4. L. Boutet de Monvel and P. Kree, *Pseudo-differential operators and Gevrey classes*, Ann. Inst. Fourier (Grenoble) **17** (1967), 295–323.

5. I. M. Gel'fand and G. E. Shilov, *Generalized functions*. Vol. 1, *Properties and operations*, Academic Press, 1964 (reprinted by Harcourt Brace Jovanovich, 1977).

6. V. Guillemin, M. Kashiwara and T. Kawai, *Seminar on microlocal analysis*, Ann. of Math. Studies, no. 93, Princeton Univ. Press, 1980.

7. V. Guillemin and S. Sternberg, *Geometric asymptotics*, Math. Surveys, no. 14, Amer. Math. Soc., Providence, R. I., 1977.

8. M. Kashiwara, *On the holonomic systems of differential equations*. II, Invent. Math. **49** (1978), 121–135.

9. ——, *Systems of microdifferential equations*, Progress in Math., vol. 34, Birkhäuser, 1983.

10. ——, *B-functions and holonomic systems*, Invent. Math. **38** (1976), 33–58.

11. M. Kashiwara and T. Kawai, *On the holonomic systems of linear differential equations* (*systems with regular singularities*). III, Publ. R.I.M.S. Kyoto Univ. **17** (1981), 813–979.

12. Le Dung Trang and Z. Mebkhout, *Introduction to linear differential equations*, Singularities, Part 2, Proc. Sympos. Pure Math., vol. 40, Amer. Math. Soc., Providence, R. I., 1983, pp. 31–63.

13. B. Malgrange, *L'involutivité des variétés caractéristiques des systemes différentiels et microdifférentiels*, Séminaire Bourbaki (1977/78) Exp. No. 522; Lecture Notes in Math., vol. 710, Springer-Verlag, Berlin and New York, 1979, pp. 277–289.

14. ——, *Polynômes de Bernstein-Sato et cohomologie évanescente*, Astérisque **101-102** (1983).

15. A. Martineau, *Les hyperfonctions de M. Sato*, Séminaire Bourbaki **214** (1961–62), mimeographed, I.H.P., Paris.

16. Z. Mebkhout, *Une autre équivalence de catégories*, Compositio Math. **51** (1984), 63–88.

17. T. Oda, *Introduction to algebraic analysis of complex manifolds*, Adv. Study Pure Math., no. 1, North-Holland, 1983.

18. M. Sato, M. Kashiwara and T. Kawai, *Microfunctions and pseudodifferential equations*, Lecture Notes in Math., vol. 287, Springer-Verlag, Berlin and New York, 1973, pp. 264–529.

19. P. Schapira, *Microdifferential systems in the complex domain*, Grundlehren Math. Wiss., Springer-Verlag, Berlin and New York, 1985.

20. T. Yano, *b-functions and exponents of hypersurface isolated singularities*, Singularities, Part 2, Proc. Sympos. Pure Math., vol. 40, Amer. Math. Soc., Providence, R. I., 1983, pp. 641–652.

J. L. BRYLINSKY

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 18, Number 1, January 1988

0273-0979/88 $1.00 + $.25 per page

Recursively enumerable sets and degrees: A study of computable functions and computably generated sets, by Robert I. Soare. Perspectives in Mathematical Logic, Springer-Verlag, Berlin, Heidelberg, New York, 1987, xviii + 437 pp., $35.00. ISBN 3-540-15299-7

One of the tantalizing aspects of twentieth century mathematical logic is the juxtaposition of the highly theoretical with the very practical. Logical investigations, aimed at giving precise mathematical definitions of "theorem", "proof", and "mathematical truth", led naturally to a study of computable processes. As a result, it is now generally recognized that recursive function theory provides the theoretical foundation for computer science.

The book under review is a contribution to classical recursive function theory. Thus it might be helpful to describe briefly some of the basic concepts of the subject.

