not cover in detail; he discusses many applications of Bessel polynomials; calls attention to some unsolved problems about Bessel polynomials; and outlines their history in a preliminary chapter (from which I have borrowed most of the historical remarks in this review).

It is too much to hope that the appearance of this book will prevent the Bessel polynomials from being reinvented, but it will be useful to anyone who comes across them, or one of their variants, and is resourceful enough to find it. Perhaps eventually someone will organize the literature of orthogonal polynomials in inverse form, listing desirable properties and typical problems, and indicating which polynomials have the properties or help solve the problems. Until the arrival of that millennial day, treatises like this one are all we can reasonably expect, and we should be duly grateful to Grosswald for making the Bessel polynomials more accessible.

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 1, Number 5, September 1979

0002-9904/79/0000-0415/$01.50

The finite element method for elliptic problems, by Philippe G. Ciarlet, North-Holland, Amsterdam, New York, Oxford, 1978, xvii + 530 pp., $56.95.

There is a wide variety of numerical techniques, particularly for the solution of partial differential equations, that go under the heading of finite element methods. The most elementary version of these methods occurs in the context of the Poisson equation where they typically are a special case of the classical variational methods developed by Galerkin, Rayleigh, and Ritz. The latter are based on the Dirichlet Principle which asserts that the solution of the boundary value problem uniquely minimizes a quadratic functional, normally called the energy functional, over a certain class U of functions. The classical idea is to obtain approximations by minimizing the energy functional over a finite-dimensional subspace of U [1], [2]. What distinguishes finite element methods in this context is the particular choice of the finite-dimensional subspace that is used in the approximation. In particular, finite element methods are typically based on spaces of piecewise polynomial functions associated with a simplical decomposition of the region.

There is not general agreement concerning the originator of these ideas although most numerical analysts quote either Courant [3] or Synge [4], both of whom had the basic ideas concerning the elementary mechanics of the method. Later in the mid 1960s engineers independently started an intensive development of the method [5], and several successful applications to large and complicated problems that were originally thought to be intractable generated almost overnight popularity in engineering circles.

The mathematical analysis of the important question of stability and convergence started later, and in particular followed the development of Sobolev Space Theory and modern treatments of partial differential equations. The work of Lions and Magenes [6] in these latter areas was particularly influential. Since that time, however, the mathematical literature on finite elements has grown to voluminous proportions. The bibliography of the present book under review, for example, covers thirty-one pages of "selected references"!

P. G. Ciarlet has attempted a comprehensive exposition of one aspect of this subject, namely elliptic boundary value problems and variational inequalities with special emphasis on problems that arise in solid mechanics such as plate bending and shells. The author acknowledges in his preface that this leaves out many prominent areas of current interest, and the reviewer finds this a very wise decision. Indeed, what we have is a unique in-depth treatment that will most certainly be regarded as definitive for the majority of the subjects treated.

All of the problems considered in this book can be put in the following concise form. We are given a Hilbert space V and a convex subset U of V along with a quadratic functional

$$J[v] = \tfrac{1}{2}a(v, v) - F(v) \tag{1}$$

defined for v in U. We seek a u in U for which

$$J[u] = \inf_{v \in U} J[v]. \tag{2}$$

In plate bending problems, J represents the strain energy and has (essentially) the form

$$J[v] = \frac{1}{2}\int_{\Omega}(\Delta v)^2 - \int_{\Omega} fv, \tag{3}$$

where Ω is the region occupied by the plate and f is the given external force. U is the set of admissible displacements. For standard problems U is a linear space and consists of functions v with finite energy which satisfy the essential boundary conditions. If an obstacle is present, however, there is an additional constraint on admissible displacements, and in this case U is properly convex. Ciarlet does an excellent job of developing this context with several examples from elasticity.

Approximations can be obtained by introducing a finite-dimensional subspace U^h of U and minimizing J over U^h:

$$J[u^h] = \inf_{v^h \in U^h} J[v^h]. \tag{4}$$

If U^h and U are linear spaces, this is equivalent to a set of linear algebraic equations; otherwise it reduces to a finite-dimensional convex programming problem. This book contains a very detailed treatment of all the important choices of spaces U^h that used in practice. In addition, there is a detailed analysis of the stability and convergence of the error $u - u^h$ both in L_2 norms as well as sup norms. Other topics such as boundary approximation and the analysis of quadrature errors receive a systematic and careful analysis.

Another excellent feature of this book is its treatment of nonconforming methods. These arise for example in plate bending problems where energy has a form similar to (3). The standard approximation thus requires that admissible displacements have square integrable second derivatives, and this can cause practical problems for piecewise polynomial functions in many cases. In the nonconforming approach one, in essence, ignores this continuity constraint thereby obtaining a space U^h which is not a subset of U. This will of course not work in general, and the approach required a careful and systematic analysis. Ciarlet has done exactly this, and his treatment of this subject is unequaled.

The only disappointing chapter is the one on mixed methods. The latter are based on variational principles where solutions emerge as stationary points rather than minima as in (2). The author's error analysis uses a generalized Lax-Milgram approach. Invariably continuity requirements of the latter lead to unusual norms that obscure important structural properties of the error (e.g., optimality or suboptimality of the rate of convergence in L_2). This, however, should not be regarded as a major defect of this book since the chapter is short and since the author in the preface acknowledges that he did not wish to stress mixed methods.

The following quotation from P. R. Halmos precedes Chapter I. "A mathematician's nightmare is a sequence n_ϵ that tends to 0 as ϵ becomes infinite." Ciarlet has heeded the message here for his choice of notation is excellent and apparently carefully planned to aid the reader through the more technical material. There are a few misprints but they are minor and do not detract. Finally the notes concluding each chapter are balanced and very informative.

In short, this is an excellent, well written, and, for the most part, carefully planned book that deserves study by anyone working in the general area of finite element methods.

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 1, Number 5, September 1979

0002-9904/79/0000-0416/$01.50

Non-Archimedean functional analysis, by A. C. M. Van Rooij, Pure and Applied Math., vol. 51, Marcel Dekker, New York, 1978, ix + 404 pp., $29.50.

There are fields that are complete, locally compact, have a nontrivial