is C^∞ near ξ whenever $P(f)$ is so. In some cases this can be seen because one can construct directly a pseudodifferential inverse; for this the large classes of pseudodifferential operators described in Chapter 19 are a very sharp tool. In other cases one cannot construct directly a pseudodifferential inverse, but one can prove microhypoellipticity by arguments using both microlocal geometry and a priori estimates. Such is the case for operators "of Kolmogorov type": $X_0 + \sum X_i^2$, where the X_j are vector fields whose Lie algebra spans the whole space at every point.

Chapter 24 contains the theory of the mixed Cauchy-Dirichlet problem for second-order differential operators (i.e., the study of the evolution in time of the solution of the wave equation in a bounded domain, with some reflection condition on the boundary). The existence of solutions has been proved by techniques using energy inequalities. The precise study of the singularities of the solutions and of their propagation requires the full arsenal of microlocal analysis.

The last chapter (30) is on scattering theory for long-range potentials (short-range scattering is dealt with in Volume II). The typical example is the theory of $H = \sum \partial^2/\partial x_j^2 + V(x)$, where the potential V does not decay fast enough at infinity (e.g. $V = O(1/|x|)$). The aim is to intertwine the part of H with continuous spectrum with the Laplace operator $\sum \partial^2/\partial x_j^2$ (i.e., prove that nonbounded particles behave at infinity as free particles). One of the key ingredients of the theory is the construction of a distorted Fourier transformation adapted to H, i.e., of a family of approximate solutions of $H(f_\xi) = -|\xi|^2 f_\xi$ which behave at infinity asymptotically as $\exp(-ix \cdot \xi)$. The same ideas of microlocal analysis are used, now applied to asymptotic expansions when $x \to \infty$.

The lines above only give a very short idea of the contents of the book. I at least hope they will be motivation to read it. Each chapter of the book also contains an introduction, which describes with more details the contents and methods of the chapter, and a bibliographical and historical notice. The book also contains a very complete bibliography. It is a superb book, which must be present in every mathematical library, and an indispensable tool for all—young and old—interested in the theory of partial differential operators.

LOUIS BOUTET DE MONVEL

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Minimal surfaces and functions of bounded variation, by Enrico Giusti, Monographs in Mathematics, Vol. 80, Birkhäuser, Boston-Basel-Stuttgart, 1984, xii + 240 pp., $39.95. ISBN 0-8176-3153-4

Among all surfaces spanning a given boundary is there one of least area? Such problems have sometimes been called collectively *the problem of Plateau* in honor of a nineteenth-century physicist who wrote a treatise on equilibrium configurations of liquids (including soap films and soap bubbles). Prior to about 1960 *surface* usually would have meant a mapping from a 2-dimensional parameter domain (typically a disk) into $\mathbf{R}^3$ and *area* would have meant the integral of the Jacobian of the mapping. Physical soap films, incidentally, frequently cannot be so described. J. Douglas was awarded one of the first two Fields medals in 1936 for showing the existence of area-minimizing mappings in this context; his basic strategy was to use Riemann's mapping theorem to replace the area integral by Dirichlet's integral, the minimization of which is much easier than area (the minima are harmonic functions)—such a device, of course, is not available for higher-dimensional domains. J. C. C. Nitsche has written a major treatise **[NJ]** devoted to minimal surfaces so understood, and R. Osserman summarizes recent advances and discusses some higher-dimensional problems as well in his new survey **[OR]**.

The year 1960 is frequently taken as a turning point in the geometric calculus of variations (and especially the study of least area) because of three seminal contributions. One was the paper of E. R. Reifenberg, *Solution of the Plateau problem for m-dimensional surfaces of varying topological type* **[RE]**. A second was the paper of H. Federer and W. H. Fleming, *Normal and integral currents* **[FF]**. The final was work of E. De Giorgi on area-minimizing hypersurfaces. His ideas, although known to experts, were never published in widely circulated journals. This present book by E. Giusti sets forth that part of the theory of area-minimizing hypersurfaces which can be studied in De Giorgi's context.

Reifenberg had been a student of A. S. Besicovitch and, in his 1960 work, introduced highly original and powerful geometric and measure-theoretic constructions both to prove the existence of m-dimensional area-minimizing sets in $\mathbf{R}^n$ spanning prescribed boundaries and also to prove that such sets were continuous manifolds in a neighborhood of almost every interior point. *Area* meant "spherical" Hausdorff measure and *spanning* was naturally defined using Čech homology (with an assist from J. F. Adams). Several years later Reifenberg showed that his least-area surfaces were almost everywhere real analytic minimal submanifolds. He died while mountain climbing in the mid-1960s without having received the general recognition his accomplishments merited.

