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Reverses of Ando's and Hölder–McCarty's inequalities

Monire Hajmohamadi[1*], Rahmatollah Lashkaripour[1] and Mojtaba Bakherad[1]

*Correspondence:
monire.hajmohamadi@yahoo.com;
lashkari@hamoon.usb.ac.ir;
mojtaba.bakherad@yahoo.com
[1]Department of Mathematics, Faculty of Mathematics, University of Sistan and Baluchestan, Zahedan, Iran

Abstract

In this paper, we give some reverse-types of Ando's and Hölder–McCarthy's inequalities for positive linear maps, and positive invertible operators. For this purpose, we use a recently improved Young inequality and its reverse.

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Keywords: Positive operator; Young inequality; Hölder–McCarthy's inequality; Ando's inequality

1 Introduction and preliminaries

Let $\mathcal{B}(\mathcal{H})$ be the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ with the operator norm $\|\cdot\|$ and the identity I; also $\mathcal{M}_n(\mathcal{C})$ denotes the space of all $n \times n$ complex matrices. For an operator $A \in \mathcal{B}(\mathcal{H})$, we write $A \geq 0$ if A is positive, and $A > 0$ if A is positive invertible. For $A, B \in \mathcal{B}(\mathcal{H})$, we say $A \geq B$ if $A - B \geq 0$. The Gelfand map $f(t) \mapsto f(A)$ is an isometrical $*$-isomorphism between the C^*-algebra $C(\mathrm{sp}(A))$ of continuous functions on the spectrum $\mathrm{sp}(A)$ of a selfadjoint operator A and the C^*-algebra generated by A and I. A linear map Φ on $\mathcal{B}(\mathcal{H})$ is positive if $\Phi(A) \geq 0$ whenever $A \geq 0$. It is said to be unital if $\Phi(I) = I$. A continuous function $f : J \to \mathcal{R}$ is operator concave if

$$f\big(\alpha A + (1-\alpha)B\big) \geq \alpha f(A) + (1-\alpha) f(B)$$

for all selfadjoint operators $A, B \in \mathcal{B}(\mathcal{H})$ with spectra in J and all $\alpha \in [0,1]$.

The well-known Young inequality says that, for positive real numbers a, b and $0 \leq t \leq 1$, we have $a^t b^{1-t} \leq ta + (1-t)b$. Refinements and reverses of this inequality are proven in [2, 9, 14–16] and the references therein. Also Kittaneh et al. in [10] obtained the following improvement of the Young inequality for any positive definite matrices $A, B \in \mathcal{M}_n(\mathcal{C})$:

$$A^{1-t}B^t + r(A + B - 2A\sharp B) \leq (1-t)A + tB \leq A^{1-t}B^t + R(A + B - 2A\sharp B), \tag{1}$$

where $t \in [0,1]$, $r = \min\{t, 1-t\}$ and $R = \max\{t, 1-t\}$.

Zhao et al. [19] obtained a refinement of Young's inequality and its reverse as follows:

(i) for $0 < t \leq \frac{1}{2}$,

$$r_0\big(\sqrt[4]{ab} - \sqrt{a}\big)^2 + r(\sqrt{a} - \sqrt{b})^2 + a^{1-t}b^t$$

$$\begin{aligned} &\leq (1-t)a+tb \\ &\leq R(\sqrt{a}-\sqrt{b})^2 - r_0\big(\sqrt[4]{ab}-\sqrt{b}\big)^2 + a^{1-t}b^t; \end{aligned} \tag{2}$$

(ii) for $\frac{1}{2} < t < 1$,

$$\begin{aligned} &r_0\big(\sqrt[4]{ab}-\sqrt{b}\big)^2 + R(\sqrt{a}-\sqrt{b})^2 + a^{1-t}b^t \\ &\quad\leq (1-t)a+tb \\ &\quad\leq r(\sqrt{a}-\sqrt{b})^2 - r_0\big(\sqrt[4]{ab}-\sqrt{a}\big)^2 + a^{1-t}b^t, \end{aligned}$$

where $r = \min\{t, 1-t\}$, $R = \max\{t, 1-t\}$ and $r_0 = \min\{2r, 1-2r\}$.

Sababheh et al. [15, 16] established some refinements and reverses of Young's inequality as follows:

(i) for $0 \leq t \leq \frac{1}{2}$,

$$S_N(t;a,b) \leq ta + (1-t)b - a^t b^{1-t} \leq (1-t)(\sqrt{a}-\sqrt{b})^2 - S_N(2t;\sqrt{ab},a); \tag{3}$$

(ii) for $\frac{1}{2} \leq t \leq 1$,

$$S_N(t;a,b) \leq ta + (1-t)b - a^t b^{1-t} \leq t(\sqrt{a}-\sqrt{b})^2 - S_N(2-2t;\sqrt{ab},b),$$

where

$$S_N(t;a,b) = \sum_{j=1}^{N} s_j(t)\Big(\sqrt[2^j]{b^{2^{j-1}-k_j}a^{k_j}} - \sqrt[2^j]{a^{k_j+1}b^{2^{j-1}-k_j-1}}\Big)^2,$$

$s_j(t) = ((-1)^{r_j}2^{j-1}t + (-1)^{r_j+1}[\frac{r_j+1}{2}])$, $r_j = [2^j t]$ and $k_j = [2^{j-1}t]$. Here $[x]$ is the greatest integer less than or equal to x.

Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive. The operator t-weighted arithmetic, geometric, and harmonic means of operators A, B are defined by $A\nabla_t B = (1-t)A + tB$, $A\sharp_t B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}B \times A^{-\frac{1}{2}})^t A^{\frac{1}{2}}$ and $A!_t B = ((1-t)A^{-1} + tB^{-1})^{-1}$, respectively. In particular, for $t = \frac{1}{2}$ we get the usual operator arithmetic mean ∇, the geometric mean $\sharp$ and the harmonic mean $!$.

2 Results and discussion

For positive real numbers a_i and b_i ($i = 1, 2, \ldots, n$) the Hölder inequality states that

$$\sum_{i=1}^{n} a_i^{1/p} b_i^{1/q} \leq \left(\sum_{i=1}^{n} a_i\right)^{1/p} \left(\sum_{i=1}^{n} b_i\right)^{1/q} \tag{4}$$

for $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. If $p = q = 2$ in (4), then we get the Cauchy–Schwarz inequality. The Hölder inequality for positive operators A_i and B_i ($i = 1, 2, \ldots, n$) is

$$\sum_{i=1}^{n} A_i \sharp_t B_i \leq \left(\sum_{i=1}^{n} A_i\right) \sharp_t \left(\sum_{i=1}^{n} B_i\right),$$

where $0 \leq t \leq 1$. In the case $t = \frac{1}{2}$, we get the operator Cauchy–Schwarz inequality. For further information as regards the Hölder and Cauchy–Schwarz inequalities we refer the reader to [3–5, 11, 12, 20] and the references therein. Ando [1] proved that if Φ is a positive linear map, then for positive operators $A, B \in \mathcal{B}(\mathcal{H})$ and $t \in [0,1]$, we have

$$\Phi(A \sharp_t B) \leq \Phi(A) \sharp_t \Phi(B).$$

Recently, some authors presented several reverse-types of Ando's inequality (see [13, 17]).

