R}^n. \tag{1.8}$$

From Theorem 1.2 and a stereographic projection, we have the sharp reversed HLS inequality on $\mathbb{R}^n$ for $p = t = 2n/(n+\alpha)$.

Corollary 1.3. *Let $1 \le n < \alpha$. For all nonnegative function $f \in L^{2n/(n+\alpha)}(\mathbb{R}^n)$,*

$$||I_\alpha f||_{L^{\frac{2n}{n-\alpha}}(\mathbb{R}^n)} \ge N^*(n,\alpha)||f||_{L^{\frac{2n}{n+\alpha}}(\mathbb{R}^n)}, \tag{1.9}$$

where $N^*(n,\alpha)$ is given by (1.7); And the equality holds if and only if

$$f(x) = c\Big(\frac{1}{|x-x_0|^2+d^2}\Big)^{\frac{n+\alpha}{2}}$$

for some $c,\ d > 0$, and $x_0 \in \mathbb{R}^n$.

We outline the strategy in proving above theorems. The proof of Theorem 1.1 is along the line of the proof for the classic HLS inequality (see, e.g. Stein [19]). The main difference is that our inequality is reversed. The reversed Hölder inequality, converse Young's inequality, as well as a new established Marcinkiewicz interpolation involving exponents less than 1 (could be negative) are used. Our proof for the existence of extremal functions is quite different to that for the sharp HLS inequality (due to Lieb [15]), and we can only obtain the result for $p = t = 2n/(n+\alpha)$. We first prove Theorem 1.2 for $F(\xi) \in L^1(\mathbb{S}^n)$. A density lemma (Lemma 3.1) will be established, which allows us to reduce the proof for all L^1 functions to continuous functions on $\mathbb{S}^n$. The extra condition for functions (i.e. $F(\xi) \in L^1(\mathbb{S}^n)$) will be removed while considering its dual form (inequality (1.4))*. In proving the existence of extremal functions, symmetrization argument is used. We point out here that for $\alpha > n$, there is a new phenomenon in proving the convergence of the minimizing sequence $\{F_i\}_{i=1}^\infty$: even F_i has a concentration mass, the mass of $\tilde{I}_\alpha F_i$ may not. In other words, the classic concentration compactness argument does not work. In fact, we show in Remark 3.3 that even a minimizing sequence $\{F_i\}_{i=1}^\infty$ pointwise converges to F, $\tilde{I}_\alpha F_j(\xi)$ may not converge to $\tilde{I}_\alpha F(\xi)$ pointwise. It is one of the main difficulties to show that there is a subsequence of $\tilde{I}_\alpha F_j(\xi)$ that is a Cauchy sequence under certain metric. Another difficulty is to classify all extremal functions in order to compute the sharp constant. This is settled via the method of moving sphere, introduced in Li and Zhu [14]. Our research certainly answers one of Y.Y. Li's open questions in [12], where he asks for the background for the study of the integral equation with negative exponents.

Quite natural question after we establish the reversed Hardy-Littlewood-Sobolev inequality is: Can we derive certain Sobolev type inequalities (such as those Sobolev inequalities with negative powers on $\mathbb{S}^1$ and on $\mathbb{S}^3$), and use these Sobolev inequalities to investigate curvature equations (for example, the prescribing $Q-$ curvature on $\mathbb{S}^3$)? It is not obvious that one can derive Sobolev inequalities from the reversed HLS inequality as in the case for HLS inequality. However, we are able to use the reversed HLS inequality directly to derive the existence of solutions to certain curvature equations, see Zhu [22]. From the view point given in Zhu [22], it seems more natural to extend Lieb's sharp HLS inequality on $\mathbb{S}^n$ to the ones on general compact Riemannian manifolds, and use them to investigate curvature equations (including a generalized Yamabe problem formulated in [22]. More details will be given in a forthcoming paper [7].

The paper is organized as follows. Theorem 1.1 is proved in Section 2, where a new Marcinkiewicz interpolation theorem is also stated and proved; Theorem 1.2 is proved in Section 3, where a Liouville theorem (Theorem 3.6) concerning an integral system is also proved.

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2. REVERSED HARDY-LITTLEWOOD-SOBOLEV INEQUALITY

In this section, we prove Theorem 1.1: the reversed HLS inequality (with a rough constant) in $\mathbb{R}^n$ for $\lambda = n - \alpha < 0$.

2.1. **Some basic inequalities.** For $p < 1$ and $p \neq 0$, if $f(x)$ satisfies $\int_{\mathbb{R}^n} |f|^p dx < \infty$, we say $f(x) \in L^p(\mathbb{R}^n)$, and call $(\int_{\mathbb{R}^n} |f|^p)^{1/p} dx$ (denoted as $||f||_{L^p}$ later) the L^p norm of $f(x)$. The L^p norm for $p < 1$ is not a norm for a vector space. Nevertheless, certain integral inequalities still hold.

Lemma 2.1 (Reversed Hölder inequality). For $p \in (0,1)$, $p' = p/(p-1)$, and nonnegative functions $f \in L^p(\mathbb{R}^n)$ and $g \in L^{p'}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} f(x)g(x)dx \geq \|f\|_{L^p}\|g\|_{L^{p'}}.$$

The reversed Hölder inequality can be derived easily from the standard Hölder inequality.

Lemma 2.2 (Converse Young's inequality). Suppose that $0 < p < 1$, and $q, r < 0$ are three parameters satisfying $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. For any nonnegative measurable functions h, g, define

$$g * h(x) = \int_{\mathbb{R}^n} g(x-y)h(y)dy.$$

Then

$$\|g * h\|_{L^r} \geq \|g\|_{L^q}\|h\|_{L^p}.$$

The proof of the above converse Young's inequality can be found, e.g. in Brascamp and Lieb [2], where they also identified the best constant for the classic Young's inequality.

Lemma 2.3 (Reversed Minkowski inequality). If $q < 0$, then for any nonnegative measurable functions $F(x,y)$,

$$\Big[\int_Y \Big(\int_X F(x,y)d\mu(x)\Big)^q d\nu(y)\Big]^{\frac{1}{q}} \geq \int_X \Big(\int_Y [F(x,y)]^q d\nu(y)\Big)^{\frac{1}{q}} d\mu(x)$$

The proof for the reversed Minkowski inequality can be found in [11] (on P_{148}).

To establish the reversed HLS inequality, we also need to extend the classic Marcinkiewicz interpolation theorem for L^p function with $p < 1$.

Recall: for a given measurable function $f(x)$ on $\mathbb{R}^n$ and $0 < p < \infty$, the weak L^p norm of $f(x)$ is defined by

$$\|f\|_{L^p_W} = \inf\{A > 0 \,:\, meas\{|f(x)| > t\} \cdot t^p \leq A^p\},$$

For $p < 0$, we define the weak L^p norm for $f(x)$ in a similar way:

$$||f||_{L^p_W} := \sup\{A > 0 :\, m\{|f(x)| < t\} \cdot t^p \leq A^p\}.$$

Thus, for $p < 0$,

$$||f||^p_{L^p_W} := \inf\{B > 0 :\, m\{|f(x)| < t\} \cdot t^p \leq B\}.$$

Let $T : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ be a linear operator. We recall that for $0 < p,\, q < \infty$, operator T is called the weak type (p,q) if there exists a constant $C(p,q) > 0$ such that for all $f \in L^p(\mathbb{R}^n)$

$$meas\{x : |Tf(x)| > \tau\} \le \big(C(p,q)\frac{\|f\|_{L^p}}{\tau}\big)^q, \ \forall\, \tau > 0.$$

Similarly, we can extend the definition of the weak type (p,q) to the case $q < 0 < p < 1$.

Definition 2.1. For $q < 0 < p < 1$, we say operator T is of the weak type (p,q), if there exists a constant $C(p,q) > 0$, such that for all $f \in L^p(\mathbb{R}^n)$,

$$meas\{x : |Tf(x)| < \tau\} \le \big(C(p,q)\frac{\|f\|_{L^p}}{\tau}\big)^q, \ \forall\, \tau > 0.$$

We now can state the extension to the classic Marcinkiewicz interpolation theorem.

Proposition 2.2. *Let T be a linear operator which maps any nonnegative function to a nonnegative function. For a pair of numbers $(p_1, q_1), (p_2, q_2)$ satisfying $q_i < 0 < p_i < 1$, $i = 1, 2$, $p_1 < p_2$ and $q_1 < q_2$, if T is weak type (p_1, q_1) and (p_2, q_2) for all nonnegative functions, then for any $\theta \in (0,1)$, and*

$$\frac{1}{p} \ = \ \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \qquad \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}, \tag{2.1}$$

then T is reversed strong type (p,q) for all nonnegative functions, that is,

$$\|Tf\|_{L^q} \ge C\|f\|_{L^p}, \quad \forall f \in L^p(\mathbb{R}^n) \quad and \quad f \ge 0, \tag{2.2}$$

for some constant $C = C(p_1, p_2, q_1, q_2, \theta) > 0$.

Proof. For any measurable function $f(x)$, denote $\tilde{m}_g(\tau) = meas\{x : |g(x)| < \tau\}$.