General *currents*, by definition, are continuous linear functionals on differential forms. The "integral currents" introduced by Federer and Fleming are those currents which can be strongly approximated by integration over Lipschitz singular chains with integer coefficients (as in algebraic topology) and, as such, have similar combinatorial structure and topological properties. Their compactness and lower semicontinuity theorems in general dimensions and codimensions implied the existence of integral currents spanning prescribed boundaries (in the sense of Stokes's theorem) or representing integral homology classes in manifolds and minimizing integrals of general convex parametric integrands such as area. At that time these surfaces were known only to have the regularity properties of general integral currents, i.e., to be carried by rectifiable sets with integer densities. Several years later, after initial work by W. P. Ziemer, Fleming invented *flat chains mod* ν (with coefficients in the integers modulo ν) with similar approximation and compactness properties —an area-minimizing Möbius band would be a 2-dimensional flat chain mod 2 in $\mathbf{R}^3$. A basic reference for currents and flat chains is Federer's treatise [**FH**].

De Giorgi worked in a framework inspired by R. Caccioppoli and studied area-minimizing hypersurfaces which were boundaries of "Caccioppoli sets". In part due to earler work of Federer and De Giorgi on the general Gauss-Green theorem, existence of relevant area-minimizing surfaces was relatively easy. De Giorgi's striking new theorem was that in a neighborhood of almost every point such an area-minimizing hypersurface is a real analytic minimal submanifold. This was the *first* such regularity result for area-minimizing hypersurfaces in general dimensions.

Since this period of the 1960s many thousands of pages of mathematics have been devoted to the study of minimal surfaces and related problems in the geometric calculus of variations in general dimensions and codimensions formulated in the context of these currents, flat chains, Caccioppoli sets, or of varifolds (the reviewer's *varifold* surfaces were introduced in 1965 to facilitate the study of the calculus of variations in the large). A cross-section of the present state of mathematical activity is suggested by the twenty-six articles in the volume, *Geometric measure theory and the calculus of variations* [**AA**], from the 1984 AMS Summer Institute. As indicated above, this book by Giusti follows that part of the overall theory which can be studied in the context of Caccioppoli sets and functions of bounded variation. His introduction alone mentions more than a dozen mathematicians who have made significant contributions. This is a lovely book to read and it facilitates rapid penetration into a beautiful and deep area of mathematical achievement and important open problems. It is fairly self-contained except for basic measure theory and some theory of elliptic partial differential equations—one of the appendices, furthermore, does summarize much of the relevant PDE material.

Giusti devotes his first four chapters to the study of spaces of BV functions (BV stands for bounded variation). These are spaces of functions whose distributional first derivatives are Radon measures of locally bounded total variation. Such spaces have especially nice compactness, semicontinuity, approximation, and trace properties. The Caccioppoli sets or "sets of locally finite perimeter" are those sets E whose characteristic functions belong to BV. The distribution first derivatives of such a characteristic function are unit exterior normal vectors of the (rectifiable) *reduced boundary* of E weighted with this boundary's Hausdorff surface area measure. Bounded Caccioppoli sets are those for which the Gauss-Green holds in its apparent most natural generality—i.e., for which volume integrals of divergences of vectorfields equal boundary surface integrals of inner products with unit exterior normal vectors. In the "parametric setting" for minimal surfaces in $\mathbf{R}^n$ in the context of this book one fixes a bounded open set Ω and a Caccioppoli set L intersecting Ω. Then among all Caccioppoli sets E which coincide with L outside Ω the compactness mentioned above guarantees the existence of one for which the boundary of $E \cap A$ has the least possible $(n-1)$-dimensional area. Intuitively, the oriented least-area surface is $\partial E \cap \Omega$, which has boundary $\partial\Omega \cap \partial L$. The reviewer does not know how to use this setup to guarantee the existence of an oriented surface of least area spanning, say, a knotted curve in $\mathbf{R}^3$. For such situations or more general ones, powerful existence and regularity theorems are fortunately available in general dimensions, at the cost of considerably more complicated mathematics than that of the present book.

