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A formalization of set theory without variables, by Alfred Tarski and Steven Givant. Colloquium Publications, vol. 41, American Mathematical Society, Providence, R.I., 1987, xxi + 318 pp., list $60.00; ind. $36.00. ISBN 0-82189-1041-3

This last book written (in part) by Tarski is typical of his life work, in that it contributes both to our understanding of what mathematics is, as well as to technical foundational research. Very briefly, the following is done. A simple equational language $\mathscr{L}^{\times}$ is introduced, and it is shown that set theory (for example, ZFC) can be translated into equations of $\mathscr{L}^{\times}$ which have no variables, so that sentences derivable in set theory are translated into equations derivable by equational rules of inference from the translates of the set-theoretical axioms. Even more briefly, one may say that it is shown that, in principle, mathematics can be developed in the very simple framework of equations and substitution of equals for equals, rather than the customary basis in set theory formalized in first-order logic. At the very least, the main result must be considered as an impressive tour de force. It will probably influence the attitude of many mathematicians concerning the nature of their discipline. Just like the authors, this reviewer will not venture into a serious philisophical discussion of the meaning of the result.

For the exposition of the mathematics involved, very little in the way of prerequisites is needed—which is not to say that the proofs are easy. For a reader who just wants to get an idea of what is going on, without investing a lot of time in checking the proofs, the proofs can be skipped without losing the drift of the ideas. A more committed reader will find the book a rich source of ideas and problems in various foundational directions.

I found the book difficult to read. I think the reason is that formalisms are emphasized over algebraic aspects of the work. In view of the philosophical purpose of the book, this is natural; and it will probably make the book attractive to many people, especially to proof-theorists, and mathematically oriented philosophers. A model-theorist or algebraic logician might prefer to read the last chapter—applications to algebra—first.

The main ideas and results are due to Tarski, and come from the period 1940–1945. In the course of writing the book with Tarski, Givant made many independent contributions to the theory. Maddux and McNulty also contributed results or proofs to this final result of the development, and very recently Andréka and Németi solved several problems which arose during the preparation. (Since the publication, they solved the problem stated before 4.8(xvi), p. 144, positively.)

The following summary of the book, partly in different terminology from the authors', is intended as further material to help the reader decide whether to look at the book, and also to aid those who wish to study it carefully. One should also mention at the outset the useful section interdependence chart and indices.

CHAPTER 1. This gives a standard formulation $\mathscr{L}$ of first-order logic, with one major and one minor difference. The major difference is that exactly one nonlogical constant is admitted—a binary relation symbol $\mathbf{E}$. The minor difference is that equality is denoted by the peculiar symbol $\mathring{1}$.

CHAPTER 2. A definitional expansion $\mathscr{L}^+$ of $\mathscr{L}$ of an unusual sort is described. One defines relation symbols by recursion, in terms of new symbols $+, ^-, \odot$, and $^\smile$, as follows. $\mathbf{E}$ and $\mathring{1}$ are relation symbols, and if A and B are relation symbols, then so are $A+B, A^-$, $A \odot B$, and $A^\smile$; all of them are binary relation symbols. (These compound expressions are, of course, not atomic relation symbols in the usual sense, but they function like relation symbols in usual developments of logic.) Then in addition to the usual atomic formulas one has an atomic formula $A = B$ for all relation symbols A, B, where $=$ is a new symbol; these new atomic formulas are called *equations*. *Models* of $\mathscr{L}^+$ are the same as those of $\mathscr{L}$. *Denotation* for relation symbols is defined as follows. Given an $\mathscr{L}$-structure $\langle A, E\rangle$, the denotation den($\mathbf{E}$) of $\mathbf{E}$ is E; den($\mathring{1}$) is $\{(a, a) \colon a \in A\}$; and

$$\begin{gathered}\operatorname{den}(A+B) = \operatorname{den}(A) \cup \operatorname{den}(B),\\ \operatorname{den}(A^-) = (A \times A) \setminus \operatorname{den}(A),\\ \operatorname{den}(A \odot B) = \{(a, b) \colon \exists c[(a, c) \in \operatorname{den}(A) \text{ and } (c, b) \in \operatorname{den}(B)]\},\\ \operatorname{den}(A^\smile) = \{(a, b) \colon (b, a) \in \operatorname{den}(A)\}.\end{gathered}$$