Easy to check that for $r < 0$, if $g \in L^r(\mathbb{R}^n)$, then

$$\tilde{m}_g(\tau) \le \frac{\|g\|^r_{L^r}}{\tau^r}, \quad \forall\, \tau > 0,$$

and

$$\|g\|^r_{L^r(X)} = |r| \int_0^\infty t^{r-1}\tilde{m}_g(t)dt. \tag{2.3}$$

For a nonnegative $f(x) \in L^{p_1}(\mathbb{R}^n) \cap L^{p_2}(\mathbb{R}^n)$ and $\gamma > 0$, write $f = f_1 + f_2$, where

$$f_1(x) = \begin{cases} f(x), & \text{if } f(x) \le \gamma, \\ 0, & \text{if } f(x) > \gamma, \end{cases}$$

and

$$f_2(x) = \begin{cases} 0, & \text{if } f(x) \le \gamma, \\ f(x), & \text{if } f(x) > \gamma. \end{cases}$$

Thus

$$\{x : Tf < \tau\} \subset \{x : Tf_1 < \frac{\tau}{2}\} \cup \{x : Tf_2 < \frac{\tau}{2}\}.$$

Since T is weak type (p_1, q_1) and (p_2, q_2), we have

$$\tilde{m}_{Tf_1}(\tau) \le c_1(\frac{\|f_1\|_{L^{p_1}}}{\tau})^{q_1}, \quad \text{and} \quad \tilde{m}_{Tf_2}(\tau) \le c_2(\frac{\|f_2\|_{L^{p_2}}}{\tau})^{q_2}.$$

It then follows from (2.3) that

$$\begin{aligned}\|Tf\|_{L^q}^q &= |q|\int_0^\infty t^{q-1}\tilde{m}_{Tf}(t)dt \\ &\le |q|\int_0^\infty t^{q-1}[\tilde{m}_{Tf_1}\{t/2\}+\tilde{m}_{Tf_2}\{t/2\}]dt \\ &\le C|q|\int_0^\infty t^{q-q_1-1}\|f_1\|_{L^{p_1}}^{q_1}dt + C|q|\int_0^\infty t^{q-q_2-1}\|f_2\|_{L^{p_2}}^{q_2}dt \\ &= C|q|V_1 + C|q|V_2, \qquad (2.4)\end{aligned}$$

where

$$V_1 = \int_0^\infty t^{q-q_1-1}\|f_1\|_{L^{p_1}}^{q_1}dt, \quad V_2 = \int_0^\infty t^{q-q_2-1}\|f_2\|_{L^{p_2}}^{q_2}dt.$$

From (2.1), we know

$$\frac{p_1}{q_1}\cdot\frac{q-q_1}{p-p_1} = \frac{p_2}{q_2}\cdot\frac{q-q_2}{p-p_2} < 0.$$

Choose $\sigma = \frac{p_1}{q_1}\cdot\frac{q-q_1}{p-p_1} = \frac{p_2}{q_2}\cdot\frac{q-q_2}{p-p_2}$, and let $k_1 = \frac{q_1}{p_1} < 0$ and $k_2 = \frac{q_2}{p_2} < 0$. We have

$$p_1 + \frac{q-q_1}{\sigma k_1} = p_2 + \frac{q-q_2}{\sigma k_2} = p.$$

Let $\gamma = (\frac{t}{A})^\sigma$ with A being a constant to be specified later. From the reversed Minkowski inequality, we have

$$\begin{aligned}V_1^{\frac{1}{k_1}} &= \Big[\int_0^\infty \big(\int_{\mathbb{R}^n} t^{\frac{q-q_1-1}{k_1}}|f_1(x)|^{p_1}dx\big)^{k_1}dt\Big]^{\frac{1}{k_1}} \\ &\ge \int_{\mathbb{R}^n}\big(\int_0^{A|f(x)|^{\frac{1}{\sigma}}} t^{q-q_1-1}|f_1(x)|^{p_1k_1}dt\big)^{\frac{1}{k_1}}dx \\ &= \big(\frac{A^{q-q_1}}{q-q_1}\big)^{\frac{1}{k_1}}\int_{\mathbb{R}^n}|f(x)|^{p_1+\frac{q-q_1}{\sigma k_1}}dx.\end{aligned}$$

That is

$$V_1 \le \frac{A^{q-q_1}}{q-q_1}\cdot\big(\int_{\mathbb{R}^n}|f(x)|^p dx\big)^{k_1}. \qquad (2.5)$$

For V_2, we have

$$\begin{aligned}V_2^{\frac{1}{k_2}} &= \Big[\int_0^\infty \big(\int_{\mathbb{R}^n} t^{\frac{q-q_2-1}{k_2}}|f_2(x)|^{p_2}dx\big)^{k_2}dt\Big]^{\frac{1}{k_2}} \\ &\ge \int_{\mathbb{R}^n}\big(\int_{A|f(x)|^{\frac{1}{\sigma}}}^\infty t^{q-q_2-1}|f_2(x)|^{p_2k_2}dt\big)^{\frac{1}{k_2}}dx \\ &= \big(\frac{A^{q-q_2}}{q_2-q}\big)^{\frac{1}{k_2}}\int_{\mathbb{R}^n}|f(x)|^{p_2+\frac{q-q_2}{\sigma k_2}}dx.\end{aligned}$$

Thus,

$$V_2 \le \frac{A^{q-q_2}}{q_2-q}\cdot\big(\int_{\mathbb{R}^n}|f(x)|^p dx\big)^{k_2}. \qquad (2.6)$$

Substituting (2.5) and (2.6) into (2.4), we have

$$\|Tf\|_{L^q}^q \le C(V_1+V_2) \le C(A^{q-q_1}\|f\|_{L^p}^{pk_1} + A^{q-q_2}\|f\|_{L^p}^{pk_2}).$$

Choosing suitable $A > 0$ such that

$$A^{q-q_1}\|f\|_{L^p}^{pk_1} = A^{q-q_2}\|f\|_{L^p}^{pk_2}.$$

We obtain

$$\|Tf\|^q_{L^q} \le C\|f\|^q_{L^p}.$$

□

2.2. **The rough reversed HLS inequality.** We are now ready to prove the reversed HLS inequality with a rough constant (Theorem 1.1).

The reversed HLS inequality with a rough constant can be derived from the following proposition.

Proposition 2.3. *For any* $1 \le n < \alpha$, $\frac{n}{\alpha} < p < 1$ *and* q *given by*

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}, \tag{2.7}$$

there exists a constant $C = C(n, \alpha, p) > 0$*, such that for all nonnegative* $f \in L^p(\mathbb{R}^n)$*,*

$$||I_\alpha f||_{L^q} \ge C||f||_{L^p}. \tag{2.8}$$

It follows from (2.8) and the reversed Hölder inequality that for nonnegative functions $f \in L^p$ and $g \in L^t$,

$$< I_\alpha f, g >\ge C||f||_{L^p} \cdot ||g||_{L^t},$$

where $t = q' = \frac{q}{q-1}$ (thus $t \in (0, 1)$); Which yields: for $\alpha > n$,

$$|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)|x-y|^{\alpha-n}g(y)dxdy| \ge C||f||_{L^p}||g||_{L^t},$$

where t, p satisfy

$$1 - \frac{1}{t} = \frac{1}{p} - \frac{\alpha}{n} \Leftrightarrow \frac{1}{p} + \frac{1}{t} + \frac{n-\alpha}{n} = 2.$$

Proof of Proposition 2.3. For $1 \le n < \alpha$, $p \in (\frac{n}{\alpha}, 1)$ and q given by (2.7), we first prove

$$\|I_\alpha f\|_{L^q_W} \ge C(n, \alpha, p)\|f\|_{L^p} \tag{2.9}$$

for some constant $C(n, \alpha, p)$. That is, we need to show that there is a constant $C(n, \alpha, p) > 0$, such that

$$meas\{x : |I_\alpha f(x)| < \tau\} \le \big(C(n, \alpha, p)\frac{\|f\|_{L^p}}{\tau}\big)^q, \ \forall f \in L^p(\mathbb{R}^n) \text{ and } f(x) \ge 0, \ \forall\, \tau > 0. \tag{2.10}$$

Inequality (2.9) then implies (2.8) via the new Marcinkiewicz interpolation (Proposition 2.2).

For any $\rho > 0$, define

$$I^1_{\alpha,\rho} f(x) = \int_{|y-x|\le\rho} \frac{f(y)}{|x-y|^{n-\alpha}} dy,$$

and

$$I^2_{\alpha,\rho} f(x) = \int_{|y-x|>\rho} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

Note both $I^1_{\alpha,\rho}$ and $I^2_{\alpha,\rho}$ map nonnegative functions to nonnegative functions. Thus, for any $\tau > 0$,

$$meas\{x : I_\alpha f(x) < 2\tau\} \le meas\{x : I^1_{\alpha,\rho} f(x) < \tau\} + meas\{x : I^2_{\alpha,\rho} f(x) < \tau\}. \tag{2.11}$$

We note that it suffices to prove inequality (2.10) with 2τ in place of τ in the left side of the inequality, and we can further assume $\|f\|_{L^p}=1$.

Using the converse Young's inequality, we have

$$\begin{aligned}\|I^1_{\alpha,\rho}f\|_{L^{r_1}} &\geq \Big(\int_{\mathbb{R}^n}\big(\frac{\chi_\rho(|x-y|)}{|x-y|^{n-\alpha}}\big)^{t_1}dy\Big)^{\frac{1}{t_1}}\|f\|_{L^p}\\ &=: D_1.\end{aligned}$$

where $\frac{1}{p}+\frac{1}{t_1}=1+\frac{1}{r_1}$ with $t_1\in(\frac{n}{n-\alpha},0), r_1<0$, $\chi_\rho(x)=1$ for $|x|\leq\rho$ and $\chi_\rho(x)=0$ for $|x|>\rho$, and

$$D_1 = \Big(\int_{B_\rho(x)}\frac{1}{|x-y|^{(n-\alpha)t_1}}dy\Big)^{\frac{1}{t_1}} = C_1(n,\alpha)\rho^{\frac{n-(n-\alpha)t_1}{t_1}}.$$

It follows that

$$meas\{x:|I^1_{\alpha,\rho}f|<\tau\}\leq\frac{\|I^1_{\alpha,\rho}f\|^{r_1}_{L^{r_1}}}{\tau^{r_1}}\leq\frac{C_2(n,\alpha)\rho^{\frac{r_1[n-(n-\alpha)t_1]}{t_1}}}{\tau^{r_1}}. \tag{2.12}$$

On the other hand, by the converse Young's inequality, we have

$$\begin{aligned}\|I^2_{\alpha,\rho}f\|_{L^{r_2}} &\geq \Big(\int_{\mathbb{R}^n}\big(\frac{1-\chi_\rho(|x-y|)}{|x-y|^{n-\alpha}}\big)^{t_2}dy\Big)^{\frac{1}{t_2}}\|f\|_{L^p}\\ &=: D_2,\end{aligned}$$

where $\frac{1}{p}+\frac{1}{t_2}=1+\frac{1}{r_2}$ with $t_2<\frac{n}{n-\alpha}, r_2<0$. Easy to see: $r_2<\frac{np}{n-\alpha p}<r_1$. Also,

$$D_2 = \Big(\int_{\mathbb{R}^n\setminus B_\rho(x)}\big(\frac{1}{|x-y|^{n-\alpha}}\big)^{t_2}dy\Big)^{\frac{1}{t_2}} = C_3(n,\alpha)\rho^{\frac{n-(n-\alpha)t_2}{t_2}}.$$

It follows that

$$meas\{x:|I^2_{\alpha,\rho}f|<\tau\}\leq\frac{\|I^2_{\alpha,\rho}f\|^{r_2}_{L^{r_2}}}{\tau^{r_2}}\leq\frac{C_4(n,\alpha)\rho^{\frac{r_2[n-(n-\alpha)t_2]}{t_2}}}{\tau^{r_2}}. \tag{2.13}$$

Bringing (2.12) and (2.13) into (2.11), we have

$$meas\{x:|I_\alpha f|<2\tau\} \leq \frac{C_2(n,\alpha)\rho^{\frac{r_1[n-(n-\alpha)t_1]}{t_1}}}{\tau^{r_1}}+\frac{C_4(n,\alpha)\rho^{\frac{r_2[n-(n-\alpha)t_2]}{t_2}}}{\tau^{r_2}}.$$

Now, choosing $\rho=\tau^{\frac{p}{p\alpha-n}}$, we have

$$\begin{aligned}\frac{p\,r_1}{p\alpha-n}\cdot[\frac{n}{t_1}-(n-\alpha)]-r_1 &= \frac{pr_1}{p\alpha-n}\Big[n\big(1+\frac{1}{r_1}-\frac{1}{p}\big)-n+\alpha\Big]-r_1\\ &= \frac{pr_1}{p\alpha-n}[\frac{np-r_1n+pr_1\alpha}{r_1p}]-r_1\\ &= -\frac{np}{n-p\alpha}\\ &= -q.\end{aligned}$$

Similarly, $\frac{p\,r_2}{p\alpha-n}\cdot[\frac{n}{t_2}-(n-\alpha)]-r_2=-\frac{np}{n-p\alpha}=-q$. we then obtain (2.10). □

3. Existence and classifications of extremal functions for sharp inequalities

We shall discuss the sharp form of the reversed HLS and prove Theorem 1.2 in this section.