The next four chapters of the book are devoted to the proof of the almost everywhere regularity of such minimal hypersurfaces S and contain the regularity theorem of De Giorgi mentioned above.

Next, three chapters are devoted to connections between tangent cones and the size of singular sets. The analysis of minimal surfaces has traditionally made extensive use of the existence of tangent cones at each point—these cones themselves are minimal surfaces which can be easier to understand. It is, incidentally, one of the major unsolved problems of the subject whether or not at every point such tangent cones are unique (this is known to hold only when one of the cones is regular everywhere except at its vertex, as shown by L. Simon [SL] following partial results by W. K. Allard and Almgren). It is a consequence of the regularity proofs that a minimal surface will be regular near any point at which any tangent cone is a plane. Following proofs in $\mathbf{R}^3$ and $\mathbf{R}^4$ by Fleming and Almgren, J. Simons showed every minimal hypercone had to be a plane up to $\mathbf{R}^7$. This meant, in particular, that oriented least-area hypersurfaces in $\mathbf{R}^n$ had no singularities for $n = 3, 4, 5, 6, 7$. Then came the striking result of E. Bombieri, De Giorgi, and Giusti that the 7-dimensional central cone over $S^3 \times S^3 \subset \mathbf{R}^4 \times \mathbf{R}^4 = \mathbf{R}^8$ is area minimizing—this cone, of course, is singular at the origin. (Bombieri received his Fields medal in 1974 in part for his contributions to minimal surfaces.) Federer then used an ingenious argument to show that, for any n, oriented hypersurfaces in $\mathbf{R}^n$ can have singularities of Hausdorff dimension no larger than $n - 8$. These singularity estimates are all proved quite efficiently in Giusti's book.

Incidentally, these interior regularity results combined with boundary regularity results of R. Hardt and Simon were an essential part of B. White's proofs **[W1, W2]** of the existence of least-area hypersurface mappings $f\colon M \to \mathbf{R}^n$ for $n = 4, 5, 6, 7$ (and by an easy extension to $n = 8$); here M is any compact $(n - 1)$-dimensional compact manifold with boundary and $f \mid \partial M$ is any smooth embedding. These are the only significant higher-dimensional least-area mapping results presently known (apparent claims of A. T. Fomenko notwithstanding).

The second part of this book is devoted to nonparametric hypersurfaces S, i.e. graphs of functions $x_n = f(x_1, \ldots, x_{n-1})$ which are area minimizing. M. Miranda showed that such an S is minimizing if and only if its subgraph $\{x\colon x_n < f(x_1, \ldots, x_{n-1})\}$ is a Caccioppoli set of least perimeter in the sense above. Together with *a priori* estimates for gradients by Bombieri, De Giorgi, and Miranda this led to the everywhere interior regularity of such nonparametric minimal surfaces. Two chapters are devoted to the classical Dirichlet problem for the minimal surface equation and boundary mean curvature conditions necessary for its general solution. Then three chapters consider the nonparametric problem in "relaxed forms" in the space of BV functions. Relaxed solutions sometimes have vertical portions above the boundary of the parameter domain—this can occur with L^1 boundary data when domain boundaries do not satisfy the relevant curvature conditions. Solutions with infinite boundary data are also considered. This sometimes produces interesting generalizations of one of Scherk's classical minimal surfaces.

The final chapter of this book is devoted to extensions of the theorem of S. Bernstein that a function $z = f(x, y)$ satisfying the minimal surface equation and defined for all (x, y) in $\mathbf{R}^2$ must be affine. The corresponding theorem for functions $f: \mathbf{R}^n \to \mathbf{R}$ is true when $n = 3, 4, 5, 6, 7$ and fails for larger n.

As indicated above, this book leads one near the frontiers of knowledge in the study of oriented area-minimizing hypersurfaces. Much more remains to be done. For example, we know very little about the structure of singularities—not even if they necessarily have integer dimensions or whether or not they can persist under small boundary deformations.

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Distribution of values of holomorphic mappings, by B. V. Shabat, translated from Russian by James R. King. Translations of Mathematical Monographs, vol. 61, American Mathematical Society, Providence, R. I., 1985, v + 225 pp., $79.00. ISBN 0-8218-4514-4

Value distribution theory has known alternating periods of quiescence and rapid progress: the classical function-theoretic work of Nevanlinna, Ahlfors' introduction of differential-geometric methods, the work of Stoll, and the work of the Griffiths school, motivated by problems in algebraic geometry.