The Hölder–McCarthy's inequality says that for any positive operator A and any unit vector $x \in \mathcal{H}$, we have

$$\langle A^t x, x \rangle \leq \langle Ax, x \rangle^t, \quad 0 \leq t \leq 1. \tag{5}$$

Furuta [8] showed that this inequality is equivalent to Young's inequality.

3 Conclusions

In this paper, we establish a reverse of Ando's inequality for positive (non-unital) linear maps and positive definite matrices by using an inequality due to Sababheh. We obtain some reverses of the matrix Hölder and Cauchy–Schwarz inequalities and a reverse of inequality (5) for $t \in (0, \frac{1}{2}]$ as follows:

$$\begin{aligned} &\langle Tx, x \rangle^t - \langle T^t x, x \rangle \\ &\quad \leq 2R\left(\langle Tx, x \rangle^{\frac{1}{2}} - \langle T^{\frac{1}{2}} x, x \rangle\right) - r_0\left(\langle T^{\frac{1}{2}} x, x \rangle + \langle Tx, x \rangle^{\frac{1}{2}} - 2\langle T^{\frac{1}{4}} x, x \rangle \langle Tx, x \rangle^{\frac{1}{4}}\right). \end{aligned}$$

4 Methods

We use the properties of inner product and the inequalities obtained in [16] and [19].

5 Main results

To prove our first result, we need the following lemmas.

Lemma 1 ([16]) *Let $A, B \in \mathcal{M}_n(\mathcal{C})$ be positive definite matrices and $t \in [0,1]$. Then*

$$\sum_{j=1}^{N} s_j(t)(A \sharp_{\alpha_j(t)} B + A \sharp_{2^{1-j}+\alpha_j(t)} B - 2A \sharp_{2^{-j}+\alpha_j(t)} B) + A \sharp_t B \leq A \nabla_t B, \tag{6}$$

where $\alpha_j(t) = \frac{k_j(t)}{2^{j-1}}$.

For $N = 2$, we have the following lemma, which is shown in [19] for positive invertible operators.

Lemma 2 *Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive invertible operators and $t \in [0,1]$.*

(i) *If $0 < t \leq \frac{1}{2}$, then*

$$r_0(A \sharp B - 2A \sharp_{\frac{1}{4}} B + A) + 2t(A \nabla B - A \sharp B) + A \sharp_t B \leq A \nabla_t B. \tag{7}$$

(ii) *If* $\frac{1}{2} < t < 1$, *then*

$$r_0(A\sharp B - 2A\sharp_{\frac{3}{4}}B + B) + 2(1-t)(A\nabla B - A\sharp B) + A\sharp_t B \le A\nabla_t B, \tag{8}$$

where $r = \min\{\nu, 1-\nu\}$ *and* $r_0 = \min\{2r, 1-2r\}$.

Lemma 3 ([16]) *Let* $A, B \in \mathcal{M}_n(\mathcal{C})$ *be positive definite matrices and* $t \in [0,1]$.

(i) *If* $0 \le t \le \frac{1}{2}$, *then*

$$\begin{aligned} A\nabla_t B &\le A\sharp_t B + 2(1-t)(A\nabla B - A\sharp B) \\ &\quad - \sum_{j=1}^{N} s_j(2t)(A\sharp_{1-\beta_j(t)}B + A\sharp_{1+2^{-j}-\beta_j(t)}B - 2A\sharp_{1-2^{-j-1}-\beta_j(t)}B). \end{aligned} \tag{9}$$

(ii) *If* $\frac{1}{2} \le t \le 1$, *then*

$$\begin{aligned} A\nabla_t B &\le A\sharp_t B + 2t(A\nabla B - A\sharp B) \\ &\quad - \sum_{j=1}^{N} s_j(2-2t)(A\sharp_{\gamma_j(t)}B + A\sharp_{\gamma_j(t)+2^{1-j}}B - 2A\sharp_{\gamma_j(t)+2^{-j}}B), \end{aligned}$$

where $\beta_j(t) = 2^{-j}k_j(2t)$ *and* $\gamma_j(t) = 2^{1-j}k_j(2-2t)$.

Remark 4 By using functional calculus and numerical inequalities in [10, 16], we can extend inequality (1), Lemmas 1 and 3 for positive invertible operators.

For $N = 2$, we have the following lemma, which is shown in [19] for positive invertible operators.

Lemma 5 *Let* $A, B \in \mathcal{B}(\mathcal{H})$ *be positive invertible operators and* $t \in [0,1]$.

(i) *If* $0 < t \le \frac{1}{2}$, *then*

$$A\nabla_t B \le A\sharp_t B + 2(1-t)(A\nabla B - A\sharp B) - r_0(A\sharp B - 2A\sharp_{\frac{3}{4}}B + B). \tag{10}$$

(ii) *If* $\frac{1}{2} < t < 1$, *then*

$$A\nabla_t B \le A\sharp_t B + 2t(A\nabla B - A\sharp B) - r_0(A\sharp B - 2A\sharp_{\frac{1}{4}}B + A), \tag{11}$$

where $r = \min\{\nu, 1-\nu\}$ *and* $r_0 = \min\{2r, 1-2r\}$.

Now, we obtain a reverse of Ando's inequality for positive invertible operators as follows.

Theorem 6 *Let* $A, B \in \mathcal{B}(\mathcal{H})$ *be positive invertible operators*, Φ *be a positive linear map and* $t \in [0,1]$.

(i) *If* $0 \le t \le \frac{1}{2}$, *then*

$$\Phi(A)\sharp_t\Phi(B) - \Phi(A\sharp_t B)$$

$$\leq 2R\Big(\Phi(A)\sharp\Phi(B) - \Phi(A\sharp B) + \frac{1}{2}\big(\Phi(A) + \Phi(B) - 2\Phi(A)\sharp\Phi(B)\big)\Big)$$
$$- \sum_{j=1}^{N} s_j(2t)\big(\Phi(A\sharp_{1-\beta_j(t)}B) + \Phi(A\sharp_{1+2^{-j}-\beta_j(t)}B) - 2\Phi(A\sharp_{1-2^{-j-1}-\beta_j(t)}B)\big)$$
$$- \sum_{j=1}^{N} s_j(t)\big(\Phi(A)\sharp_{\alpha_j(t)}\Phi(B) + \Phi(A)\sharp_{2^{1-j}+\alpha_j(t)}\Phi(B)$$
$$- 2\Phi(A)\sharp_{2^{-j}+\alpha_j(t)}\Phi(B)\big). \tag{12}$$

(ii) *If* $\frac{1}{2} \leq t \leq 1$, *then*

$$\Phi(A)\sharp_t\Phi(B) - \Phi(A\sharp_t B)$$
$$\leq 2R\Big(\Phi(A)\sharp\Phi(B) - \Phi(A\sharp B) + \frac{1}{2}\big(\Phi(A) + \Phi(B) - 2\Phi(A)\sharp\Phi(B)\big)\Big)$$
$$- \sum_{j=1}^{N} s_j(2-2t)\big(\Phi(A\sharp_{\gamma_j(t)}B) + \Phi(A\sharp_{\gamma_j(t)+2^{1-j}}B) - 2\Phi(A\sharp_{\gamma_j(t)+2^{-j}}B)\big)$$
$$- \sum_{j=1}^{N} s_j(t)\big(\Phi(A)\sharp_{\alpha_j(t)}\Phi(B) + \Phi(A)\sharp_{2^{1-j}+\alpha_j(t)}\Phi(B)$$
$$- 2\Phi(A)\sharp_{2^{-j}+\alpha_j(t)}\Phi(B)\big), \tag{13}$$

where $\alpha_j(t) = \frac{k_j(t)}{2^{j-1}}$, $\beta_j(t) = 2^{-j}k_j(2t)$, $\gamma_j(t) = 2^{1-j}k_j(2-2t)$ *and* $R = \max\{t, 1-t\}$.