3.1. **Existence of extremal functions.** In the case of $p = t$, we are able to show the existence of extremal functions. In this regard, it is relatively easier to state the sharp form of the inequality on the standard sphere $\mathbb{S}^n$.

Let $\mathcal{S} : x \in \mathbb{R}^n \to \xi \in \mathbb{S}^n \backslash (0, 0, \cdots, -1)$ be the inverse of a stereographic projection, defined by

$$\xi^j := \frac{2x^j}{1 + |x|^2}, \quad \text{for } j = 1, 2, \cdots, n; \quad \xi^{n+1} := \frac{1 - |x|^2}{1 + |x|^2}.$$

Easy to check (or, see e.g. [15, 16]): for $x, y \in \mathbb{R}^n, \xi \in \mathbb{S}^n$,

$$|\mathcal{S}(x) - \mathcal{S}(y)| = \big[\frac{4|x - y|^2}{(1 + |x|^2)(1 + |y|^2)}\big]^{\frac{1}{2}}, \quad d\xi = \big(\frac{2}{1 + |x|^2}\big)^n dx.$$

The area of the unit sphere in $\mathbb{R}^{n+1}$ is given by

$$|\mathbb{S}^n| := \int_{\mathbb{S}^n} d\xi = 2\pi^{\frac{n+1}{2}} \big[\Gamma\big(\frac{n+1}{2}\big)\big]^{-1} = 2^n \pi^{\frac{n}{2}} \frac{\Gamma(n/2)}{\Gamma(n)}.$$

For $1 \le n < \alpha$, let $p = \frac{2n}{n+\alpha}, q = \frac{2n}{n-\alpha}$ throughout whole subsection 3.1. For any $F(\xi) \in L^p(\mathbb{S}^n)$, let $x = \mathcal{S}^{-1}(\xi) \in \mathbb{R}^n$, and define $f(x) := \big(\frac{2}{1+|x|^2}\big)^{\frac{n+\alpha}{2}} F(\xi)$. Also recall

$$\tilde{I}_\alpha F(\xi) = \int_{\mathbb{S}^n} \frac{F(\eta)}{|\xi - \eta|^{n-\alpha}} d\eta.$$

Direct computation yields

$$\int_{\mathbb{S}^n} |F(\xi)|^p d\xi = \int_{\mathbb{R}^n} \big(\frac{2}{1 + |x|^2}\big)^{-n} |f(\mathcal{S}^{-1}(\xi))|^p \big(\frac{2}{1 + |x|^2}\big)^n dx = \int_{\mathbb{R}^n} |f(x)|^p dx;$$

And

$$\begin{aligned}
\|\tilde{I}_\alpha F\|^q_{L^q(\mathbb{S}^n)} &= \int_{\mathbb{S}^n} \big(\int_{\mathbb{S}^n} \frac{F(\eta)}{|\xi - \eta|^{n-\alpha}} d\eta\big)^q d\xi \\
&= \int_{\mathbb{R}^n} \big(\int_{\mathbb{R}^n} \big(\frac{2}{1 + |y|^2}\big)^{-\frac{n+\alpha}{2}} f(\mathcal{S}^{-1}(\eta)) \big[\frac{4|x - y|^2}{(1 + |x|^2)(1 + |y|^2)}\big]^{-\frac{n-\alpha}{2}} \\
&\quad \times \big(\frac{2}{1 + |y|^2}\big)^n dy\big)^q \big(\frac{2}{1 + |x|^2}\big)^n dx \\
&= \int_{\mathbb{R}^n} \big(\int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} dy\big)^q dx.
\end{aligned}$$

Thus

$$\|F\|_{L^p(\mathbb{S}^n)} = \|f\|_{L^p(\mathbb{R}^n)}, \quad \text{and} \quad \|\tilde{I}_\alpha F\|_{L^q(\mathbb{S}^n)} = \|I_\alpha f\|_{L^q(\mathbb{R}^n)}. \tag{3.1}$$

We hereby have an equivalent sharp reversed HLS inequality (1.4) for all non-negative $F(\xi),\ G(\xi) \in L^p(\mathbb{S}^n)$. The sharp constant to inequality (1.6) is classified by

$$\begin{aligned}
N^*(n, \alpha) &= \inf\{\|\tilde{I}_\alpha F\|_{L^q(\mathbb{S}^n)} : F \ge 0, \ \|F\|_{L^p(\mathbb{S}^n)} = 1\} \\
&= \inf\{\|I_\alpha f\|_{L^q(\mathbb{R}^n)} : f \ge 0, \ \|f\|_{L^p(\mathbb{R}^n)} = 1\}.
\end{aligned} \tag{3.2}$$

We remark that we only need to show that sharp inequality (1.6) holds for all nonnegative $F \in L^1(\mathbb{S}^n)$. In fact, if for all nonnegative $F, G \in L^1(\mathbb{S}^n)$,

$$\int_{\mathbb{S}^n} \int_{\mathbb{S}^n} F(\xi) |\xi - \eta|^{\alpha - n} G(\eta) dS_\xi dS_\eta \ge N^*(n, \alpha) ||F||_{L^p(\mathbb{S}^n)} ||G||_{L^p(\mathbb{S}^n)},$$

then for any nonnegative $u, v \in L^p(\mathbb{S}^n)$, we consider $u_A = \min(u, A) \in L^1(\mathbb{S}^n)$ and $v_A = \min(v, A) \in L^1(\mathbb{S}^n)$, thus

$$\int_{\mathbb{S}^n}\int_{\mathbb{S}^n} u_A(\xi)|\xi-\eta|^{\alpha-n}v_A(\eta)dS_\xi dS_\eta \geq N^*(n,\alpha)||u_A||_{L^p(\mathbb{S}^n)}||v_A||_{L^p(\mathbb{S}^n)}.$$

Sending $A \to \infty$, we obtain via the monotone convergence theorem the desired sharp inequality for $u, v \in L^p(\mathbb{S}^n)$.

Since we are dealing with a reserved inequality, the usual density argument does not work here. More specifically, even one can prove inequality (1.3) for all $f,\ g \in C_0^\infty(\mathbb{R}^n)$, it is not obvious that the inequality also holds for general function $f \in L^p(\mathbb{R}^n)$ and $g \in L^t(\mathbb{R}^n)$. So we need to establish the following density lemma on $\mathbb{S}^n$.

Lemma 3.1. (Density Lemma) *Let $F(\xi) \in L^1(\mathbb{S}^n)$ be a nonnegative function with $\|F\|_{L^p(\mathbb{S}^n)} = 1$. For any $\epsilon > 0$, there is a nonnegative $G(\xi) \in C^0(\mathbb{S}^n)$, such that*

$$||F-G||_{L^p(\mathbb{S}^n)} + \left|||\tilde{I}_\alpha F||_{L^q(\mathbb{S}^n)} - ||\tilde{I}_\alpha G||_{L^q(\mathbb{S}^n)}\right| < \epsilon.$$

Proof. Let $\{G_i\}_{i=1}^\infty$ be a sequence of nonnegative, continuous functions such that $||G_i - F||_{L^1(\mathbb{S}^n)} \to 0$ as $i \to \infty$. Then, for any $\xi \in \mathbb{S}^n$, as $i \to \infty$,

$$\begin{aligned} |\tilde{I}_\alpha G_i(\xi) - \tilde{I}_\alpha F(\xi)| &\leq \int_{\mathbb{S}^n} |G_i(\eta) - F(\eta)| \cdot |\xi-\eta|^{\alpha-n} d\eta \\ &\leq C\int_{\mathbb{S}^n} |G_i(\eta) - F(\eta)| d\eta \to 0. \end{aligned}$$

Since $||F||_{L^p(\mathbb{S}^n)} = 1$, we know that $||F||_{L^1(\mathbb{S}^n)} \geq C > 0$. This implies that $\tilde{I}_\alpha F(\xi)$ is a continuous and positive function. Thus $\tilde{I}_\alpha G_i(\xi) \geq C > 0$ for large i. From the dominant convergent theorem, we have

$$\liminf_{i\to\infty} \int_{\mathbb{S}^n} |\tilde{I}_\alpha G_i|^q = \int_{\mathbb{S}^n} |\tilde{I}_\alpha F|^q.$$

Lemma 3.1 then follows from the above. □

Next, we prove that the infimum in (3.2) is attained. Due to Lemma 3.1 and the remark before it, we can choose $\{F_j\}_{j=1}^\infty \in C^0(\mathbb{S}^n)$ to be a nonnegative minimizing sequence with $\|F_j\|_{L^p} = 1$. Let $f_j(x) := \left(\frac{2}{1+|x|^2}\right)^{\frac{n+\alpha}{2}} F_j(\mathcal{S}(x))$, then $\{f_j\}_{j=1}^\infty \in C^0(\mathbb{R}^n)$ is the nonnegative corresponding minimizing sequence with $\|f_j\|_{L^p} = 1$ on $\mathbb{R}^n$.