Let us begin by recalling Turing's pioneering work on computability. In 1936, Turing gave a definition of a digital computer of a most general type, the Turing machine [**17**]. This led naturally to a precise definition of "computable function from nonnegative integers into nonnegative integers". As is well known, Turing went further and demonstrated the existence of a universal Turing machine, a computer which could simulate all other such computers. Essentially, a computer of today differs from a Turing machine by the fact that it has limited storage capacity, whereas the Turing machine does not.

Turing's work was not the only approach to computability. At about the same time, several other mathematicians—e.g. Church, Herbrand/Gödel, Kleene, Post—also gave definitions of the concept [**2**, **4**, **11**]. Post's approach turned out to have practical significance both in computer science and in linguistics. The Post Production, a rule for manipulating strings of symbols, found its way into programming languages, the work of Noam Chomsky on the grammatical analysis of language, and other topics.

It is worth noting that the set of functions obtained from each of these different approaches is the same. (The functions so defined are called *recursive* functions.) The equivalence of these definitions provides evidence that the recursive function does, in fact, capture the intuitive notion of "computable function from nonnegative integers into nonnegative integers". There is, of course, additional evidence, which we shall not be concerned with here.

In the intervening years, the theory of recursive functions has been applied to other branches of mathematics. For example, in number theory and algebra, we have Matijasevič's solution of Hilbert's tenth problem, and Novikov's theorem on the unsolvability of the word problem for finitely presented groups [**8**, **10**]. There are many other problems which are not so well known. Some, for example, occur in analysis. By suitable coding one can define "recursive real", "recursive function of a real variable", and beyond. One can ask and answer such questions as: Is the unique solution of the wave equation with computable initial data computable? (Answer: No, not necessarily [**13**, **14**].)

The book under review is not concerned with applications. Rather it is concerned with a detailed classification of "recursively enumerable set", a concept closely associated with that of a "recursive function". Intuitively, a set of nonnegative integers is "recursively enumerable" if its members can be effectively listed. Imagine a Turing Machine programmed to compute the values of a recursive function a: $a(0), a(1), a(2), \ldots$. The set of these values is a recursively enumerable set. Thus a set A is *recursively enumerable* if either $A = \varnothing$ or A is the range of a recursive function.

The classification comes about in the following way. There are recursively enumerable sets A which have the property that the complement A^c is also recursively enumerable. (This fact provides an effective procedure for determining whether or not $n \in A$. Merely enumerate the elements of the set A and the elements of A^c until n turns up.) Such sets are called *recursive*. However, many recursively enumerable sets do not have this property. In fact, it turns out that, by suitable identification, the recursively enumerable sets form an "upper semilattice" whose bottom element consists of the recursive sets. The elements of the semilattice are called degrees. The book is concerned with the structure of this semilattice.

The *impetus* for such a classification comes from E. L. Post's famous paper of 1944 [**12**]. Let us consider any axiomatizable mathematical theory—e.g., group theory. By suitable coding, each statement of such a theory can be associated with a nonnegative integer. For a decidable theory, the set of theorems is a recursive set. For an undecidable theory the set of theorems is recursively enumerable, but *not* recursive. Now the proof of Gödel's famous incompleteness theorem provides a method for showing that number theory is undecidable. In Post's time, there were only two known methods for showing that a given axiomatizable theory is undecidable. They were: (1) use the method of Gödel directly or (2) "embed" a theory which was known to be undecidable into the given theory. This produced only one kind of recursively enumerable, nonrecursive set. Roughly, the question (often called Post's problem) was: Are there others? For a long time Post's problem was open. It was finally solved independently by R. M. Friedberg and A. A. Muchnik in 1956 and 1957 [**3**, **9**]. The method which Friedberg and Muchnik employed became known as the "priority argument".