In addition to the usual logical axioms one has the following:

$$\begin{gathered}\forall x, y[x(A+B)y \leftrightarrow (xAy \vee xBy)],\\ \forall x, y[xA^-y \leftrightarrow \neg(xAy)],\\ \forall x, y[x(A \odot B)y \leftrightarrow \exists z(xAz \wedge zBy)],\\ \forall x, y[xA^\smile y \leftrightarrow yAx],\\ A = B \leftrightarrow \forall x, y(xAy \leftrightarrow xBy).\end{gathered}$$

Since $\mathscr{L}^+$ is equivalent to $\mathscr{L}$ but is easier to work with, it is emphasized in the rest of this review.

CHAPTER 3. Here one considers the fragment $\mathscr{L}^\times$ of $\mathscr{L}$ which syntactically consists just of the equations $A = B$ in that language. Models of $\mathscr{L}^\times$ are the same as those for $\mathscr{L}$ and $\mathscr{L}^+$. For axioms one takes the equations formulated in $\mathscr{L}^+$ corresponding to a standard set of equations for relation algebras (in Tarski's sense), and one allows the usual simple kind of inference with equations alone—substitution of equals for equals. Note that no variables appear in the equations. It is shown that not as much can be expressed in $\mathscr{L}^\times$ as in $\mathscr{L}^+$. For example, for the following sentence ϕ of $\mathscr{L}^+$ there is no equation $A = B$ of $\mathscr{L}^+$ which has the same models as ϕ:

$$\forall x, y, z \exists u[\neg(x\mathring{1}u) \wedge \neg(y\mathring{1}u) \wedge \neg(z\mathring{1}u)].$$

(This is an old result of Korselt, but Tarski and Givant extend it considerably, showing that for any reasonable way of expanding the primitive notions, nonequivalence in means of expression still holds.) It is still an open question whether there is a reasonable proof theory for an extension of $\mathscr{L}^{\times}$ equivalent to the appropriate extension of $\mathscr{L}^{+}$.

Next, it is shown that $\mathscr{L}^{\times}$ is equivalent to the weakenings $\mathscr{L}_3$ and $\mathscr{L}_3^{+}$ of $\mathscr{L}$ and $\mathscr{L}^{+}$ obtained by using only the first three variables in them. Less formally, the equational calculus above is equivalent to the fragment of first-order logic in which only three variables are used.

The final section of the chapter contains a brief discussion of what is expressible in $\mathscr{L}_3$, and extension of notions to similar languages $\mathscr{L}_n$, $n > 3$.

CHAPTER 4. With relation symbols, A, B in $\mathscr{L}^{+}$ one associates an equation Q_{AB}:

$$\left([(A^{\smile} \odot A) + (B^{\smile} \odot B)]^{-} + \dot{1}\right) \cdot (A^{\smile} \odot B) = 1.$$

In a model, Q_{AB} expresses that A and B are functions, and for any x and y there is a z such that $Az = x$ and $Bz = y$; that is, A and B are *pairing functions* (called *conjugated quasiprojections* in the book). Deferring until later conditions under which Q_{AB} is derivable, the authors derive many consequences from the hypothesis Q_{AB} in the proof-theory of $\mathscr{L}^{\times}$ mentioned in Chapter 3. Then the following basic results are shown.

(1) For every sentence X of $\mathscr{L}^{+}$ there is an equation Y of $\mathscr{L}^{\times}$ such that $Q_{AB} \vdash^{+} X \leftrightarrow Y$.

(2) For every collection Ψ of equations of $\mathscr{L}^{\times}$ and every equation X of $\mathscr{L}^{\times}$ we have $\Psi \cup \{Q_{AB}\} \vdash^{+} X$ iff $\Psi \cup \{Q_{AB}\} \vdash^{\times} X$.