For a given nonnegative measurable function $u(x)$ on $\mathbb{R}^n$ decaying at infinity, we can define its radially symmetric, non-increasing rearrangement function u^*. $u^*(x)$ is a nonnegative lower-semicontinuous function and has the same distribution as u. Define $v_* = ((v^{-1})^*)^{-1}$, then v_* is radially symmetric, increasing rearrangement function. It is known (see, e.g. the proof of Proposition 9 in Brascamp and Lieb [2]) that

$$\begin{aligned} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(x)w(x-y)v(y)dydx &\geq \textstyle\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u^*(x)w^*(x-y)v_*(y)dydx \\ &= \textstyle\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u^*(x)v_*(x-y)w^*(y)dydx. \end{aligned}$$

Suppose that $\|w\|_{L^{q'}(\mathbb{R}^n)} = \|w^*\|_{L^{q'}(\mathbb{R}^n)} = 1$ for $0 < q' < 1$. Then for $q < 0$ and $q' = q/(q-1)$, we have

$$\begin{aligned}
\|u * v\|_{L^q} &= \inf_{\|w\|_{L^{q'}}=1} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u(x)v(x-y)w(y)dydx \\
&\geq \inf_{\|w*\|_{L^{q'}}=1} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u^*(x)v_*(x-y)w^*(y)dydx \\
&\geq \inf_{\|w*\|_{L^{q'}}=1} \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} (u^*(x)v_*(x-y))^q dy\Big)^{\frac{1}{q}} \Big(\int_{\mathbb{R}^n} (w^*(y))^{q'} dy\Big)^{\frac{1}{q'}} dx \\
&= \|u^* * v_*\|_{L^q}. \qquad (3.3)
\end{aligned}$$

Let f_j^* be the non-increasing radial symmetric rearrangement of f_j. Since

$$\|f_j^*\|_{L^p} = \|f_j\|_{L^p} = 1,$$

and

$$\begin{aligned}
\|I_\alpha(f_j)\|_{L^q}^q &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} \frac{f_j(y)}{|x-y|^{n-\alpha}} dy\Big)^q dx \\
&\leq \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} \frac{f_j^*(y)}{|x-y|^{n-\alpha}} dy\Big)^q dx \qquad \text{(by (3.3))} \\
&= \|I_\alpha(f_j^*)\|_{L^q}^q,
\end{aligned}$$

we know that $\{f_j^*\}_{j=1}^\infty$ is also a minimizing sequence. Without loss of generality, we can assume that $\{f_j\}_{j=1}^\infty$ is a nonnegative radially symmetric and non-increasing minimizing sequence.

For $\alpha \in (0, n)$, to avoid that f_j converges to a trivial function, Lieb modified his maximizing sequence via a technical lemma (Lemma 2.4 in Lieb [15]). In our case, we need to modify the minimizing sequence in a similar way so that both $I_\alpha f_j$ and f_j will stay away from the trivial function via the following lemma.

Lemma 3.2. *Let $p_1 \in (0, 2n/(n+\alpha))$, and $s \in (\frac{n}{n-\alpha}, 0)$ be two parameters satisfying $\frac{1}{p_1} + \frac{1}{s} - 1 = \frac{n-\alpha}{2n}$. Suppose that $f \in L^p(\mathbb{R}^n)$ is a nonnegative, radially symmetric function satisfying $f(|x|) \leq \varepsilon |x|^{-\frac{n}{p}}$ for all $|x| > 0$. Then, there exists a constant C_n independent of f and ε such that*

$$\|I_\alpha f\|_{L^q(\mathbb{R}^n)} \geq C_n \varepsilon^{1-\frac{p}{p_1}} \|f\|_{L^p(\mathbb{R}^n)}^{\frac{p}{p_1}}. \qquad (3.4)$$

Proof. Our proof is similar to that of Lemma 2.4 in Lieb [15].

Define $F : \mathbb{R} \to \mathbb{R}$ by

$$F(u) = e^{\frac{un}{p}} f(e^u).$$

We can easily see that

$$(n\omega_n)^{\frac{1}{p}} \|F\|_{L^p(\mathbb{R})} = \|f\|_{L^p(\mathbb{R}^n)}, \quad \text{and} \quad \|F\|_{L^\infty(\mathbb{R})} \leq \varepsilon,$$

where $\omega_n = \frac{2\pi^{\frac{n}{2}}}{n} \Gamma(\frac{n}{2})$ denotes the volume of the n-dimensional unit ball. Define $h = I_\alpha f$. Easy to see that h is radially symmetric. Define $H : \mathbb{R} \to \mathbb{R}$ by

$$H(u) = e^{\frac{un}{q}} h(e^u).$$

Then

$$(n\omega_n)^{\frac{1}{q}} \|H\|_{L^q(\mathbb{R})} = \|h\|_{L^q(\mathbb{R}^n)}.$$

By integrating dx over angles in $\mathbb{R}^n$, an explicit form for H can be obtained as follows.

$$H(u) = \int_{-\infty}^{+\infty} L_n(u-v)F(v)dv,$$

where

$$\begin{aligned} L_n(u) &= \big(\frac{1}{2}\big)^{\frac{n-\alpha}{2}} e^{u(\frac{n}{q}-\frac{n-\alpha}{2})} Z_n(u), \\ Z_n(u) &= \begin{cases} (n-1)\omega_{n-1}\int_0^\pi (\cosh u - \cos\theta)^{\frac{\alpha-n}{2}} (\sin\theta)^{n-2} d\theta, & n \geq 2, \\ (\cosh u + 1)^{\frac{\alpha-n}{2}} + (\cosh u - 1)^{\frac{\alpha-n}{2}}, & n = 1. \end{cases} \end{aligned}$$

We have $L_n \in L^s(\mathbb{R})$ for any given $s < 0$.

Now, by the converse Young inequality (Lemma 2.2), for given $p_1 \in (0, 2n/(n+\alpha))$, and $s \in (\frac{n}{n-\alpha}, 0)$ satisfying $\frac{1}{p_1} + \frac{1}{s} - 1 = \frac{n-\alpha}{2n} = \frac{1}{q}$, we have

$$\|H\|_{L^q(\mathbb{R})} \geq \|L_n\|_{L^s(\mathbb{R})} \|F\|_{L^{p_1}(\mathbb{R})}. \tag{3.5}$$

On the other hand, since $p_1 < p < 1$, we have

$$\begin{aligned} \|F\|_{L^{p_1}(\mathbb{R})} &= \big(\int_{-\infty}^{+\infty} |F(v)|^p |F(v)|^{p_1 - p} dv\big)^{\frac{1}{p_1}} \\ &\geq \|F\|_{L^\infty(\mathbb{R})}^{1-\frac{p}{p_1}} \|F\|_{L^p(\mathbb{R})}^{\frac{p}{p_1}} \\ &\geq \varepsilon^{1-\frac{p}{p_1}} \|F\|_{L^p(\mathbb{R})}^{\frac{p}{p_1}}. \end{aligned}$$

Combining the above with (3.5), we obtain (3.4). □

For convenience, denote $e_1 = (1, 0, \cdots, 0, 0) \in \mathbb{R}^n$, and define

$$a_j := \sup_{\lambda > 0} \lambda^{-\frac{n}{p}} f_j(\frac{e_1}{\lambda}).$$

Note that for $y \in \mathbb{R}^n$,

$$f_j(y) = f_j(|y|e_1) = |y|^{-\frac{n}{p}} |y|^{\frac{n}{p}} f_j(|y|e_1) \leq a_j |y|^{-\frac{n}{p}},$$

and $\|I_\alpha f_j\|_{L^q} \to N^*(n, \alpha) < \infty$. We know from Lemma 3.2 that $a_j \geq 2c_0 > 0$.

For any given nonnegative function $g(x)$ and $\lambda > 0$, define $g^\lambda(x) = \lambda^{-\frac{n}{p}} g(\frac{x}{\lambda})$. Easy to check that

$$I_\alpha g^\lambda(x) = \lambda^{\alpha - \frac{n}{p}} (I_\alpha g)(\frac{x}{\lambda});$$

and

$$\|g^\lambda\|_{L^p} = \|g\|_{L^p}, \qquad \|I_\alpha(g^\lambda)\|_{L^q} = \|I_\alpha g\|_{L^q}. \tag{3.6}$$

For each j, choose λ_j so that $f_j^{\lambda_j}(e_1) \geq c_0$. Due to (3.6), we know that $\{f_j^{\lambda_j}\}_{j=1}^\infty$ is also a minimizing sequence. Therefore, we can further assume that there is a nonnegative, radially symmetric and non-increasing minimizing sequence $\{f_j\}_{j=1}^\infty$ with $\|f_j\|_{L^p} = 1$ and $f_j(e_1) \geq c_0$. Similar to Lieb's argument, we know, up to a subsequence, that $f_j \to f_\circ$ a.e. in $\mathbb{R}^n$.

Consider the corresponding minimizing sequence $F_j(\xi) = \big(\frac{1+|\mathcal{S}^{-1}(\xi)|^2}{2}\big)^{\frac{n+\alpha}{2}} f_j(\mathcal{S}^{-1}(\xi))$, and $F_\circ(\xi) = \big(\frac{1+|\mathcal{S}^{-1}(\xi)|^2}{2}\big)^{\frac{n+\alpha}{2}} f_\circ(\mathcal{S}^{-1}(\xi))$. We know $F_j(\xi) = F_j(\xi^{n+1})$. Denote $\mathfrak{N} = (0, ..., 0, 1)$ as the north pole of the sphere, and $\xi_1 = \mathcal{S}(e_1)$. So $F_j(\xi_1) \geq c_0$, and $F_j(\xi) \ge 2^{-(n+\alpha)/2}c_0$ for all ξ in the geodesic ball $B_{r_0}(\mathfrak{N})$ where $r_0 = dis(\xi_1, \mathfrak{N})$ on $\mathbb{S}^n$. Thus, there is a positive universal constant $C > 0$, such that

$$\tilde{I}_\alpha F_j(\xi) \ge C, \quad \forall\, \xi \in \mathbb{S}^n. \tag{3.7}$$

If $\tilde{I}_\alpha F_j(\xi) \to +\infty$ almost everywhere, then the dominant convergent theorem (using (3.7)) yields that $\lim_{j\to\infty}\int_{\mathbb{S}^n} |\tilde{I}_\alpha F_j(\xi)|^q = 0$. But $\lim_{j\to\infty}\int_{\mathbb{S}^n} |\tilde{I}_\alpha F_j(\xi)|^q = (N^*(n,\alpha))^q > 0$. Contradiction. Thus for $\eta = (0, ..., 0, \eta^{n+1})$ and $\eta^{n+1} \in (a,b) \subset (-1,1)$, $\tilde{I}_\alpha F_j(\eta) < C(a,b)$ for certain constant $-1 \le a < b \le 1$ and a constant $C(a,b)$ depending only on $a,\ b$. This yields

$$\int_{\mathbb{S}^n} F_j \le C_{a,b} \tag{3.8}$$

for some constant $C_{a,b}$ only depending on $a,\ b$. From (3.8) we know that sequence $\{\tilde{I}_\alpha F_j\}$ is uniformly bounded and equicontinuous on $\mathbb{S}^n$. Up to a subsequence, $\tilde{I}_\alpha F_j(\xi) \to L(\xi) \in C^0(\mathbb{S}^n)$. Using Fatou Lemma and the reversed HLS, we have, up to a further subsequence, for $m \in \mathbb{N}$, that

$$\begin{aligned} 0 \ &\ge\ \big(\lim_{j\to\infty}\int_{\mathbb{S}^n} |\tilde{I}_\alpha F_j(\xi) - \tilde{I}_\alpha F_{j+m}(\xi)|^q\big)^{1/q} \\ &\ge\ C(\lim_{j\to\infty} ||F_j - F_{j+m}||^q_{L^p})^{1/q}. \end{aligned}$$

Thus $||F_j - F_{j+m}||_{L^p} \to 0$. Since $F_j \to F_\circ$ pointwise, we know that $||F_j - F_\circ||_{L^p} \to 0$, thus $||F_\circ||_{L^p} = 1$.