Proof The proof of inequality (13) is similar to the proof of inequality (12). Thus, we only prove inequality (12).

Let $0 \leq t \leq \frac{1}{2}$. Applying inequalities (10) and (9), we have

$$\sum_{j=1}^{N} s_j(t)(A\sharp_{\alpha_j(t)}B + A\sharp_{2^{1-j}+\alpha_j(t)}B - 2A\sharp_{2^{-j}+\alpha_j(t)}B)$$
$$\leq A\nabla_t B - A\sharp_t B$$
$$\leq 2R(A\nabla B - A\sharp B) - \sum_{j=1}^{N} s_j(2t)(A\sharp_{1-\beta_j(t)}B + A\sharp_{1+2^{-j}-\beta_j(t)}B - 2A\sharp_{1-2^{-j-1}-\beta_j(t)}B). \tag{14}$$

Now, using the positive linear map Φ on (14), we get

$$\sum_{j=1}^{N} s_j(t)\big(\Phi(A\sharp_{\alpha_j(t)}B) + \Phi(A\sharp_{2^{1-j}+\alpha_j(t)}B) - 2\Phi(A\sharp_{2^{-j}+\alpha_j(t)}B)\big) + \Phi(A\sharp_t B)$$
$$\leq \Phi(A)\nabla_t\Phi(B)$$
$$\leq 2R\big(\Phi(A)\nabla\Phi(B) - \Phi(A\sharp B)\big) + \Phi(A\sharp_t B)$$
$$- \sum_{j=1}^{N} s_j(2t)\big(\Phi(A\sharp_{1-\beta_j(t)}B) + \Phi(A\sharp_{1+2^{-j}-\beta_j(t)}B) - 2\Phi(A\sharp_{1-2^{-j-1}-\beta_j(t)}B)\big). \tag{15}$$

Moreover, if we replace A and B by $\Phi(A)$ and $\Phi(B)$ in inequality (14), respectively, then

$$\begin{aligned}
&\sum_{j=1}^{N} s_j(t)\bigl(\Phi(A)\sharp_{\alpha_j(t)}\Phi(B) + \Phi(A)\sharp_{2^{1-j}+\alpha_j(t)}\Phi(B) - 2\Phi(A)\sharp_{2^{-j}+\alpha_j(t)}\Phi(B)\bigr) \\
&\qquad + \Phi(A)\sharp_t\Phi(B) \\
&\quad \le \Phi(A)\nabla_t\Phi(B) \\
&\quad \le 2R\bigl(\Phi(A)\nabla\Phi(B) - \Phi(A\sharp B)\bigr) + \Phi(A)\sharp_t\Phi(B) \\
&\qquad - \sum_{j=1}^{N} s_j(2t)\bigl(\Phi(A)\sharp_{1-\beta_j(t)}\Phi(B) + \Phi(A)\sharp_{1+2^{-j}-\beta_j(t)}\Phi(B) \\
&\qquad - 2\Phi(A)\sharp_{1-2^{-j-1}-\beta_j(t)}\Phi(B)\bigr).
\end{aligned} \tag{16}$$

From the first inequality of (16) and the second inequality of (15), we have

$$\begin{aligned}
&\sum_{j=1}^{N} s_j(t)\bigl(\Phi(A)\sharp_{\alpha_j(t)}\Phi(B) + \Phi(A)\sharp_{2^{1-j}+\alpha_j(t)}\Phi(B) - 2\Phi(A)\sharp_{2^{-j}+\alpha_j(t)}\Phi(B)\bigr) \\
&\qquad + \Phi(A)\sharp_t\Phi(B) \\
&\quad \le \Phi(A)\nabla_t\Phi(B) \\
&\quad \le 2R\bigl(\Phi(A)\nabla\Phi(B) - \Phi(A\sharp B)\bigr) + \Phi(A\sharp_t B) \\
&\qquad - \sum_{j=1}^{N} s_j(2t)\bigl(\Phi(A\sharp_{1-\beta_j(t)}B) + \Phi(A\sharp_{1+2^{-j}-\beta_j(t)}B) - 2\Phi(A\sharp_{1-2^{-j-1}-\beta_j(t)}B)\bigr),
\end{aligned}$$

which implies that

$$\begin{aligned}
&\sum_{j=1}^{N} s_j(t)\bigl(\Phi(A)\sharp_{\alpha_j(t)}\Phi(B) + \Phi(A)\sharp_{2^{1-j}+\alpha_j(t)}\Phi(B) - 2\Phi(A)\sharp_{2^{-j}+\alpha_j(t)}\Phi(B)\bigr) \\
&\qquad + \Phi(A)\sharp_t\Phi(B) \\
&\quad \le 2R\bigl(\Phi(A)\nabla\Phi(B) - \Phi(A\sharp B)\bigr) + \Phi(A\sharp_t B) \\
&\qquad - \sum_{j=1}^{N} s_j(2t)\bigl(\Phi(A\sharp_{1-\beta_j(t)}B) + \Phi(A\sharp_{1+2^{-j}-\beta_j(t)}B) - 2\Phi(A\sharp_{1-2^{-j-1}-\beta_j(t)}B)\bigr).
\end{aligned}$$

Therefore, applying inequality (1), we get

$$\begin{aligned}
&\Phi(A)\sharp_t\Phi(B) - \Phi(A\sharp_t B) \\
&\quad \le 2R\bigl(\Phi(A)\nabla\Phi(B) - \Phi(A\sharp B)\bigr) \\
&\qquad - \sum_{j=1}^{N} s_j(2t)\bigl(\Phi(A\sharp_{1-\beta_j(t)}B) + \Phi(A\sharp_{1+2^{-j}-\beta_j(t)}B) - 2\Phi(A\sharp_{1-2^{-j-1}-\beta_j(t)}B)\bigr) \\
&\qquad - \sum_{j=1}^{N} s_j(t)\bigl(\Phi(A)\sharp_{\alpha_j(t)}\Phi(B) + \Phi(A)\sharp_{2^{1-j}+\alpha_j(t)}\Phi(B) - 2\Phi(A)\sharp_{2^{-j}+\alpha_j(t)}\Phi(B)\bigr)
\end{aligned}$$

$$\leq 2R\Big(\Phi(A)\sharp\Phi(B)-\Phi(A\sharp B)+\frac{1}{2}\big(\Phi(A)+\Phi(B)-2\Phi(A)\sharp\Phi(B)\big)\Big)$$
$$-\sum_{j=1}^{N}s_j(2t)\big(\Phi(A\sharp_{1-\beta_j(t)}B)+\Phi(A\sharp_{1+2^{-j}-\beta_j(t)}B)-2\Phi(A\sharp_{1-2^{-j-1}-\beta_j(t)}B)\big)$$
$$-\sum_{j=1}^{N}s_j(t)\big(\Phi(A)\sharp_{\alpha_j(t)}\Phi(B)+\Phi(A)\sharp_{2^{1-j}+\alpha_j(t)}\Phi(B)-2\Phi(A)\sharp_{2^{-j}+\alpha_j(t)}\Phi(B)\big).\qquad\square$$

Similarly for $N=2$ by applying Lemma 2 and Lemma 5, we can obtain a reverse of Ando's inequality for positive invertible operators.

