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REVERSIBILITY OF THE TIME-DEPENDENT SHORTEST PATH PROBLEM

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ABSTRACT

Time-dependent shortest path problems arise in a variety of applications; e.g., dynamic traffic assignment (DTA), network control, automobile driver guidance, ship routing and airplane dispatching. In the majority of cases one seeks the cheapest (least generalized cost) or quickest (least time) route between an origin and a destination for a given time of departure. This is the "forward" shortest path problem. In some applications, however, e.g., when dispatching airplanes from airports and in DTA versions of the "morning commute problem", one seeks the cheapest or quickest routes for a given arrival time. This is the "backward" shortest path problem. It is shown that an algorithm that solves the forward quickest path problem on a network with first-in-first-out (FIFO) links also solves the backward quickest path problem on the same network. More generally, any algorithm that solves forward (or backward) problems of a particular type is shown also to solve backward (forward) problems of a conjugate type.

The problem of finding the shortest path from an origin to a destination over a network in which the link travel times are time-dependent is of central importance in dynamic traffic assignment (DTA) and many other applications. In some DTA problems (e.g., as originally formulated in Merchant and Nemhauser, 1978) one looks for the earliest arrival time at a destination (or set of destinations) from a given origin for a given departure time. This (forward) shortest path problem also arises in other applications. In DTA versions of the "morning commute problem" (first formulated in Hendrickson et. al., 1983) one looks for the quickest routes for a given arrival time. This is the "backward" shortest path problem, which also arises in vehicle dispatching problems. The connexion between these two problems is examined below. It is assumed initially that the link travel times satisfy a first-in-first-out (FIFO) rule, which prevents anyone to depart a link earlier by arriving later.

A compact formulation of the forward time-dependent shortest path problem with FIFO is given by (1) below. These equations pertain to a generic origin "o" that is left at time t_0 ($t_0 = 0$, without loss of generality) and a generic destination, "d". We wish to find a sequence of nodes {d, in, in-1, ..., i2, i1, o} for which the following telescoping series is minimized:

$$\min\{E_{d,in}(E_{in,in-1}(\ldots E_{i2,i1}(E_{i1,o}(0)) \ldots))\}, \qquad (1a)$$

where $E_{ij}(t)$ is the "exit function" that gives the arrival time at downstream node i when upstream node node j is left at time t. The exit function is defined only for node pairs for which there is a link from j (the second subscript) to i. Thus, (1a) must be complemented by a feasible region of possible sequences:

$$\text{for all sequences } \{d, in, in\text{-}1, \ldots, i2, i1, o\} \text{ such that } \delta_{ij,ij-1} = 1 \qquad (1b)$$

where δ_{ij} is the "upstream" link indicator function which is "1" if and only if j (its first subscript) is an *exit* node of i. Equation (1b) is understood to include pairs (i1, o) and (d, in) for j = 1 and j = n+1; the lablel "n" is a variable . The FIFO condition is simply:

$$E_{ij}(t) > t\,, \quad \text{and} \quad E_{ij}(t) \text{ increases.} \tag{1c}$$

Likewise, a compact formulation of the backward time-dependent shortest path problem, where we look for the latest time(s) at which one can depart a (set of) origin(s) "o" so as to arrive at a single destination "d" at a given time ($t_d = 0$, without loss of generality), is in terms of link input functions $I_{ij}(t)$ that give the time of departure from upstream node i (first subscript) that is required to arrive at j at time t. Now we have:

$$\max\{I_{o,i1}(I_{i1,i2}(\ldots (I_{in,d}(0))\ldots))\} \tag{2a}$$

$$\text{for all sequences } \{o, i1, i2, \ldots in\text{-}1, in, d\} \text{ such that } \Delta_{ij-1,ij} = 1, \tag{2b}$$

and assuming that:

$$I_{ij}(t) < t\,, \quad \text{and} \quad I_{ij}(t) \text{ increases.} \tag{2c}$$

Here, Δ_{ij} is the "downstream" link indicator function that is "1" if and only if its first subscript is an *entry* node for the second subscript.