On the other hand, the dominant convergent theorem (using (3.7)) yields $\lim_{j\to\infty}\int_{\mathbb{S}^n} |\tilde{I}_\alpha F_j(\xi)|^q = \int_{\mathbb{S}^n} |L(\xi)|^q$, and we know from Fatou Lemma that

$$L(\xi) = \lim_{j\to\infty} \tilde{I}_\alpha F_j(\xi) \ge \tilde{I}_\alpha F_\circ(\xi).$$

It follows that

$$\big(\int_{\mathbb{S}^n} |L(\xi)|^q\big)^{1/q} \ge \big(\int_{\mathbb{S}^n} |\tilde{I}_\alpha F_\circ(\xi)|^q\big)^{1/q}.$$

Thus the $\inf\{\|\tilde{I}_\alpha F\|_{L^q(\mathbb{S}^n)} \,:\, F \ge 0,\ \ \|F\|_{L^p(\mathbb{S}^n)} = 1\}$ is achieved by $F_\circ(\xi)$.

Remark 3.3. It is not clear whether the pointwise convergence

$$\tilde{I}_\alpha F_j(\xi) \to \tilde{I}_\alpha F_\circ(\xi) \qquad a.e.\text{on}\ \ \mathbb{S}^n \tag{3.9}$$

is true or not. Even though we tend to believe this is the case, we do not know how to prove it. On the other hand, we point out here that a new phenomenon does arise while dealing with a concentrating minimizing sequence for $q < 0$. We will show that without assuming that $f_j(e_1) \ge c_0$ for the corresponding minimizing sequence $\{f_j(x)\}$ defined on $\mathbb{R}^n$, the pointwise convergence (3.9) may not be true. This is opposite to the case for $q > 0$, where the pointwise convergence (3.9) usually holds for extremal sequences.

In fact, for

$$f_j(x) = \big(\frac{\epsilon_j}{\epsilon_j^2 + |x|^2}\big)^{\frac{n+\alpha}{2}}$$

where $\epsilon_j \to 0$ as $j \to \infty$, we know $f_j \to f_\circ = 0$ a.e. in $\mathbb{R}^n$. One may check directly that $I_\alpha f_j$ does not converge to 0 a.e. in $\mathbb{R}^n$ (in fact, $I_\alpha f_j \to \infty$ a.e. in $\mathbb{R}^n$). This can also be observed from the reversed HLS inequality:

$$\|I_\alpha|f_j - f_\circ|\|_{L^q} \ge C(n,\alpha,p)\|f_j - f_\circ\|_{L^p}. \tag{3.10}$$

if $I_\alpha f_j \to 0$ pointwise, from Fatou Lemma, we know that the left side in (3.10) will go to 0, but the right side $||f_j||_{L^p} = constant > 0$. Impossible.

Let $F_\circ \in L^p(\mathbb{S}^n)$ be a nonnegative minimizer. After normalization, we can assume $||F_\circ||_{L^p} = 1$. Easy to see $\tilde{I}_\alpha F_\circ(\xi) \geq C > 0$. Thus, for any positive smooth test function $\phi \in C^\infty(\mathbb{S}^n)$, we have

$$\int_{\mathbb{S}^n} F_\circ^{p-1}(\xi)\phi(\xi)d\xi \leq C \int_{\mathbb{S}^n}\int_{\mathbb{S}^n} \frac{(\tilde{I}_\alpha F_\circ(\eta))^{q-1}}{|\eta-\xi|^{n-\alpha}}\phi(\xi)d\eta d\xi \leq C_1 < \infty. \tag{3.11}$$

Since $p < 1$, we conclude that there is a positive constant $c_0 > 0$ such that $F_\circ(\xi) > c_0$ everywhere on $\mathbb{S}^n$. Thus $F_\circ(\xi)$ is a weak positive solution to

$$F_\circ^{p-1}(\xi) = \int_{\mathbb{S}^n} \frac{(\tilde{I}_\alpha F_\circ(\eta))^{q-1}}{|\xi-\eta|^{n-\alpha}}d\eta, \qquad \forall\, \xi \in \mathbb{S}^n. \tag{3.12}$$

To complete the proof of Theorem 1.2, we need to classify all positive solutions to (3.12), and to computer the best constant next.

Let $f(x) := \big(\frac{2}{1+|x|^2}\big)^{\frac{n+\alpha}{2}} F_\circ(\mathcal{S}(x))$, then $f(x)$ is a measurable positive function, satisfying:

$$f^{p-1}(x) = \int_{\mathbb{R}^n} \frac{(I_\alpha f(y))^{q-1}}{|x-y|^{n-\alpha}}dy, \qquad \forall\, x \in \mathbb{R}^n. \tag{3.13}$$

3.2. **Extremal functions and best constant.** We will classify all positive, measurable solutions to equation (3.13) via the method of moving sphere for $p = 2n/(n+\alpha)$ and $q = 2n/(n-\alpha)$, and compute the best constant $N^*(n,\alpha)$.

For $R > 0, x \in \mathbb{R}^n$, denote

$$B_R(x) = \{y \in \mathbb{R}^n \,:\, |y-x| < R\}, \quad \text{and } \Sigma_{x,R} = \mathbb{R}^n \backslash \overline{B_R(x)}.$$

For $x = 0$, we write $B_R = B_R(0), \Sigma_R = \Sigma_{0,R}$.

3.2.1. *Regularity.* First, we show that positive solutions to (3.13) are smooth except the case that the function $f(x)$ (thus $I_\alpha f(x)$) is infinity everywhere. Throughout this subsection, we always assume that f is a positive measurable function satisfying (3.13) such that both f and $I_\alpha f \not\equiv \infty$.

Define $u(y) = f^{p-1}(y)$, $v(x) = I_\alpha f(x)$, $\theta = \frac{1}{p-1} < 0$ and $\kappa = q-1 < 0$. Then u, v are also positive measurable functions and the single equation (3.13) can be rewritten as an integral system

$$\begin{cases} u(y) = \int_{\mathbb{R}^n} |x-y|^{\alpha-n} v^\kappa(x)dx, & y \in \mathbb{R}^n, \\ v(x) = \int_{\mathbb{R}^n} |x-y|^{\alpha-n} u^\theta(y)dy, & x \in \mathbb{R}^n. \end{cases} \tag{3.14}$$

Lemma 3.4. *For $1 \le n < \alpha$ and $\theta, \kappa < 0$, if (u, v) is a pair of positive Lebesgue measurable solutions to* (3.14)*, then*

$$(i) \qquad \int_{\mathbb{R}^n} (1+|y|^{\alpha-n}) u^\theta(y) dy < \infty, \quad \text{and} \quad \int_{\mathbb{R}^n} (1+|x|^{\alpha-n}) v^\kappa(x) dx < \infty;$$

$$(ii) \qquad a := \lim_{|y|\to\infty} |y|^{n-\alpha} u(y) = \int_{\mathbb{R}^n} v^\kappa(x) dx < \infty,$$

$$b := \lim_{|x|\to\infty} |x|^{n-\alpha} v(x) = \int_{\mathbb{R}^n} u^\theta(y) dy < \infty;$$

(iii) *for some constants $C_1, C_2 > 0$,*

$$\frac{1+|y|^{\alpha-n}}{C_1} \le u(y) \le C_1(1+|y|^{\alpha-n}), \ \forall\, y \in \mathbb{R}^n,$$

$$\frac{1+|x|^{\alpha-n}}{C_2} \le v(x) \le C_2(1+|x|^{\alpha-n}), \ \forall\, x \in \mathbb{R}^n.$$

Proof. The proof is the same as that of Lemma 5.1 in Li [12]. We include details for the completion of the paper.

Since $u,\ v \not\equiv \infty$, we know that

$$meas\{y \in \mathbb{R}^n : u(y) < \infty\} > 0, \quad \text{and} \quad meas\{x \in \mathbb{R}^n : v(x) < \infty\} > 0.$$

Thus, there exist $R > 1$ and some measurable set E such that

$$E \subset \{x \in \mathbb{R}^n : v(x) < R\} \cap B_R$$

with $|E| > \frac{1}{R}$. It follows, for any $y \in \mathbb{R}^n$, that

$$\begin{aligned} u(y) &= \int_{\mathbb{R}^n} |x-y|^{\alpha-n} v^\kappa(x) dx \\ &\ge \int_E |x-y|^{\alpha-n} v^\kappa(x) dx \\ &\ge R^\kappa \int_E |x-y|^{\alpha-n} dy. \end{aligned}$$

And then,

$$\lim_{|y|\to\infty} \frac{u(y)}{(1+|y|^{\alpha-n})} \quad \ge \quad \lim_{|y|\to\infty} \frac{R^\kappa}{(1+|y|^{\alpha-n})} \int_E |x-y|^{\alpha-n} dx = CR^{\kappa-1}.$$

This shows

$$u(y) \ge \frac{(1+|y|^{\alpha-n})}{C_1}.$$

Similarly, for any $x \in \mathbb{R}^n$, we have

$$v(x) \ge \frac{(1+|x|^{\alpha-n})}{C_2}.$$

This implies that the left hand side inequalities in (iii) hold.

On the other hand, for some $y_0 \in \mathbb{R}^n$ with $1 \le |y_0| \le 2$,

$$\int_{\mathbb{R}^n} |x-y_0|^{\alpha-n} v^\kappa(x) dx \quad = \quad u(y_0) < \infty;$$

And, for some $x_0 \in \mathbb{R}^n$ with $1 \le |x_0| \le 2$,

$$\int_{\mathbb{R}^n} |x_0-y|^{\alpha-n} u^\theta(y) dy \quad = \quad v(x_0) < \infty.$$

From the left hand side inequalities in (iii) and the above, we obtain (i).