Corollary 7 *Let $A,B\in\mathcal{B}(\mathcal{H})$ be positive invertible operators, Φ be a positive linear map and $t\in[0,1]$.*

(i) *If $0<t\leq\frac{1}{2}$, then*

$$\begin{aligned}&\Phi(A)\sharp_t\Phi(B)-\Phi(A\sharp_t B)\\&\quad\leq 2R\Big(\Phi(A)\sharp\Phi(B)-\Phi(A\sharp B)+\frac{1}{2}\big(\Phi(A)+\Phi(B)-2\Phi(A)\sharp\Phi(B)\big)\Big)\\&\qquad-r_0\big(\Phi(A\sharp B)+\Phi(B)-2\Phi(A\sharp_{\frac{3}{4}}B)\big)\\&\qquad-r_0\big(\Phi(A)\sharp\Phi(B)+\Phi(A)-2\big(\Phi(A)\sharp_{\frac{1}{4}}\Phi(B)\big)\big)\\&\quad\leq 2R\Big(\Phi(A)\sharp\Phi(B)-\Phi(A\sharp B)+\frac{1}{2}\big(\Phi(A)+\Phi(B)-2\Phi(A)\sharp\Phi(B)\big)\Big);\end{aligned}\tag{17}$$

(ii) *if $\frac{1}{2}<t<1$, then*

$$\begin{aligned}&\Phi(A)\sharp_t\Phi(B)-\Phi(A\sharp_t B)\\&\quad\leq 2R\Big(\Phi(A)\sharp\Phi(B)-\Phi(A\sharp B)+\frac{1}{2}\big(\Phi(A)+\Phi(B)-2\Phi(A)\sharp\Phi(B)\big)\Big)\\&\qquad-r_0\big(\Phi(A)\sharp\Phi(B)+\Phi(B)-2\big(\Phi(A)\sharp_{\frac{3}{4}}\Phi(B)\big)\big)\\&\qquad-r_0\big(\Phi(A\sharp B)+\Phi(A)-2\Phi(A\sharp_{\frac{1}{4}}B)\big)\\&\quad\leq 2R\Big(\Phi(A)\sharp\Phi(B)-\Phi(A\sharp B)+\frac{1}{2}\big(\Phi(A)+\Phi(B)-2\Phi(A)\sharp\Phi(B)\big)\Big),\end{aligned}\tag{18}$$

where $r=\min\{t,1-t\}$, $R=\max\{t,1-t\}$ and $r_0=\min\{2r,1-2r\}$.

We want to establish some inequalities for positive invertible operators.

Theorem 8 *Let $A,B\in\mathcal{B}(\mathcal{H})$ be positive invertible. If $t\in[0,1]$ and Φ, Ψ are two unital positive linear maps, then for any unit vector $x\in\mathcal{H}$*

(i) *for $0<t\leq\frac{1}{2}$,*

$$\begin{aligned}&2r\big(\langle\Phi(A)x,x\rangle\nabla\langle\Psi(B)x,x\rangle-\langle\Phi(A^{\frac{1}{2}})x,x\rangle\langle\Psi(B^{1/2})x,x\rangle\big)\\&\qquad+r_0\big(\langle\Phi(A^{\frac{1}{2}})x,x\rangle\big)\langle\Psi(B^{\frac{1}{2}})x,x\rangle+\langle\Phi(A)x,x\rangle-2\langle\Phi(A^{\frac{3}{4}})x,x\rangle\langle\Psi(B^{\frac{1}{4}})x,x\rangle\big)\end{aligned}$$

$$\begin{aligned}
&\le (1-t)\langle\Phi(A)x,x\rangle + t\langle\Psi(B)x,x\rangle - \langle\Psi(B^t)x,x\rangle\langle\Phi(A^{1-t})x,x\rangle \\
&\le 2R\big(\langle\Phi(A)x,x\rangle\nabla\langle\Psi(B)x,x\rangle - \langle\Phi(A^{\frac{1}{2}})x,x\rangle\langle\Psi(B^{\frac{1}{2}})x,x\rangle\big) \\
&\quad - r_0\big(\langle\Phi(A^{\frac{1}{2}})x,x\rangle\langle\Psi(B^{\frac{1}{2}})x,x\rangle + \langle\Psi(B)x,x\rangle - 2\langle\Phi(A^{\frac{1}{4}})x,x\rangle\langle\Psi(B^{\frac{3}{4}})x,x\rangle\big); \quad (19)
\end{aligned}$$

(ii) *for* $\frac{1}{2} < t < 1$,

$$\begin{aligned}
&R\big(\langle\Phi(A)x,x\rangle + \langle\Psi(B)x,x\rangle - 2\langle\Phi(A^{\frac{1}{2}})x,x\rangle\langle\Psi(B^{1/2})x,x\rangle\big) \\
&\quad + r_0\big(\langle\Phi(A^{\frac{1}{2}})x,x\rangle\big)\langle\Psi(B^{\frac{1}{2}})x,x\rangle + \langle\Phi(A)x,x\rangle - 2\langle\Phi(A^{\frac{1}{4}})x,x\rangle\langle\Psi(B^{\frac{3}{4}})x,x\rangle\big) \\
&\le (1-t)\langle\Phi(A)x,x\rangle + t\langle\Psi(B)x,x\rangle - \langle\Psi(B^t)x,x\rangle\langle\Phi(A^{1-t})x,x\rangle \\
&\le r\big(\langle\Phi(A)x,x\rangle + \langle\Psi(B)x,x\rangle - 2\langle\Phi(A^{\frac{1}{2}})x,x\rangle\langle\Psi(B^{\frac{1}{2}})x,x\rangle\big) \\
&\quad - r_0\big(\langle\Phi(A^{\frac{1}{2}})x,x\rangle\langle\Psi(B^{\frac{1}{2}})x,x\rangle + \langle\Phi(A)x,x\rangle - 2\langle\Phi(A^{\frac{3}{4}})x,x\rangle\langle\Psi(B^{\frac{1}{4}})x,x\rangle\big),
\end{aligned}$$

where $r = \min\{t, 1-t\}$, $R = \max\{t, 1-t\}$, $r_0 = \min\{2r, 1-2r\}$.