These results say that, under the hypothesis Q_{AB}, the systems $\mathscr{L}^{+}$ and $\mathscr{L}^{\times}$ are equivalent after all, even though they are not equivalent in general.

Now the authors turn to the vital question concerning conditions under which Q_{AB} is derivable. In fact, a set Γ of sentences of $\mathscr{L}^{+}$ is called a *Q-system* provided that there exist relation symbols A, B such that $\Gamma \vdash Q_{AB}$. With the help of the translation used in proving (1) and (2), they show that if Γ is a Q-system, then the translated set $\Gamma^{\times}$ of equations in $\mathscr{L}^{\times}$ is a Q-system. This shows the method for constructing equational Q-systems.

The culminating point of this chapter, and indeed of the whole book, is then the application of this procedure to set theory. Let **P** be the pairing axiom

$$\forall x, y \exists z \forall u (u \mathbf{E} z \leftrightarrow u \dot{1} x \vee u \dot{1} y).$$

If Γ is a set of sentences of $\mathscr{L}^{+}$ such that $\mathbf{P} \in \Gamma$ (for example, if Γ is ZF or ZFC), then Γ is a Q-system; for example, one can define A and B as follows:

$$D = \mathbf{E}^{\smile} \odot [\mathbf{E}^{\smile} \cdot (\mathbf{E}^{\smile -} \oplus \dot{1})], \qquad F = \mathbf{E}^{\smile} \odot \mathbf{E}^{\smile},$$
$$A = D \cdot (D^{-} \oplus \dot{1}), \qquad B = F \cdot (F^{-} + (A \oplus \dot{1})).$$

(Here $X \oplus Y$ abbreviates $(X^{-} \odot Y^{-})^{-}$.) Similar but more complicated considerations apply to systems of set theory admitting proper classes, or individuals (Urelemente). The methods do not seem to apply to the well-known systems of Mostowski and of Ackermann.

The final portion of the chapter is concerned with showing that Q-systems can be formalized in three-variable logic.

CHAPTER 5. This is concerned with (relatively minor) improvements of the preceeding results: more elegant translation functions for proving (1) and (2) above; reducing the number of primitive notions of $\mathscr{L}^{\times}$, in particular eliminating $\mathring{1}$ or $=$; and applications to undecidable subsystems of sentential logic.

CHAPTER 6. This chapter is concerned with applications of the main results to the foundations of set theory. There is a general result as to the finite axiomatizability of predicative set theories with power classes (thus including the well-known Bernays-Gödel system as a special case). The same kind of theorem holds for the less well-known predicative set theories without proper classes.

CHAPTER 7. First of all, extensions of the main results to an arbitrary finite number n of binary relation constants, rather than just one, **E**, are discussed. Q-systems in this setting still give rise to equivalent $^{\times}$- and $^{+}$-notions, as in Chapter 4. More is true: under some mild restrictions on the new Q-system, an equivalent system can be constructed in the original language $\mathscr{L}^{\times}$.

The results extend in a modified form to every first-order language. This naturally leads to a discussion of languages in general with finitely many variables.

True number theory, Peano arithmetic, and real arithmetic all prove to be Q-systems, and therefore have equivalent formulations in $\mathscr{L}^{\times}$.

CHAPTER 8. Many of the main results in the book can be given a purely algebraic formulation, and that is done in this chapter. Thus a *Q-relation algebra* is defined to be a relation algebra having two pairing elements a, b, i.e., elements satisfying

$$\Big([(a^{\smile} \odot a) + (b^{\smile} \odot b)]^{-} + \mathring{1}\Big) \cdot (a^{\smile} \odot b) = 1.$$

The basic results (1), (2) in Chapter 4 now have the following algebraic formulation: Every Q-relation algebra is representable.

Another major application of the results of the book to algebra concerns decision problems: certain equational theories are shown to be undecidable, or essentially undecidable.

The chapter closes with some interesting historical remarks about these results.

J. DONALD MONK
UNIVERSITY OF COLORADO