The symmetry arises because equations (1) transform into equations (2), and viceversa, if we reverse the direction of every arc in the network (changing δ to δ' and Δ to Δ' by transposing subscripts in both cases) and also reverse the direction of time (changing t to -t'). We expect this to be true because the time-dependent shortest path problem arises in nature, and we know that the basic equations of physics are invariant to inversions of the space-time coordinates. To verify this statement, define a set of E' functions for the new network as follows:

$$E'_{ij}(t) = -\, I_{ij}(-t). \tag{3}$$

If we now replace t by -t' in (2c) and then use (3) to express the relation in terms of the E', we find:

$$E'_{ij}(t') > t' , \quad \text{and} \quad E'_{ij}(t') \text{ is increasing.} \tag{4c}$$

Thus, the E' are proper exit functions, as in (1c). If we now change t to -t' in (2a) by changing the sign of the output of every function, the equation becomes

$$\max \{-I_{o,i1}(- I_{i1,i2}(\ldots (-I_{in,d}(-0)) \ldots))\}$$

and we find, using Eq.(3), that this becomes:

$$\max \{-E'_{o,i1}(- [-E'_{i1,i2}(\ldots (-[-E'_{in,d}(0)]) \ldots)])\},$$

which in turn reduces to:

$$\min \{E'_{o,i1}(E'_{i1,i2}(\ldots (E'_{in,d}(0)) \ldots))\} \tag{4a}$$

which is analogous to (1a). This must happen for all sequences:

$$\{o, i1, i2, \ldots , in\text{-}1, in, d\} \text{ such that } \delta'_{ij-1,ij} = 1, \tag{4b}$$

which is analogous to (1b). [The order of the subscripts in the last equality is justified because the subscripts have to be transposed twice in going from (2b) to (4b)--once when changing "downstream" to "upstream" notation (Δ to δ) and once again when changing the network by reversing the direction of all the arcs (δ to δ').]

Note that Eqs.(4) are identical to (1). They define a "forward" shortest path problem with FIFO for departures from a single origin "d" at time zero, to the (set of possible) destination(s) "o". Thus, an algorithm that would solve (1) would solve (4)-(2), and viceversa. Note too that there is nothing inherently more difficult in extracting the $I_{ij}(t)$ than the $E_{ij}(t)$ from normally available information. For example, simulations of the form suggested in Daganzo (1994) readily give the $E_{ij}(t)$ and $I_{ij}(t)$ curves in parametric form, indexed by vehicle number.

More general problems. The above reversibility results are a manifestation of a more general principle. Let **F** denote a network and the data that define a forward problem, i.e., the connectivity matrix and the associated exit functions (whether FIFO or not), and let f denote a generalized cost mapping that returns a cost $f(\mathrm{p},\mathbf{F})$ for every path, p, and network **F**, if $\mathrm{p} \in \mathbf{F}$. (In the main body of this note, the role of f was played by (1a) and path feasibility was enforced by (1b).) Likewise, let **B** denote a network and the data for a backward problem, i.e., a connectivity matrix and the associated input functions, and assume that the goal in this case is to find a feasible path $\mathrm{p} \in \mathbf{B}$ that minimizes a generalized cost mapping $b(\mathrm{p}, \mathbf{B})$, such as the negative of (2a).

Construct now a conjugate network of **B**, $\mathbf{B}^{\mathrm{R}}$, by reversing all the arcs and paths in **B** (so that $\mathrm{p} \in \mathbf{B}^{\mathrm{R}}$ if and only if $\mathrm{p}^{\mathrm{R}} \in \mathbf{B}$) and by converting the input functions of **B** into forward exit functions as per (3), reversing time. A conjugate generalized cost mapping b^{R} is also defined for this forward network by means of the relation: $b^{\mathrm{R}}(\mathrm{p}, \mathbf{B}^{\mathrm{R}}) = b(\mathrm{p}^{\mathrm{R}}, \mathbf{B})$, $\forall\ \mathrm{p} \in \mathbf{B}^{\mathrm{R}}$. (Note that this is possible since $\mathrm{p} \in \mathbf{B}^{\mathrm{R}}$ if and only if $\mathrm{p}^{\mathrm{R}} \in \mathbf{B}$.) Obviously then, a forward algorithm F can be used to solve the conjugate problem (solving the backward problem in the process) if $\mathbf{B}^{\mathrm{R}}$ and b^{R} satisfy the properties required by F. (Similarly, a backward algorithm B can be used to solve the conjugate of a forward problem if the latter satisfies the requirements of B.) It was shown in the main body of this note that conjugation (in either direction) preserved the properties required by FIFO quickest route algorithms.

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