Proof The proof of part (ii) is similar to the proof of part (i). Thus we only prove (i).

Applying inequality (2) for any positive real numbers k, s, we have

$$\begin{aligned}
&r(k+s-2\sqrt{ks}) + r_0\big(k^{\frac{1}{2}}s^{\frac{1}{2}} + k - 2k^{\frac{3}{4}}s^{\frac{1}{4}}\big) \\
&\quad \le (1-t)k + ts - k^{1-t}s^t \\
&\quad \le R(k+s-2\sqrt{ks}) - r_0\big(k^{\frac{1}{2}}s^{\frac{1}{2}} + s - 2k^{\frac{1}{4}}s^{\frac{3}{4}}\big). \quad (20)
\end{aligned}$$

Fix s in (20). Then applying functional calculus to the operator A, we have

$$\begin{aligned}
&r\big(A + sI - 2\sqrt{s}A^{\frac{1}{2}}\big) + r_0\big(A^{\frac{1}{2}}s^{\frac{1}{2}} + A - 2A^{\frac{3}{4}}s^{\frac{1}{4}}\big) \\
&\quad \le (1-t)A + tsI - s^tA^{1-t} \\
&\quad \le R\big(A + sI - 2\sqrt{s}A^{\frac{1}{2}}\big) - r_0\big(A^{\frac{1}{2}}s^{\frac{1}{2}} + sI - 2A^{\frac{1}{4}}s^{\frac{3}{4}}\big). \quad (21)
\end{aligned}$$

If we apply the positive linear map Φ and inner product for $x \in \mathcal{H}$ with $\|x\| = 1$ in inequality (21), we have

$$\begin{aligned}
&r\big(\langle\Phi(A)x,x\rangle + s - 2\sqrt{s}\langle\Phi(A^{\frac{1}{2}})x,x\rangle\big) \\
&\qquad + r_0\big(\langle\Phi(A^{\frac{1}{2}})x,x\rangle\big)s^{\frac{1}{2}} + \langle\Phi(A)x,x\rangle - 2\langle\Phi(A^{\frac{3}{4}})x,x\rangle s^{\frac{1}{4}}\big) \\
&\quad \le (1-t)\langle\Phi(A)x,x\rangle + ts - s^t\langle\Phi(A^{1-t})x,x\rangle \\
&\quad \le R\big(\langle\Phi(A)x,x\rangle + s - 2\sqrt{s}\langle\Phi(A^{\frac{1}{2}})x,x\rangle\big) - r_0\big(\langle\Phi(A^{\frac{1}{2}})x,x\rangle s^{\frac{1}{2}} + s - 2\langle\Phi(A^{\frac{1}{4}})x,x\rangle s^{\frac{3}{4}}\big).
\end{aligned}$$

Now, using the functional calculus to the operator B, we have

$$\begin{aligned}
&r\big(\langle\Phi(A)x,x\rangle + B - 2\langle\Phi(A^{\frac{1}{2}})x,x\rangle B^{1/2}\big) \\
&\qquad + r_0\big(\langle\Phi(A^{\frac{1}{2}})x,x\rangle\big)B^{\frac{1}{2}} + \langle\Phi(A)x,x\rangle - 2\langle\Phi(A^{\frac{3}{4}})x,x\rangle B^{\frac{1}{4}}\big)
\end{aligned}$$

$$\begin{aligned}&\le (1-t)\langle\Phi(A)x,x\rangle + tB - B^t\langle\Phi(A^{1-t})x,x\rangle\\&\le R\big(\langle\Phi(A)x,x\rangle + B - 2\langle\Phi(A^{\frac{1}{2}})x,x\rangle B^{\frac{1}{2}}\big) - r_0\big(\langle\Phi(A^{\frac{1}{2}})x,x\rangle B^{\frac{1}{2}} + B - 2\langle\Phi(A^{\frac{1}{4}})x,x\rangle B^{\frac{3}{4}}\big).\end{aligned}$$

Taking the positive linear map Ψ and the inner product for $y \in \mathcal{H}$ with $\|y\| = 1$, we get

$$\begin{aligned}&r\big(\langle\Phi(A)x,x\rangle + \langle\Psi(B)y,y\rangle - 2\langle\Phi(A^{\frac{1}{2}})x,x\rangle\langle\Psi(B^{1/2})x,x\rangle\big)\\&\quad + r_0\big(\langle\Phi(A^{\frac{1}{2}})x,x\rangle)\langle\Psi(B^{\frac{1}{2}})x,x\rangle + \langle\Phi(A)x,x\rangle - 2\langle\Phi(A^{\frac{3}{4}})x,x\rangle\langle\Psi(B^{\frac{1}{4}})x,x\rangle\big)\\&\le (1-t)\langle\Phi(A)x,x\rangle + t\langle\Psi(B)x,x\rangle - \langle\Psi(B^t)x,x\rangle\langle\Phi(A^{1-t})x,x\rangle\\&\le R\big(\langle\Phi(A)x,x\rangle + \langle\Psi(B)x,x\rangle - 2\langle\Phi(A^{\frac{1}{2}})x,x\rangle\langle(B^{\frac{1}{2}})x,x\rangle\big)\\&\quad - r_0\big(\langle\Phi(A^{\frac{1}{2}})x,x\rangle\langle\Psi(B^{\frac{1}{2}})x,x\rangle + \langle\Psi(B)x,x\rangle - 2\langle\Phi(A^{\frac{1}{4}})x,x\rangle\langle\Psi(B^{\frac{3}{4}})x,x\rangle\big).\end{aligned}$$

Now, if we put $x = y$, then we get the desired result. □

Theorem 9 *Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive invertible. If $t \in [0,1]$ and Φ, Ψ are two unital positive linear maps, then for any unit vector $x \in \mathcal{H}$*

(i) *for* $0 < t \le \frac{1}{2}$,

$$\begin{aligned}&2r\big(\langle\Phi(A)x,x\rangle\nabla\langle\Psi(B)x,x\rangle - \langle\Phi(A^{\frac{1}{2}})x,x\rangle\langle\Psi(B)x,x\rangle^{1/2}\big)\\&\quad + r_0\big(\langle\Phi(A^{1/2})x,x\rangle\langle\Psi(B)x,x\rangle^{1/2} + \langle\Phi(A)x,x\rangle - 2\langle\Phi(A^{3/4})x,x\rangle\langle\Psi(B)x,x\rangle^{1/4}\big)\\&\le (1-t)\langle\Phi(A)x,x\rangle + t\langle\Psi(B)x,x\rangle - \langle\Psi(B)x,x\rangle^t\langle\Phi(A^{1-t})x,x\rangle\\&\le 2R\big(\langle\Phi(A)x,x\rangle\nabla\langle\Psi(B)x,x\rangle - \langle\Phi(A^{1/2})x,x\rangle\langle\Psi(B)x,x\rangle^{1/2}\big)\\&\quad - r_0\big(\langle\Phi(A^{1/2})x,x\rangle\langle\Psi(B)x,x\rangle^{1/2} + \langle\Psi(B)x,x\rangle - 2\langle\Phi(A^{1/4})x,x\rangle\langle\Psi(B)x,x\rangle^{3/4}\big);\end{aligned}$$