For $|x| \geq 1$,

$$\frac{|x-y|^{\alpha-n}}{|x|^{\alpha-n}} u^{\theta}(y) \leq (1+|y|^{\alpha-n}) u^{\theta}(y),$$

and for $|y| \geq 1$,

$$\frac{|x-y|^{\alpha-n}}{|y|^{\alpha-n}} v^{\kappa}(x) \leq (1+|x|^{\alpha-n}) v^{\kappa}(x).$$

Combining these with (i) and using the dominated convergence theorem we have (ii) :

$$a = \lim_{|y|\to\infty} |y|^{n-\alpha} u(y) = \lim_{|y|\to\infty} \int_{\mathbb{R}^n} \frac{|x-y|^{\alpha-n}}{|y|^{\alpha-n}} v^{\kappa}(x)dx = \int_{\mathbb{R}^n} v^{\kappa}(x)dx < \infty,$$

and

$$b = \lim_{|x|\to\infty} |x|^{n-\alpha} v(x) = \lim_{|x|\to\infty} \int_{\mathbb{R}^n} \frac{|x-y|^{\alpha-n}}{|x|^{\alpha-n}} u^{\theta}(y)dy = \int_{\mathbb{R}^n} u^{\theta}(y)dy < \infty.$$

Combining (i) and (ii) with (3.14), we have the right side inequality in (iii). □

Lemma 3.5. *For $1 \leq n < \alpha$ and $\theta, \kappa < 0$, if (u, v) is a pair of positive Lebesgue measurable solutions to* (3.14)*, then $u, v \in C^{\infty}(\mathbb{R}^n)$.*

Proof. Again, we adopt the proof given in Li [12]. For $R > 0$, we can split u into following two parts

$$\begin{aligned} u(y) &= \int_{|y|\leq 2R} |x-y|^{\alpha-n} v^{\kappa}(x)dx + \int_{|y|>2R} |x-y|^{\alpha-n} v^{\kappa}(x)dx \\ &= J_1(x) + J_2(x). \end{aligned}$$

From Lemma 3.4 (i) we know that $J_2(x)$ can be differentiated under the integral for $|y| < R$, so $J_2 \in C^{\infty}(B_R)$. On the other hand, by Lemma 3.4 (iii), we have $v^{\kappa} \in L^{\infty}(B_{2R})$, it is obvious that J_1 is at least Hölder continuous in B_R. Since $R > 0$ is arbitrary, u is Hölder continuous in $\mathbb{R}^n$. Thus, u^{θ} is Hölder continuous in B_{2R}. Similarly, we have v, v^{κ} are Hölder continuous in B_{2R}. By bootstrap, we conclude that $u, v \in C^{\infty}(\mathbb{R}^n)$. □

3.2.2. *Classification of solutions to* (3.14)*.* In this part, we classify all nonnegative, non-infinity solutions to integral system (3.14) for $\theta = \kappa = (n+\alpha)/(n-\alpha)$ (that is: for $1 \leq n < \alpha$, $p = 2n/(n+\alpha)$, $q = 2n/(n-\alpha)$ in (3.13)).

From the above discussion, we know that if (u, v) is a pair of positive measurable solutions to system (3.14) which is not identical infinity, then u, $v \in C^{\infty}(\mathbb{R}^n)$.

Theorem 3.6. *For $1 \leq n < \alpha$ and $\theta = \kappa = (n+\alpha)/(n-\alpha)$, if (u, v) is a pair of positive finite smooth solutions to system* (3.14)*, then u, v must be the following forms on $\mathbb{R}^n$:*

$$u(\xi) = c_1 \Big(\frac{1}{|\xi-\xi_0|^2 + d^2}\Big)^{\frac{n-\alpha}{2}},$$

$$v(\xi) = c_2 \Big(\frac{1}{|\xi-\xi_0|^2 + d^2}\Big)^{\frac{n-\alpha}{2}},$$

where $c_1, c_2 > 0, d > 0, \xi_0 \in \mathbb{R}^n$.

Remark 3.7. If one can prove that u is proportional to v first, then system (3.14) can be reduced to a single equation, and the classification result for the single equation was early obtained by Li [12]. However, it is not obvious to us that u is proportional to v, even though one can show that it is the case for the classic HLS inequality. In the meantime, our current work certainly gives an answer to Li's open question 1 in [12].

The above theorem will be proved via the method of moving sphere, following the proof for a single equation given in Li [12].

For $x \in \mathbb{R}^n$ and $\lambda > 0$, we define the following transform:

$$\omega_{x,\lambda}(\xi) = \big(\frac{\lambda}{|\xi - x|}\big)^{n-\alpha}\omega(\xi^{x,\lambda}), \quad \forall \xi \in \mathbb{R}^n \setminus \{x\},$$

where

$$\xi^{x,\lambda} = x + \frac{\lambda^2(\xi - x)}{|\xi - x|^2}$$

is the Kelvin transformation of ξ with respect to $B_\lambda(x)$. Also we write $\omega^k_{x,\lambda}(\xi) := (\omega_{x,\lambda}(\xi))^k$ for any give power k.

Lemma 3.8. *Let $1 \le n < \alpha$ and $\theta, \kappa < 0$. If (u, v) is a pair of positive solutions to system (3.14), then, for any $x \in \mathbb{R}^n$,*

$$u_{x,\lambda}(\xi) = \int_{\mathbb{R}^n} \frac{v^\kappa_{x,\lambda}(\eta)}{|\xi - \eta|^{n-\alpha}}\big(\frac{\lambda}{|\eta - x|}\big)^{\tau_1} d\eta, \quad \forall \xi \in \mathbb{R}^n, \tag{3.15}$$

$$v_{x,\lambda}(\eta) = \int_{\mathbb{R}^n} \frac{u^\theta_{x,\lambda}(\xi)}{|\xi - \eta|^{n-\alpha}}\big(\frac{\lambda}{|\xi - x|}\big)^{\tau_2} d\xi, \quad \forall \eta \in \mathbb{R}^n, \tag{3.16}$$

where $\tau_1 = n + \alpha - \kappa(n - \alpha), \tau_2 = n + \alpha - \theta(n - \alpha)$. Moreover,

$$u_{x,\lambda}(\xi) - u(\xi) = \int_{\Sigma_{x,\lambda}} K(x, \lambda; \xi, \eta)\big[v^\kappa(\eta) - \big(\frac{\lambda}{|\eta - x|}\big)^{\tau_1} v^\kappa_{x,\lambda}(\eta)\big] d\eta, \tag{3.17}$$

$$v_{x,\lambda}(\eta) - v(\eta) = \int_{\Sigma_{x,\lambda}} K(x, \lambda; \eta, \xi)\big[u^\theta(\xi) - \big(\frac{\lambda}{|\xi - x|}\big)^{\tau_2} u^\theta_{x,\lambda}(\xi)\big] d\xi, \tag{3.18}$$

where

$$K(x, \lambda; \xi, \eta) = \big(\frac{\lambda}{|\xi - x|}\big)^{n-\alpha}\frac{1}{|\xi^{x,\lambda} - \eta|^{n-\alpha}} - \frac{1}{|\xi - \eta|^{n-\alpha}},$$

and

$$K(x, \lambda; \xi, \eta) > 0, \ \textit{for} \ \forall \xi, \eta \in \Sigma_{x,\lambda}, \lambda > 0.$$

Proof. The proof is similar to that of Lemma 5.3 in [12]. See also our early work [3]. We skip details here.

□

It is clear in Lemma 3.8 that $\tau_1 = \tau_2 = 0$ if and only if $\theta = \kappa = \frac{n+\alpha}{n-\alpha}$. From now on in this subsection, we assume that $\theta = \kappa = \frac{n+\alpha}{n-\alpha}$.

The next lemma indicates that the procedure of moving sphere can be started.

Lemma 3.9. *Assume the same conditions on n, α, θ and κ as those in Theorem 3.6. Then for any $x \in \mathbb{R}^n$, there exists $\lambda_0(x) > 0$ such that: $\forall\ 0 < \lambda < \lambda_0(x)$,*

$$u_{x,\lambda}(\xi) \ge u(\xi), \quad \forall \xi \in \Sigma_{x,\lambda},$$
$$v_{x,\lambda}(\eta) \ge v(\eta), \quad \forall \eta \in \Sigma_{x,\lambda}.$$

Proof. The proof is similar to that of Lemma 5.4 in [12]. For simplicity, we assume $x = 0$, and write $u_\lambda = u_{0,\lambda}$.

Since $n < \alpha$ and $u \in C^1(\mathbb{R}^n)$ is a positive function, there exists $r_0 \in (0,1)$, such that

$$\nabla_\xi \big(|\xi|^{\frac{n-\alpha}{2}} u(\xi)\big) \cdot \xi < 0, \ \forall\, 0 < |\xi| < r_0.$$

Thus,

$$u_\lambda(\xi) > u(\xi), \ \forall\, 0 < \lambda < |\xi| < r_0. \tag{3.19}$$

Using Lemma 3.4 (iii), we have

$$u(\xi) \le C(r_0)|\xi|^{\alpha-n}, \ \forall \ |\xi| \ge r_0.$$

For small $\lambda_0 \in (0, r_0)$ and any $0 < \lambda < \lambda_0$, by (iii) of Lemma 3.4 and (3.19)

$$u_\lambda(\xi) = \big(\frac{\lambda}{|\xi|}\big)^{n-\alpha} u\big(\frac{\lambda^2 \xi}{|\xi|^2}\big) \ge \big(\frac{|\xi|}{\lambda_0}\big)^{\alpha-n} \inf_{B_{r_0}} u \ge u(\xi), \quad\ |\xi| \ge r_0.$$

Combining the above with (3.19), we conclude

$$u_{x,\lambda}(\xi) \ge u(\xi), \quad\ \forall \xi \in \ \Sigma_{x,\lambda}$$

with $x = 0$ and $\lambda_0(x) = \lambda_0$. In the same way, we can prove the inequality for $v(\eta)$. $\square$

For a given $x \in \mathbb{R}^n$, define

$$\bar\lambda(x) = sup\{\mu > 0 \,|\, u_{x,\lambda}(\xi) \ge u(\xi), \text{and } v_{x,\lambda}(\eta) \ge v(\eta), \ \forall \lambda \in (0,\mu), \forall\, \xi, \eta \in \Sigma_{x,\lambda}\}.$$

The next lemma shows: if the sphere stops, then we have conformal invariant properties for solutions.

