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The geometry and dynamics of magnetic monopoles, by Michael Atiyah and Nigel Hitchin. Princeton University Press, Princeton, New Jersey, 1988, 133 pp., $25.00. ISBN 0-691-08480-7

The monopoles of this book are "lump-like" solutions of a nonlinear system of hyperbolic partial differential equations. A snapshot of such a solution, at any moment in time, reveals a finite number of lumps (or solitons), localized in space. As time evolves, these solitons interact with one another in a manner determined by the equations, while maintaining their separate identities.

This whole subject of solitons is truly interdisciplinary; it has caught the interest and imagination of analysts and geometers as well as of applied mathematicians, mathematical physicists, and theoretical physicists. Sometimes the word "soliton" is reserved for the solutions of the rather special class of equations described as completely integrable: in these cases, the scattering of solitons is essentially trivial (their velocities are unaffected by the collision, and no radiation is emitted). The best-known examples of these involve just one space dimension, and include such (sometimes bizarrely-named) systems as the Korteweg-de Vries (KdV), nonlinear Schrödinger, and sine-Gordon equations. By contrast, monopoles live in three spatial dimensions, and scatter nontrivially; this book studies that scattering, in a slow-motion approximation. So, whereas for (say) the KdV equation one can write down an explicit solution representing a two-soliton collision, in this case one only has an approximate solution for small relative velocities. But the authors' point of view is that an equation such as KdV is in any event only an approximation to some more realistic physical model. As far as the physics is concerned, an approximate solution to an exact equation amounts to the same thing as an exact solution to an approximate equation.

But considerations of physics are, in any case, of only limited relevance. The main theme of the book is mathematical: the beautiful interplay between nonlinear systems and geometry.

In fact, geometry enters in a number of different ways. First of all, the monopoles occur as solutions of an SU(2) gauge theory, which means (mathematically) that we have an SU(2)-connection on a vector bundle over (four-dimensional) space-time $\mathbf{R}^4$, and the components of this connection satisfy a hyperbolic system of partial differential equations. (Actually, the system also includes three scalar functions comprising the Higgs field; this, together with the connection, is called a Yang-Mills-Higgs field in physics language.) The equations are geometrical, in the sense that they involve the curvature of the connection and the covariant derivative of the Higgs field.

Since 1974, physicists had known of one solution to the equations, representing a single static monopole. It was strongly suspected that static

k-monopole solutions should exist, for any positive integer k (there is no net force between static monopoles). That this was indeed so had been established by 1980, in particular by C. H. Taubes. The parameter space (moduli space) of all static k-monopole solutions is a $4k$-dimensional manifold M_k (for each monopole, three parameters determine its position in space, and the remaining parameter is an "internal phase" angle).

At first, this was simply an existence theorem: only for $k = 1$ was the monopole solution known explicitly. But then it was discovered how to construct (more-or-less explicitly) k-monopole solutions, by making use of Penrose's twistor theory. Here geometry enters a second time, as the complex geometry of twistors.

All this, however, was for static systems (of course, the equations are then elliptic rather than hyperbolic, which is why the solution space, for a given monopole number k, is finite-dimensional). When the monopoles move, then forces between them arise as a result of this motion, and so their trajectories need not be straight lines. The question is: how does a moving k-monopole system evolve? Solving the evolution equations could only be done numerically, and even then would require a great deal of computing power. But in 1982, N. S. Manton pointed out that geodesics on the parameter space M_k give a good approximation to the evolution of the system, as long as the relative speeds between monopoles remain small. (The metric on M_k is a natural one, coming from the expression for the kinetic energy of the field.)

So geometry makes its third appearance, in order to describe the slow-motion approximation of monopole dynamics. Most of the book concentrates on this aspect, and deals with constructing the metric of M_k, and then understanding the geodesics on M_2. It is too complicated to try to compute the metric directly, in the way that Manton introduced it. But one can argue indirectly, and deduce that the metric has certain symmetries. In particular, it has a hyperkähler structure; this property brings in another variant of twistor geometry. (So twistor theory appears in the problem in two completely independent ways.)

In the simplest nontrivial case $k = 2$, one has enough information to determine the metric completely, in terms of elliptic integrals. The geodesics on M_2 can then be found (a complete description requires some numerical integration of the geodesic equations), and interpreted, to reveal the way in which two monopoles scatter when they collide (at least, when they collide slowly).

This book is an expanded version of a series of lectures given by M. F. Atiyah at Rice Unviersity. It requires some background knowledge of algebraic and differential geometry, but not of the mathematical physics that originally inspired the problem. It should be read by any mathematician who wants to see something of the exciting connections between geometry and the nonlinear systems of mathematical physics.

R. S. WARD
UNIVERSITY OF DURHAM