(ii) *for* $\frac{1}{2} < t < 1$,

$$\begin{aligned}&R\big(\langle\Phi(A)x,x\rangle + \langle\Psi(B)x,x\rangle - 2\langle\Phi(A^{1/2})x,x\rangle\langle\Psi(B)x,x\rangle^{1/2}\big)\\&\quad + r_0\big(\langle\Phi(A^{1/2})x,x\rangle)\langle\Psi(B)x,x\rangle^{1/2} + \langle\Phi(A)x,x\rangle - 2\langle\Phi(A^{1/4})x,x\rangle\langle\Psi(B)x,x\rangle^{3/4}\big)\\&\le (1-t)\langle\Phi(A)x,x\rangle + t\langle\Psi(B)x,x\rangle - \langle\Psi(B)x,x\rangle^t\langle\Phi(A^{1-t})x,x\rangle\\&\le r\big(\langle\Phi(A)x,x\rangle + \langle\Psi(B)x,x\rangle - 2\langle\Phi(A^{1/2})x,x\rangle\langle\Psi(B)x,x\rangle^{1/2}\big)\\&\quad - r_0\big(\langle\Phi(A^{1/2})x,x\rangle\langle\Psi(B)x,x\rangle^{1/2} + \langle\Phi(A)x,x\rangle - 2\langle\Phi(A^{3/4})x,x\rangle\langle\Psi(B)x,x\rangle^{1/4}\big),\end{aligned}$$

where $r = \min\{t, 1-t\}$, $R = \max\{t, 1-t\}$, $r_0 = \min\{2r, 1-2r\}$.

Proof The proof of part (ii) is similar to the proof of part (i). Thus we just prove (i). For any positive real number k and any unit vector $x \in \mathcal{H}$, we have

$$\begin{aligned}&r\big(k + \langle\Psi(B)x,x\rangle - 2\sqrt{k}\langle\Psi(B)x,x\rangle\big) + r_0\big(k^{1/2}\langle\Psi(B)x,x\rangle^{1/2} + k - 2k^{3/4}\langle\Psi(B)x,x\rangle^{1/4}\big)\\&\le (1-t)k + t\langle\Psi(B)x,x\rangle - k^{1-t}\langle\Psi(B)x,x\rangle^t\\&\le R\big(k + \langle\Psi(B)x,x\rangle - 2\sqrt{k}\langle\Psi(B)x,x\rangle^{1/2}\big)\\&\quad - r_0\big(k^{1/2}\langle\Psi(B)x,x\rangle^{1/2} + \langle\Psi(B)x,x\rangle - 2k^{1/4}\langle\Psi(B)x,x\rangle^{3/4}\big).\end{aligned}\tag{22}$$

Applying inequality (22) and the functional calculus for the operator A, we have

$$\begin{aligned} &r\big(A + \langle\Psi(B)x,x\rangle I_{\mathcal{H}} - 2\sqrt{A}\langle\Psi(B)x,x\rangle\big) \\ &\quad + r_0\big(A^{1/2}\langle\Psi(B)x,x\rangle^{1/2} + A - 2A^{3/4}\langle\Psi(B)x,x\rangle^{1/4}\big) \\ &\quad\le (1-t)A + t\langle\Psi(B)x,x\rangle I_{\mathcal{H}} - A^{1-t}\langle\Psi(B)x,x\rangle^{t} \\ &\quad\le R\big(A + \langle\Psi(B)x,x\rangle I_{\mathcal{H}} - 2\sqrt{A}\langle\Psi(B)x,x\rangle^{1/2}\big) \\ &\quad\quad - r_0\big(A^{1/2}\langle\Psi(B)x,x\rangle^{1/2} + \langle\Psi(B)x,x\rangle I_{\mathcal{H}} - 2A^{1/4}\langle\Psi(B)x,x\rangle^{3/4}\big). \end{aligned} \tag{23, 24}$$

Now, using the unital positive operator Φ and the inner product for $y \in \mathcal{H}$ with $\|y\| = 1$ in inequality (23), we get

$$\begin{aligned} &r\big(\langle\Phi(A)y,y\rangle + \langle\Psi(B)x,x\rangle I_{\mathcal{H}} - 2\langle\Phi(A)y,y\rangle^{1/2}\langle\Psi(B)x,x\rangle\big) \\ &\quad + r_0\big(\langle\Phi(A^{1/2})y,y\rangle\langle\Psi(B)x,x\rangle^{1/2} + \langle\Phi(A)y,y\rangle - 2\langle\Phi(A^{3/4})y,y\rangle\langle\Psi(B)x,x\rangle^{1/4}\big) \\ &\quad\le (1-t)\langle\Phi(A)y,y\rangle + t\langle\Psi(B)x,x\rangle I_{\mathcal{H}} - \langle\Phi(A^{1-t})y,y\rangle\langle\Psi(B)x,x\rangle^{t} \\ &\quad\le R\big(\langle\Phi(A)y,y\rangle + \langle\Psi(B)x,x\rangle I_{\mathcal{H}} - 2\langle\Phi(A)y,y\rangle^{1/2}\langle\Psi(B)x,x\rangle^{1/2}\big) \\ &\quad\quad - r_0\big(\langle\Phi(A^{1/2})y,y\rangle\langle\Psi(B)x,x\rangle^{1/2} + \langle\Psi(B)x,x\rangle I_{\mathcal{H}} - 2\langle\Phi(A^{1/4})y,y\rangle\langle\Psi(B)x,x\rangle^{3/4}\big). \end{aligned}$$

Now, putting $y = x$, we get the desired result. □

Corollary 10 *Let $A \in \mathcal{B}(\mathcal{H})$ be positive, Φ be a unital positive linear map and $t \in [0,1]$. Then for any unit vector $x \in \mathcal{H}$*

(i) *for* $0 < t \le \frac{1}{2}$,

$$\begin{aligned} &2r\langle\Phi(A)x,x\rangle^{t-\frac{1}{2}}\big(\langle\Phi(A)x,x\rangle^{\frac{1}{2}} - \langle\Phi(A^{\frac{1}{2}})x,x\rangle\big) \\ &\quad + r_0\langle\Phi(A)x,x\rangle^{t-\frac{1}{2}}\big(\langle\Phi(A^{1/2})x,x\rangle + \langle\Phi(A)x,x\rangle^{1/2} \\ &\quad - 2\langle\Phi(A^{3/4})x,x\rangle\langle\Phi(A)x,x\rangle^{-1/4}\big) \\ &\le \langle\Phi(A)x,x\rangle^{t} - \langle\Phi(A^{t})x,x\rangle \\ &\le 2R\langle\Phi(A)x,x\rangle^{t-\frac{1}{2}}\big(\langle\Phi(A)x,x\rangle^{\frac{1}{2}} - \langle\Phi(A^{\frac{1}{2}})x,x\rangle\big) \\ &\quad - r_0\langle\Phi(A)x,x\rangle^{t-\frac{1}{2}}\big(\langle\Phi(A^{1/2})x,x\rangle + \langle\Phi(A)x,x\rangle^{1/2} \\ &\quad - 2\langle\Phi(A^{1/4})x,x\rangle\langle\Phi(A)x,x\rangle^{1/4}\big); \end{aligned}$$