Lemma 3.10. *For some $x_0 \in \mathbb{R}^n$, if $\bar\lambda(x_0) < \infty$, then*

$$\begin{aligned} u_{x_0,\bar\lambda(x_0)}(\xi) &= u(\xi), \quad \forall \xi \in \ \mathbb{R}^n, \\ v_{x_0,\bar\lambda(x_0)}(\eta) &= v(\eta), \quad \forall \eta \in \ \mathbb{R}^n. \end{aligned}$$

Proof. Again, the proof is similar to that of Lemma 5.5 in [12].

Without loss of generality, we assume that $x_0 = 0$, and write $\bar\lambda = \bar\lambda(0)$, $u_\lambda = u_{0,\lambda}$, $v_\lambda = v_{0,\lambda}$, $\xi^\lambda = \xi^{0,\lambda}$, $\eta^\lambda = \eta^{0,\lambda}$. By the definition of $\bar\lambda$,

$$u_{\bar\lambda}(\xi) \ge u(\xi), \quad v_{\bar\lambda}(\eta) \ge v(\eta), \quad \forall\, |\xi|, |\eta| \ge \bar\lambda.$$

Noting that $\tau_1 = \tau_2 = 0$ for $\theta = \kappa = \frac{n+\alpha}{n-\alpha}$. Thus, using (3.17) and (3.18) with $x = 0, \lambda = \bar\lambda$, and the positivity of the kernel, we know that there are following two cases:

(**a**) $u_{\bar\lambda}(\xi) = u(\xi)$ and $v_{\bar\lambda}(\eta) = v(\eta)$ for all $|\xi|, |\eta| \ge \bar\lambda$; or (**b**) $u_{\bar\lambda}(\xi) > u(\xi)$ and $v_{\bar\lambda}(\eta) > v(\eta)$ for all $|\xi|, |\eta| \ge \bar\lambda$.

We show that case (**b**) can not happen. More precisely, supposing that $u_{\bar\lambda}(\xi) > u(\xi)$ and $v_{\bar\lambda}(\eta) > v(\eta)$ for all $|\xi|, |\eta| \ge \bar\lambda$, we will show that there is a $\varepsilon_* > 0$, such that, for any $\lambda \in (\bar\lambda, \bar\lambda + \varepsilon_*)$, $u_\lambda(\xi) \ge u(\xi)$ and $v_\lambda(\eta) \ge v(\eta)$ for any $|\xi|, |\eta| > \lambda$. This contradicts to the definition of $\bar\lambda$. We will show this via two steps.

Step 1. There is a $\varepsilon_1 \in (0,1)$, such that for any $\varepsilon < \varepsilon_1$, $\bar\lambda \le \lambda \le \bar\lambda + \varepsilon$, if $|\xi|, |\eta| \ge \bar\lambda + 1$, then

$$u_\lambda(\xi) - u(\xi) \ge \frac{\varepsilon_1}{2}|\xi|^{\alpha-n} \quad \text{and} \quad v_\lambda(\eta) - v(\eta) \ge \frac{\varepsilon_1}{2}|\eta|^{\alpha-n}.$$

From Lemma 3.8, we know that $K(x,\lambda,\xi,z)>0\ \forall\,\xi,\eta\in\Sigma_{x,\lambda}$. By (3.17) with $x=0,\lambda=\bar\lambda$, and Fatou Lemma, we know, for all $|\xi|\ge\bar\lambda$, that

$$\begin{aligned}&\liminf_{|\xi|\to\infty}|\xi|^{n-\alpha}(u_{\bar\lambda}(\xi)-u(\xi))\\ \ge\ &\int_{\Sigma_\lambda}\liminf_{|\xi|\to\infty}|\xi|^{n-\alpha}K(0,\bar\lambda,\xi,\eta)[v^\kappa(\eta)-v^\kappa_{\bar\lambda}(\eta)]d\eta\\ =\ &\int_{\Sigma_\lambda}\big((\frac{\bar\lambda}{|\eta|})^{n-\alpha}-1\big)[v^\kappa(\eta)-v^\kappa_{\bar\lambda}(\eta)]d\eta.\end{aligned}$$

Thus, using the positivity of $v_{\bar\lambda}-v$, we know that there exists $\varepsilon_2\in(0,1)$, such that

$$u_{\bar\lambda}(\xi)-u(\xi)\ge\varepsilon_2|\xi|^{\alpha-n},\ \forall\,|\xi|\ge\bar\lambda+1.$$

Due to the continuity of u, there exists a $\varepsilon_3\in(0,\varepsilon_2)$ such that for $|\xi|\ge\bar\lambda+1$ and $\bar\lambda\le\lambda\le\bar\lambda+\varepsilon_3$,

$$\begin{aligned}|u_\lambda(\xi)-u_{\bar\lambda}(\xi)| &= |(\frac{\lambda}{|\xi|})^{n-\alpha}u(\frac{\lambda^2\xi}{|\xi|^2})-(\frac{\bar\lambda}{|\xi|})^{n-\alpha}u(\frac{\bar\lambda^2\xi}{|\xi|^2})|\\ &\le \frac{\varepsilon_3}{2}|\xi|^{\alpha-n}.\end{aligned}$$

Thus

$$u_\lambda(\xi)-u(\xi)=u_{\bar\lambda}(\xi)-u(\xi)+u_\lambda(\xi)-u_{\bar\lambda}(\xi)\ge\frac{\varepsilon_2}{2}|\xi|^{\alpha-n},$$

for all $|\xi|\ge\bar\lambda+1,\bar\lambda\le\lambda\le\bar\lambda+\varepsilon_2$.

Similarly, there exists $\varepsilon_4\in(0,\varepsilon_3)$ such that

$$v_\lambda(\eta)-v(\eta)\ge\frac{\varepsilon_4}{2}|\eta|^{\alpha-n},$$

for all $|\eta|\ge\bar\lambda+1,\bar\lambda\le\lambda\le\bar\lambda+\varepsilon_4$. Choosing $\varepsilon_1=\varepsilon_4$, we complete the proof for Step 1.

Step 2. There is a $\varepsilon_*<\varepsilon_1$, such that for any $\varepsilon<\varepsilon_*$, $\bar\lambda\le\lambda\le\bar\lambda+\varepsilon$, if $\xi,\eta\in\mathbb{R}^n$ satisfy $\lambda\le|\xi|,|\eta|\le\bar\lambda+1$, then $u_\lambda(\xi)-u(\xi)\ge0$ and $v_\lambda(\eta)-v(\eta)\ge0$.

Let $\varepsilon_*\in(0,\varepsilon_1)$.We have, for $\bar\lambda\le\lambda\le\bar\lambda+\varepsilon_*$ and $\lambda\le|\xi|\le\bar\lambda+1$,

$$\begin{aligned}u_\lambda(\xi)-u(\xi) &= \int_{\Sigma_\lambda}K(0,\lambda;\xi,\eta)\big(v^\kappa(\eta)-v^\kappa_\lambda(\eta)\big)d\eta\\ &\ge \int_{\Sigma_\lambda\setminus\Sigma_{\bar\lambda+1}}K(0,\lambda;\xi,\eta)\big(v^\kappa(\eta)-v^\kappa_\lambda(\eta)\big)d\eta\\ &\quad+\int_{\Sigma_{\bar\lambda+2}\setminus\Sigma_{\bar\lambda+3}}K(0,\lambda;\xi,\eta)\big(v^\kappa(\eta)-v^\kappa_\lambda(\eta)\big)d\eta\\ &\ge \int_{\Sigma_\lambda\setminus\Sigma_{\bar\lambda+1}}K(0,\lambda;\xi,\eta)\big(v^\kappa_{\bar\lambda}(\eta)-v^\kappa_\lambda(\eta)\big)d\eta\\ &\quad+\int_{\Sigma_{\bar\lambda+2}\setminus\Sigma_{\bar\lambda+3}}K(0,\lambda;\xi,\eta)\big(v^\kappa(\eta)-v^\kappa_\lambda(\eta)\big)d\eta. \qquad (3.20)\end{aligned}$$

By Step 1, there exists $\delta_1>0$ such that

$$v^\kappa(\eta)-v^\kappa_\lambda(\eta)\ge\delta_1,\ \forall\eta\in\Sigma_{\bar\lambda+2}\setminus\Sigma_{\bar\lambda+3}.$$

Since

$$K(0,\lambda;\xi,\eta)=0,\ \forall\,|\xi|=\lambda,$$

$$\nabla_\xi K(0,\lambda;\xi,\eta)|_{|\xi|=\lambda}=(\alpha-n)|\xi-\eta|^{\alpha-n-2}\big(|\eta|^2-|\xi|^2\big)>0,\ \forall\,\eta\in\Sigma_{\bar\lambda+2}\setminus\Sigma_{\bar\lambda+3},$$

and the function is smooth in the relevant region, it follows, also based on the positivity of kernel, that

$$K(0,\lambda;\xi,\eta) \geq \delta_2(|\xi|-\lambda),\ \forall\, \lambda \leq |\xi| \leq \bar\lambda+1, \forall \eta \in \Sigma_{\bar\lambda+2}\backslash\Sigma_{\bar\lambda+3}.$$

where $\delta_2 > 0$ is some constant independent of ε_*. It is easy to see that for some constant $C > 0$ (independent of ε_*), and $\bar\lambda \leq \lambda \leq \bar\lambda + \varepsilon_*$,

$$|v^\kappa_{\bar\lambda}(\eta) - v^\kappa_\lambda(\eta)| \leq C\varepsilon,\ \forall\, \eta \in \Sigma_{\bar\lambda+2}\backslash\Sigma_{\bar\lambda+3}, \lambda \leq |\xi| \leq \bar\lambda+1.$$

Using the mean value theorem, we have, for $\lambda \leq |\xi| \leq \bar\lambda + 1$, that

$$\begin{aligned}
&\int_{\Sigma_\lambda\backslash\Sigma_{\bar\lambda+1}} K(0,\lambda;\xi,\eta)d\eta = \int_{\Sigma_\lambda\backslash\Sigma_{\bar\lambda+1}} \Big(\big(\frac{|\xi|}{\lambda}\big)^{\alpha-n}|\xi^\lambda-\eta|^{\alpha-n} - |\xi-\eta|^{\alpha-n}\Big)d\eta \\
= &\int_{\Sigma_\lambda\backslash\Sigma_{\bar\lambda+1}} \Big[\big(\big(\frac{|\xi|}{\lambda}\big)^{\alpha-n}-1\big)|\xi^\lambda-\eta|^{\alpha-n} + \big(|\xi^\lambda-\eta|^{\alpha-n} - |\xi-\eta|^{\alpha-n}\big)\Big]d\eta \\
\leq\ & C(|\xi|-\lambda) + \int_{\Sigma_\lambda\backslash\Sigma_{\bar\lambda+1}} \big(|\xi^\lambda-\eta|^{\alpha-n} - |\xi-\eta|^{\alpha-n}\big)d\eta \\
\leq\ & C(|\xi|-\lambda) + C|\xi^\lambda - \xi| \\
\leq\ & C(|\xi|-\lambda).
\end{aligned}$$