Over the years, the priority argument has become an important tool in the study of recursively enumerable sets. It seems pointless, in this expository account written for nonlogicians, to describe it in any detail. Let us merely say that the priority argument is an intricate combinatorial technique, usually containing an infinite number of requirements $R_0, R_1, R_2, \ldots$ which must be satisfied. In general, later elements of the sequence have lower "priority" than earlier ones.

We turn now to a discussion of Soare's book. The work is divided into four parts. Parts A and B are introductory, and contain the basic material included in standard texts on recursive function theory (cf. Kleene [**5**] and Rogers [**15**]). Part A discusses the elementary facts of recursion theory—e.g. recursive and partial recursive functions, recursively enumerable sets, the Normal Form Theorem, the S^m_n-theorem, and the Recursion Theorem. Part B is motivated by the work of Post which was discussed earlier. It contains the formulation and solution of Post's problem, thus giving an introduction to the "priority argument".

The main thrust of the book, however, does not lie in its account of introductory material. Rather it lies in the careful, systematic presentation of the priority argument as it applies to recursively enumerable sets. Beginning with the last chapter of Part B, and continuing through Parts C and D, we progress from "finite injury priority arguments", through "infinite injury priority arguments", up to the latest intricacies in the method. Part C opens with a discussion of the infinite injury priority argument and presents many well-known results based on it (e.g., the Density Theorem). Part D is concerned with advanced topics and areas of current research. Among the topics considered are: promptly simple sets and degrees, the tree method and $0'''$-priority arguments, and automorphisms of the lattice of recursively enumerable sets.

The technical complications of the priority argument lead naturally to an open problem. First, a bit of background. For about thirty years, the only known solution to Post's problem was via a priority argument. Not long ago, A. Kučera obtained a proof which does not use this method at all [**6**]. The question, of course, is: to what extent can the priority argument be dispensed with in the classification of recursively enumerable sets?

We turn now to a different topic. Since recursion theory provides the theoretical foundation for computer science, it seems natural to discuss briefly the relation between these two disciplines and the contents of this book. In general, the recursion-theoretic concepts used in computer science are not new: they have been known since the late 1950s. This is obviously true of the work of Post and Turing referred to earlier. It is equally true of very recent work. Consider, for example, the $P \stackrel{?}{=} NP$ problem, probably the most famous open problem in theoretical computer science today. The formulation of this problem borrows heavily from pre-1960 recursion theory. More specifically, the concepts of "reducibility" and of a "complete problem" are polynomial time analogs of the notions of "reducibility" and a "complete set"—two notions which are discussed in Part A of this book. Going further, the polynomial time hierarchy [**16**] is an analog of the arithmetic hierarchy, also discussed in Part A. For this reason the introductory part of the book may be useful to those interested in theoretical computer science. The account given here is well written and can serve as a reference. However, since some of the proofs are merely sketched, the reader may find it helpful to consult either experts or other texts.

There is another aspect to the matter of the interconnection between theoretical computer science and recursion theory which ought to be mentioned. Although the formulation of problems in theoretical computer science owes much to recursion-theoretic concepts, it is not at all clear that recursion-theoretic techniques (in particular, the priority argument)—in their present form—will be useful in the solution of such problems. This can be seen by considering the work of Baker, Gill, and Solovay [**1**]. Indeed one of the well-known facts of recursive function theory is that the relativized theorem—i.e. the theorem relativized to an oracle—follows immediately once the theorem itself is proved. This certainly does not hold for the $P \stackrel{?}{=} NP$ problem. For although the $P \stackrel{?}{=} NP$ problem is open, Baker, Gill, and Solovay have proved that there are oracles A and B such that $P^A = NP^A$, but $P^B \neq NP^B$. Perhaps computer scientists and recursion theorists, working together, can find more appropriate techniques.

In the opinion of the reviewer, Soare's book is essentially a specialized monograph on the priority argument. The bibliography, however, has broader scope, and lists references on topics not discussed in the text. The book itself differs from the work of Manuel Lerman in that Lerman is concerned mainly with the priority argument as applied to arbitrary sets of nonnegative integers [**7**]. By contrast, Soare's book deals mainly with recursively enumerable sets.

