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Gaussian processes, function theory, and the inverse spectral problem, by H. Dym and H. P. McKean, Probability and Mathematical Statistics, vol. 31, Academic Press, New York, San Francisco, London, 1976, xi + 333 pp., $35.00.

A stationary Gaussian process is a continuous map $t \to \xi_t$ from the real line into the real L^2 space of a probability measure, P, with the following properties:

(i) $\int \xi_t \, dP = 0$ for all t;

(ii) $\int \xi_s \xi_t \, dP$ depends only on the difference $t - s$ (and so can be written as $Q(t-s)$, where Q is a continuous positive definite function on the line, known as the covariance function of the process);

(iii) every function in the linear span of the functions ξ_t is normally distributed.

By Bochner's theorem, the covariance function Q admits a representation

$$Q(t) = \int e^{itx} \, d\Delta(x),$$

where Δ is a positive measure on the line, symmetric with respect to the origin. This leads to what is called the spectral representation of the process: the map sending ξ_t to the function e^{itx} on the line extends to an isometry sending the span, in complex $L^2(P)$, of the functions ξ_t onto the space $Z = L^2(\Delta)$.

The Gaussian condition (that is, condition (iii)) enables one to give geometric interpretations to various probabilistic aspects of the process. The simplest instance is the statement that, in the L^2 span of the functions ξ_t, orthogonality is equivalent to stochastic independence. Because of the spectral representation, one can go a step further, translating probabilistic questions about the process into questions in analysis. The questions in analysis that arise usually involve the theory of Hardy spaces in the upper half-plane and the theory of entire functions of exponential type. It is to them that the book under review is devoted.

The process is called deterministic if its past determines its future. This means, in probabilistic terms, that every function ξ_t is measurable with respect to the σ-algebra generated by the family $\{\xi_s: s < 0\}$. Because the process is Gaussian, the latter reduces to the condition that every ξ_t belong to the span, in $L^2(P)$, of the family $\{\xi_s: s < 0\}$. Application of the spectral representation now shows that the process is deterministic if and only if Z is spanned by the functions e^{isx}, $s < 0$. A criterion is provided by a theorem which originated, in a slightly different context, with G. Szegö in 1920: a necessary and sufficient condition for the process to be nondeterministic is $\int_{-\infty}^{\infty}(1 + x^2)^{-1}\log \Delta'(x)\,dx > -\infty$, where Δ' is the derivative of Δ with respect to Lebesgue measure. Moreover, the preceding condition holds if and only if one can write $\Delta' = |h|^2$ where h is a function of a special kind–a so-called outer function–in the Hardy space H^2. This result is the starting point of the analysis the authors develop. For most questions it is natural to limit one's attention to the case where $d\Delta = |h|^2\,dx$ with h an outer function.

The prediction problem of Kolmogorov and Wiener is that of finding the conditional distribution of ξ_t $(t > 0)$ given the σ-algebra generated by $\{\xi_s: s \leqslant 0\}$. The spectral representation transforms this to the problem of projecting the function e^{itx} onto $Z^{-\infty 0}$, the span in Z of the functions e^{isx}, $s \leqslant 0$. The problem can be handled by a suitable application of the Fourier transformation. One applies the Fourier transformation to the function $\bar{h}e^{itx}$, multiples the transformed function by the characteristic function of $(-\infty, 0)$, and transforms back.

A related but more complicated problem is the prediction problem of M. G. Krein. In analytic terms, this is the problem of projecting the function e^{itx} $(t > T)$ onto Z^T, the span in Z of the functions e^{isx}, $|s| \leqslant T$. For Krein's problem one needs much more elaborate machinery, namely, Krein's theory of strings, to which more than half of the book under review is devoted.

A string in the sense of Krein is, basically, a positive measure m on an interval $[0, l)$ or $[0, l]$ ($l = \infty$ is allowed in the former case). Given such a string, one introduces a selfadjoint operator in the space $L^2(m)$ corresponding to the formal expression $d^2/dm\,dx$. (In the case of a string of finite length and finite mass, there is actually a one-parameter family of such operators.) Associated with the operator is a certain positive measure Δ on the line, symmetric with respect to the origin and satisfying $\int(1 + x^2)^{-1}\,d\Delta(x) < \infty$. The measure Δ is related to the spectral measure of the operator. Its precise connection with the operator is too involved to spell out here; suffice it to say that each measure Δ with the above properties corresponds to precisely one string operator. (The problem of going from Δ to the string is the "inverse spectral problem" of the book's title.) Connected with Δ is a pair of transforms, an even transform and an odd transform, analogous to the Fourier cosine and sine transforms. To deal with Krein's problem, one introduces the string associated with the spectral measure Δ of the process. The space Z^T, in case it is not all of Z, can be recognized as a space of entire functions of exponential type of the kind studied by L. de Branges. Whether a function in Z belongs to Z^T is easily expressed in terms of the images of the even and odd components of the functions under the inverses of the even and odd transforms mentioned above. The upshot is that, once the machinery has been developed, the problem of Krein can be handled in much the same way as the problem of Kolmogorov and Wiener.

The authors also apply the theory of strings to the problem of interpolation. In analytic terms, this is the problem of projecting the function e^{itx} $(|t| < T)$ onto the orthogonal complement in Z of the span of the functions e^{isx}, $|s| \geqslant T$. It receives its first full solution in the final two sections of the book. One requires a wider class of strings than originally considered by

Krein, and the technicalities are more involved, but the general features of the solutions are similar to those for Krein's problem.

The reviewer is in the uncomfortable position of not being an expert in prediction theory, the main topic of the book under review. Rather, I am someone who was brought up in Hardy spaces and developed a curiosity about how they get involved with prediction theory. For such a person the book is almost ideal. I imagine the same would be true for someone reared in probability theory who developed the complementary curiosity to mine. The book begins with three short but intense preparatory chapters which provide the needed background in function theory, Hardy spaces, and probability. The fourth chapter deals with various prediction problems, beginning with the Kolmogorov-Wiener problem mentioned above. The central theme is an effort to express in terms of the spectral measure Δ the amount of dependence between the past and the future of the process. In the two remaining chapters, Krein's theory of strings and its connection with de Branges spaces of entire functions are developed in detail and applied in the manner sketched above.

I found the comparatively informal style of the book congenial and effective. Many details of proofs are left to the reader in the form of carefully prepared exercises. The authors have clearly made an effort to write a book that will be of value to the learner. If my experience is typical, they have succeeded.

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Probability methods for approximations in stochastic control and for elliptic equations, by Harold J. Kushner, Academic Press, New York, San Francisco, London, 1977, xvii + 243 pp. $23.00.

The analysis of the transition from Markov chains to diffusions, the convergence of solutions of difference equations to corresponding ones for differential equations and related approximation problems have been studied intensively for many years and appear frequently in so many different specialized contexts that it is practically impossible today to have a comprehensive idea of what goes on in the field. Kushner's work aims directly at a specific class of approximations for optimal diffusion processes which are associated with partial differential equations (PDE). In this way he limits the material to manageable size which one can divide, roughly, into two parts.

The first one is the content of Chapters one to seven and Chapter ten and deals with background material, the theory of weak convergence of measures (without details), and the convergence of (nonoptimal) chains to diffusions. The second part, the main point of the book, is the content of Chapters eight and nine and deals with the approximation of optimal diffusions. Chapter eleven deals with a special topic, the separation theorem of stochastic control.

Let us look into part one in some detail. The beginning of the theory of approximations of Markov chains by diffusions is probably the well-known work of Khinchine [1]. The analysis here is simple and direct. It is based on