Thus, for $\varepsilon \in (0,\varepsilon_*)$, $\bar\lambda \leq \lambda \leq \bar\lambda+\varepsilon$, $\lambda \leq |\xi| \leq \bar\lambda+1$, from (3.20) it follows

$$\begin{aligned}
u_\lambda(\xi) - u(\xi) &\geq -C\varepsilon \int_{\Sigma_\lambda\backslash\Sigma_{\bar\lambda+1}} K(0,\lambda;\xi,\eta)d\eta + \delta_1\delta_2(|\xi|-\lambda)\int_{\Sigma_{\bar\lambda+2}\backslash\Sigma_{\bar\lambda+3}} d\eta \\
&\geq \Big(\delta_1\delta_2 \int_{\Sigma_{\bar\lambda+2}\backslash\Sigma_{\bar\lambda+3}} d\eta - C\varepsilon\Big)(|\xi|-\lambda) \geq 0.
\end{aligned}$$

Along the same line, we can show

$$v_\lambda(\eta) - v(\eta) \geq 0$$

for $\lambda \leq |\xi|, |\eta| \leq \bar\lambda + 1$. Step 2 is established. Hence we complete the proof for Lemma 3.10. □

The following two key calculus lemmas are needed for carrying out moving sphere procedure. Under a stronger assumption ($f \in C^1(\mathbb{R}^n)$), these lemmas were early proved by Li and Zhu [14] (see, also, Li and Zhang [13]). The current forms, due to Li and Nirenberg, are adopted from Li [12]. See Frank and Lieb [4] for further extension to nonnegative measures.

Lemma 3.11. *(Lemma 5.7 in [12]) For $n \geq 1, \mu \in \mathbb{R}$, let f be a function defined on $\mathbb{R}^n$ and valued in $(-\infty,+\infty)$ satisfying*

$$\big(\frac{\lambda}{|y-x|}\big)^\mu f\big(x + \frac{\lambda^2(y-x)}{|y-x|^2}\big) \geq f(y), \quad \forall\, |y-x| \geq \lambda > 0, x, y \in \mathbb{R}^n,$$

then $f(x) =$ constant.

Lemma 3.12. *(Lemma 5.8 in [12]) For $n \geq 1, \mu \in \mathbb{R}$, let $f \in C^0(\mathbb{R}^n)$, and $\mu \in \mathbb{R}$. Suppose that for every $x \in \mathbb{R}^n$, there exists $\lambda \in \mathbb{R}$ such that*

$$\big(\frac{\lambda}{|y-x|}\big)^\mu f\big(x + \frac{\lambda^2(y-x)}{|y-x|^2}\big) = f(y), \quad \forall y \in \mathbb{R}^n \setminus \{x\}.$$

Then there are $a \geq 0, d > 0$ and $\bar{x} \in \mathbb{R}^n$, such that

$$f(x) \equiv \pm a\Big(\frac{1}{d + |x - \bar{x}|^2}\Big)^{\frac{\mu}{2}}.$$

We are now ready to give a proof to Theorem 3.6.

Proof of Theorem 3.6.

We first show that $\bar{\lambda}(x)$ is finite for some $x \in \mathbb{R}^n$. Otherwise, $\bar{\lambda}(x) = \infty$ for all $x \in \mathbb{R}^n$, then for $\xi, \eta \in \mathbb{R}^n$,

$$u_{x,\lambda}(\xi) \geq u(\xi), \text{ and } v_{x,\lambda}(\eta) \geq v(\eta) \quad \forall\, |\xi - x|,\, |\eta - x| > \lambda,$$

By Lemma 3.11, we know that $u = v = \textit{constant}$, which can not satisfy (3.14).

Now, for a fixed $x \in \mathbb{R}^n$, we know from the definition of $\bar{\lambda}(x)$, that,

$$u_{x,\lambda}(\xi) \geq u(\xi),\ \forall\, 0 < \lambda < \bar{\lambda}(x),\ \forall\, |\xi - x| \geq \lambda.$$

From Lemma 3.4 (ii), we have, for any $\lambda \in (0,\, \bar{\lambda}(x))$, that

$$0 \quad < \quad a = \lim_{|\xi|\to\infty} |\xi|^{n-\alpha} u(\xi) \leq \lim_{|\xi|\to\infty} |\xi|^{n-\alpha} u_{x,\lambda}(\xi) = \lambda^{n-\alpha} u(x).$$

This shows $\bar{\lambda}(x) < \infty$ for all $x \in \mathbb{R}^n$. Applying Lemma 3.10, we know

$$u_{x,\bar{\lambda}}(\xi) = u(\xi), \quad \text{ and } v_{x,\bar{\lambda}}(\eta) = v(\eta),\ \forall\, x, \xi, \eta \in \mathbb{R}^n.$$

It then follows from Lemma 3.12, that

$$u(\xi) = c_1\Big(\frac{1}{|\xi - \xi_0|^2 + d^2}\Big)^{\frac{n-\alpha}{2}}$$

and

$$v(\xi) = c_2\Big(\frac{1}{|\xi - \xi_0|^2 + d^2}\Big)^{\frac{n-\alpha}{2}}.$$

for some $c_1,\, c_2 > 0, d > 0$ and $\xi_0 \in \mathbb{R}^n$. □

3.2.3. *The Best constant $N^*(n, \alpha)$.* It follows from Theorem 3.6 and direct computation that all extremal functions to inequality (1.6) can be represented by

$$F(\xi) = a(1 - \xi \cdot \eta)^{-\frac{n+\alpha}{2}}$$

for some $a > 0$ and $\eta \in \mathbb{R}^{n+1}$ with $|\eta| < 1$. In particular, $F(\xi) = 1$ is an extremal function.

Note that

$$\int_{\mathbb{S}^n} |\xi - \eta|^{\alpha - n} d\eta = 2^{\alpha-1}|\mathbb{S}^{n-1}|\frac{\Gamma(n/2)\Gamma(\alpha/2)}{\Gamma((n+\alpha)/2)}.$$

We have

$$\begin{aligned} N^*(n, \alpha) &= |\mathbb{S}^n|^{\frac{1}{q} - \frac{1}{p}} \int_{\mathbb{S}^n} |\xi - \eta|^{\alpha-n} d\eta \\ &= |\mathbb{S}^n|^{-\frac{\alpha}{n}} 2^{\alpha-1} |\mathbb{S}^{n-1}| \frac{\Gamma(n/2)\Gamma(\alpha/2)}{\Gamma((n+\alpha)/2)} \\ &= \pi^{\frac{n-\alpha}{2}} \frac{\Gamma(\alpha/2)}{\Gamma((n+\alpha)/2)} \Big(\frac{\Gamma(n/2)}{\Gamma(n)}\Big)^{-\frac{\alpha}{n}}. \end{aligned}$$

We hereby complete the proof of Theorem 1.2.

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References

[1] W. Beckner, Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality, Ann. of Math. 138 (1993), 213-242.
[2] H. J. Brascamp, E. H. Lieb, Best constants in Young's inequality, its converse and its generalization to more than three functions, Adv. Math. 20 (1976), 151-173.
[3] J. Dou, M. Zhu, Sharp Hardy-Littlewood -Sobolev inequality on the upper half space, ArXiv: 1309.2341, to appear in IMRN, 2013.
[4] R. L. Frank, E. H. Lieb, Inversion positivity and the sharp Hardy-Littlewood-Sobolev inequality, Calc. Var. Partial Differential Equations 39 (2010), 85-99.
[5] R. L. Frank, E. H. Lieb, Sharp constants in several inequalities on the Heisenberg group, Ann. of Math. 176 (2012), 349-381.
[6] L. Gross, Logarithmic Sobolev inequalities, Amer. J. Math. 97 (1976), 1061-1083.
[7] Y. Han, M. Zhu, Hardy-Littlewood-Sobolev inequalities on compact Riemannian manifolds and applications.
[8] F. Hang, P. Yang, The Sobolev inequality for Paneitz operator on three manifolds, Calc. Vari. PDE, 21 (2004), 57-83.
[9] G. H. Hardy, J. E. Littlewood, Some properties of fractional integrals (1), Math. Zeitschr. 27 (1928), 565-606.
[10] G. H. Hardy, J. E. Littlewood, On certain inequalities connected with the calculus of variations, J. London Math. Soc. 5 (1930), 34-39.
[11] G. H. Hardy, J. E. Littlewood, G. Polya, Inequalities, 2nd edition, Cambridge University Press, 1967.
[12] Y. Y. Li, Remark on some conformally invariant integral equations: the method of moving spheres, J. Eur. Math. Soc. 6 (2004), 153-180.
[13] Y. Y. Li, L. Zhang, Liouville type theorems and Harnack type inequalities for semilinear elliptic equations, J. D'Anal. Math. 90 (2003), 27-87.
[14] Y. Y. Li, M. Zhu, Uniqueness theorems through the method of moving spheres, Duke Math. J. 80 (1995), 383-417.
[15] E. H. Lieb, Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities, Ann. of Math. 118 (1983), 349-374.
[16] E. H. Lieb, M. Loss, Analysis, 2nd edition, Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, R. I. 2001.
[17] Y. Ni, M. Zhu, Steady states for one dimensional curvature flows, Comm. Contemp. Math. 10 (2008), 155-179.
[18] S. L. Sobolev, On a theorem of functional analysis, Mat. Sb. (N.S.) 4 (1938), 471-479; A. M. S. Transl. Ser. 2, 34 (1963), 39-68.
[19] E. M. Stein, Singular integrals and differentiability properties of functions, Princeton Mathematical Series, 30. Princeton University Press, Princeton, N.J. 1970.
[20] E. M. Stein, G. Weiss, On the theory of harmonic functions of several variables I: The theory of H^p spaces, Acta Math. 103 (1960), 25-62.

[21] P. Yang, M. Zhu, Sharp inequality for Paneitz operator on $\mathbb{S}^3$, ESAIM: Control, Optimisation and Calculus of Variations, 10 (2004), 211-223.
[22] M. Zhu, On a prescribing integral curvature equation, preprint, 2014.

Jingbo Dou, School of Statistics, Xi'an University of Finance and Economics, Xi'an, Shaanxi, 710100, China and Department of Mathematics, The University of Oklahoma, Norman, OK 73019, USA
E-mail address: jbdou@xaufe.edu.cn

Meijun Zhu, Department of Mathematics, The University of Oklahoma, Norman, OK 73019, USA
E-mail address: mzhu@math.ou.edu