Soare's monograph is well written, and should appeal to those persons with a knowledge of basic recursion theory who wish to study the priority argument, as it applies to recursively enumerable sets. For such persons it is an excellent source of information.

BIBLIOGRAPHY

1. T. Baker, J. Gill, and R. Solovay, *Relativizations of the* $P \stackrel{?}{=} NP$ *question*, Siam J. Comput. **4** (1975), 431–442.

2. A. Church, *A set of postulates for the foundation of logic* (second paper), Ann. of Math. (2) **34** (1933), 839–864.

3. R. M. Friedberg, *Two recursively enumerable sets of incomparable degrees of unsolvability*, Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 236–238.

4. S. C. Kleene, *A theory of positive integers in formal logic*, Amer. J. Math. **57** (1935), 153–173; 219–244.

5. ——, *Introduction to metamathematics*, van Nostrand, New York, 1952.

6. A. Kučera, *An alternative, priority-free, solution to Post's problem*, Lecture Notes in Computer Science (J. Gruska, B. Rovan and J. Wiederman, eds.), No. 233, Springer-Verlag, Berlin and New York, 1986, pp. 493–500.

7. M. Lerman, *Degrees of unsolvability*, Perspectives in Mathematical Logic, Omega Series, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983.

8. Ju. V. Matijasevič, *Enumerable sets are diophantine*, Dokl. Akad. Nauk. SSSR, N.S. **191** (1970), 279–282; English transl. in Soviet Math. Dokl. **11** (1970), 354–357.

9. A. A. Muchnik, *On the unsolvability of the problem of reducibility in the theory of algorithms*, Dokl. Akad. Nauk SSSR N.S. **108** (1956), 194–197. (Russian)

10. P. S. Novikov, *On the algorithmic unsolvability of the word problem in group theory*, Trudy Mat. Inst. Steklov. **44** (1955). (Russian)

11. E. L. Post, *Formal reductions of the general combinatorial decision problem*, Amer. J. Math. **65** (1943), 197–215.

12. ——, *Recursively enumerable sets of positive integers and their decision problems*, Bull. Amer. Math. Soc. **50** (1944), 284–316.

13. M. B. Pour-El and I. Richards, *Noncomputability in analysis and physics: A complete determination of the class of noncomputable linear operators*, Adv. in Math. **48** (1983), 44–74.

14. ——, *The eigenvalues of an effectively determined self-adjoint operator are computable, but the sequence of eigenvalues is not*, Adv. in Math. **63** (1987), 1–41.

15. H. Rogers, Jr., *Theory of recursive functions and effective computability*, McGraw-Hill, New York, 1967.

16. L. J. Stockmeyer, *The polynomial-time hierarchy*, Theoret. Comput. Sci. **3** (1977), 1–22.

17. A. M. Turing, *On computable numbers, with an application to the Entscheidungsproblem*, Proc. London Math. Soc. **42** (1936), 230–265; *A correction*, ibid. **43** (1937), 544–546.

MARIAN BOYKAN POUR-EL

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 18, Number 1, January 1988

0273-0979/88 $1.00 + $.25 per page

Modern geometry—methods and applications. Part II, *The geometry and topology of manifolds*, by B. A. Dubrovin, A. T. Fomenko, and S. P. Novikov, Graduate Texts in Mathematics, vol. 104, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1985, xv + 430 pp., $54.00. ISBN 0-387-96162-3

This is the second volume of an excellent series of books on modern aspects of geometry. A review of the English translation of Part I of the series, also published by Springer-Verlag, appeared in this Bulletin, vol. 13 (1985), 62–65. In this volume the modern conceptions and ramifications of manifolds are treated. Perhaps no other concept in modern mathematics has been so