(ii) *for* $\frac{1}{2} < t < 1$,

$$\begin{aligned} &2R\langle\Phi(A)x,x\rangle^{t-\frac{1}{2}}\big(\langle\Phi(A)x,x\rangle^{\frac{1}{2}} - \langle\Phi(A^{\frac{1}{2}})x,x\rangle\big) \\ &\quad + r_0\langle\Phi(A)x,x\rangle^{t-\frac{1}{2}}\big(\langle\Phi(A^{1/2})x,x\rangle + \langle\Phi(A)x,x\rangle^{1/2} \\ &\quad - 2\langle\Phi(A^{1/4})x,x\rangle\langle\Phi(A)x,x\rangle^{1/4}\big) \\ &\le \langle\Phi(A)x,x\rangle^{t} - \langle\Phi(A^{t})x,x\rangle \end{aligned}$$

$$\begin{aligned}
&\le 2r\langle\Phi(A)x,x\rangle^{t-\frac{1}{2}}\left(\langle\Phi(A)x,x\rangle^{\frac{1}{2}}-\langle\Phi(A^{\frac{1}{2}})x,x\rangle\right)\\
&\quad -r_0\langle\Phi(A)x,x\rangle^{t-\frac{1}{2}}\left(\langle\Phi(A^{1/2})x,x\rangle+\langle\Phi(A)x,x\rangle^{1/2}\right.\\
&\quad \left.-2\langle\Phi(A^{3/4})x,x\rangle\langle\Phi(A)x,x\rangle^{-1/4}\right),
\end{aligned}$$

where $r=\min\{t,1-t\}$, $R=\max\{t,1-t\}$, $r_0=\min\{2r,1-2r\}$.

Proof Letting $\Psi=\Phi$ and $B=A$ in Theorem 9, we get the desired inequalities. □

In the next result, we obtain a refinement of inequality (5) for $t\in(0,\frac{1}{2}]$.

Corollary 11 *Let $T\in\mathcal{B}(\mathcal{H})$ be positive operator and $x\in\mathcal{H}$ be a unit vector. Then, for $0<t\le\frac{1}{2}$, we have*

$$\begin{aligned}
&\langle Tx,x\rangle^t-\langle T^tx,x\rangle\\
&\quad\le 2R\langle Tx,x\rangle^{t-\frac{1}{2}}\left(\langle Tx,x\rangle^{\frac{1}{2}}-\langle T^{\frac{1}{2}}x,x\rangle\right)\\
&\qquad -r_0\langle Tx,x\rangle^{t-\frac{1}{2}}\left(\langle T^{\frac{1}{2}}x,x\rangle+\langle Tx,x\rangle^{\frac{1}{2}}-2\langle T^{\frac{1}{4}}x,x\rangle\langle Tx,x\rangle^{\frac{1}{4}}\right)\\
&\quad\le 2R\left(\langle Tx,x\rangle^{\frac{1}{2}}-\langle T^{\frac{1}{2}}x,x\rangle\right)-r_0\left(\langle T^{\frac{1}{2}}x,x\rangle+\langle Tx,x\rangle^{\frac{1}{2}}-2\langle T^{\frac{1}{4}}x,x\rangle\langle Tx,x\rangle^{\frac{1}{4}}\right),
\end{aligned}$$

where $r=\min\{t,1-t\}$, $R=\max\{t,1-t\}$, $r_0=\min\{2r,1-2r\}$.

Proof If we replace $\Phi(A)=A$, $A\in\mathcal{B}(\mathcal{H})$ and t with $1-t$ in Corollary 10, then we get the desired result. □

6 Some new results

In this section, we prove some difference reverse-types of the Hölder and Cauchy–Schwarz inequalities.

Theorem 12 *Let $A_i,B_i\in\mathcal{B}(\mathcal{H})$ $(1\le i\le n)$ be positive invertible and $t\in[0,1]$.*

(i) *If $0<t\le\frac{1}{2}$, then*

$$\begin{aligned}
&\left(\sum_{i=1}^n A_i\right)\sharp_t\left(\sum_{i=1}^n B_i\right)-\left(\sum_{i=1}^n A_i\sharp_t B_i\right)\\
&\quad\le R\left(\sum_{i=1}^n A_i+\sum_{i=1}^n B_i-2\sum_{i=1}^n(A_i\sharp B_i)\right)\\
&\qquad -r_0\left(\sum_{i=1}^n(A_i\sharp B_i)+\sum_{i=1}^n B_i-2\sum_{i=1}^n(A_i\sharp_{\frac{3}{4}}B_i)\right)\\
&\qquad -r_0\left(\sum_{i=1}^n A_i\sharp\sum_{i=1}^n B_i+\sum_{i=1}^n A_i-2\left(\sum_{i=1}^n A_i\sharp_{\frac{1}{4}}\sum_{i=1}^n B_i\right)\right).
\end{aligned}$$

(ii) *If $\frac{1}{2}<t<1$, then*

$$\left(\sum_{i=1}^n A_i\right)\sharp_t\left(\sum_{i=1}^n B_i\right)-\left(\sum_{i=1}^n A_i\sharp_t B_i\right)$$

$$\begin{aligned}
&\leq R\left(\sum_{i=1}^n A_i + \sum_{i=1}^n B_i - 2\left(\sum_{i=1}^n A_i \sharp B_i\right)\right) \\
&\quad - r_0\left(\left(\sum_{i=1}^n A_i\right)\sharp\left(\sum_{i=1}^n B_i\right) + \sum_{i=1}^n B_i - 2\left(\left(\sum_{i=1}^n A_i\right)\sharp_{\frac{3}{4}}\left(\sum_{i=1}^n B_i\right)\right)\right) \\
&\quad - r_0\left(\sum_{i=1}^n (A_i \sharp B_i) + \sum_{i=1}^n A_i - 2\sum_{i=1}^n (A_i \sharp_{\frac{1}{4}} B_i)\right),
\end{aligned}$$

where $r = \min\{t, 1-t\}$, $R = \max\{t, 1-t\}$ *and* $r_0 = \min\{2r, 1-2r\}$.

Proof Taking $A = \operatorname{diag}(A_1, \ldots, A_n)$, $B = \operatorname{diag}(B_1, \ldots, B_n)$ and $\Phi([C_{ij}]_{1\leq i,j\leq n}) = \sum_{i=1}^n C_{ii}$ in equalities (17) and (18), we get the desired inequality. □

Since the function $f(x) = x^t$ ($t \in [0,1]$) is an operator concave function, $\sum_{i=1}^n w_i T_i^t \leq (\sum_{i=1}^n w_i T_i)^t$ for positive operators T_i and positive real numbers w_i such that $\sum_{i=1}^n w_i = 1$. Now, Theorem 12 yields a reverse of this inequality as follows.

Example 13 If for positive operators T_i ($1 \leq i \leq n$), we take $A_i = w_i I$ and $B_i = w_i T_i$ ($1 \leq i \leq n$), in Theorem 12, where w_i's are positive real numbers such that $\sum_{i=1}^n w_i = 1$, we obtain the following inequalities:

(i) If $0 \leq t \leq \frac{1}{2}$, then

$$\begin{aligned}
\left(\sum_{i=1}^n w_i T_i\right)^t - \sum_{i=1}^n w_i T_i^t &\leq R\left(I + \sum_{i=1}^n w_i T_i - 2\sum_{i=1}^n w_i T_i^{1/2}\right) \\
&\quad - r_0\left(\sum_{i=1}^n w_i T_i^{1/2} + \sum_{i=1}^n w_i T_i - 2\sum_{i=1}^n w_i T_i^{3/4}\right) \\
&\quad - r_0\left(\left(\sum_{i=1}^n w_i T_i\right)^{1/2} + I - 2\left(\sum_{i=1}^n w_i T_i\right)^{1/4}\right).
\end{aligned}$$

(ii) If $\frac{1}{2} < t \leq 1$, then

$$\begin{aligned}
\left(\sum_{i=1}^n w_i T_i\right)^t - \sum_{i=1}^n w_i T_i^t &\leq R\left(I + \sum_{i=1}^n w_i T_i - 2\sum_{i=1}^n w_i T_i^{1/2}\right) \\
&\quad - r_0\left(\left(\sum_{i=1}^n w_i T_i\right)^{1/2} + \sum_{i=1}^n w_i T_i - 2\left(\sum_{i=1}^n w_i T_i\right)^{3/4}\right) \\
&\quad - r_0\left(\sum_{i=1}^n w_i T_i^{1/2} + I - 2\sum_{i=1}^n w_i T_i^{1/4}\right).
\end{aligned}$$

In [18], the Tsallis relative operator entropy $T_t(A|B)$ for positive invertible operators A, B and $0 < t \leq 1$ is defined as follows:

$$T_t(A,B) = \frac{A\sharp_t B - A}{t}.$$

For further information as regards the Tsallis relative operator entropy see [6] and the references therein. In [7, Proposition 2.3], it is shown that for any unital positive linear map Φ the following inequality holds:

$$\Phi\big(T_t(A|B)\big) \le T_t\big(\Phi(A)|\Phi(B)\big). \tag{25}$$

In (25), by similar techniques of Theorem 12, for positive operators A_i, B_i ($i = 1, 2, \dots, n$), we have

$$\sum_{i=1}^{n}\big(T_t(A_i|B_i)\big) \le T_t\left(\sum_{i=1}^{n} A_i \middle| \sum_{i=1}^{n} B_i\right). \tag{26}$$

In the next theorem, we show a reverse of inequality (26).

Theorem 14 *Let $A_i, B_i \in \mathcal{B}(\mathcal{H})$ $(1 \le i \le n)$ be positive invertible and $t \in (0,1)$.*
(i) *If $0 < t \le \frac{1}{2}$, then*

$$\begin{aligned} &T_t\left(\sum_{i=1}^{n} A_i \middle| \sum_{i=1}^{n} B_i\right) - \sum_{i=1}^{n}\big(T_t(A_i|B_i)\big) \\ &\quad\le \frac{1}{t}\Bigg[R\left(\sum_{i=1}^{n} A_i + \sum_{i=1}^{n} B_i - 2\sum_{i=1}^{n}(A_i\sharp B_i)\right) \\ &\qquad - r_0\left(\sum_{i=1}^{n}(A_i\sharp B_i) + \sum_{i=1}^{n} B_i - 2\sum_{i=1}^{n}(A_i\sharp_{\frac{3}{4}} B_i)\right) \\ &\qquad - r_0\left(\sum_{i=1}^{n} A_i\sharp\sum_{i=1}^{n} B_i + \sum_{i=1}^{n} A_i - 2\left(\sum_{i=1}^{n} A_i\sharp_{\frac{1}{4}}\sum_{i=1}^{n} B_i\right)\right)\Bigg]. \end{aligned}$$

(ii) *If $\frac{1}{2} < t < 1$, then*

$$\begin{aligned} &T_t\left(\sum_{i=1}^{n} A_i \middle| \sum_{i=1}^{n} B_i\right) - \sum_{i=1}^{n}\big(T_t(A_i|B_i)\big) \\ &\quad\le \frac{1}{t}\Bigg[R\left(\sum_{i=1}^{n} A_i + \sum_{i=1}^{n} B_i - 2\left(\sum_{i=1}^{n} A_i\sharp B_i\right)\right) \\ &\qquad - r_0\left(\left(\sum_{i=1}^{n} A_i\right)\sharp\left(\sum_{i=1}^{n} B_i\right) + \sum_{i=1}^{n} B_i - 2\left(\sum_{i=1}^{n} A_i\sharp_{\frac{3}{4}}\sum_{i=1}^{n} B_i\right)\right) \\ &\qquad - r_0\left(\sum_{i=1}^{n}(A_i\sharp B_i) + \sum_{i=1}^{n} A_i - 2\sum_{i=1}^{n}(A_i\sharp_{\frac{1}{4}} B_i)\right)\Bigg]. \end{aligned}$$

Proof Applying Theorem 12 for $0 < t \le \frac{1}{2}$, we have

$$\begin{aligned} &T_t\left(\sum_{i=1}^{n} A_i \middle| \sum_{i=1}^{n} B_i\right) - \sum_{i=1}^{n}\big(T_t(A_i|B_i)\big) \\ &\quad= \frac{(\sum_{i=1}^{n} A_i)\sharp_t(\sum_{i=1}^{n} B_i) - \sum_{i=1}^{n} A_i}{t} - \sum_{i=1}^{n}\frac{A_i\sharp_t B_i - A_i}{t} \end{aligned}$$

$$\leq \frac{1}{t}\Bigg[R\left(\sum_{i=1}^{n}A_i+\sum_{i=1}^{n}B_i-2\sum_{i=1}^{n}(A_i\sharp B_i)\right)-r_0\left(\sum_{i=1}^{n}(A_i\sharp B_i)+\sum_{i=1}^{n}B_i-2\sum_{i=1}^{n}(A_i\sharp_{\frac{3}{4}}B_i)\right)$$
$$-r_0\left(\sum_{i=1}^{n}A_i\sharp\sum_{i=1}^{n}B_i+\sum_{i=1}^{n}A_i-2\left(\sum_{i=1}^{n}A_i\sharp_{\frac{1}{4}}\sum_{i=1}^{n}B_i\right)\right)\Bigg],$$

whence we get the first inequality. The proof of the second inequality is similar. □

Remark 15 We can present our results for non-invertible operators; see [6]. It is a direct consequence of the definition of the mean in the sense of Kubo–Ando [11] that $A\sharp_t(B+\varepsilon)$ is a monotone increasing net. Let B be a non-invertible operator and $\epsilon > 0$. It follows from the set $\{A\sharp_t(B+\epsilon):\epsilon>0\}$ being bounded above for $0<\varepsilon<1$ that the limit

$$A\sharp_t B=\lim_{\epsilon\downarrow 0}A\sharp_t(B+\epsilon) \tag{27}$$

exists in the strong operator topology. So by (27), $A\sharp_t B$ exists.

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