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SUNY AT STONY BROOK

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Infinite-dimensional dynamical systems in mechanics and physics, by Roger Temam. Applied Mathematical Sciences, vol. 68, Springer-Verlag, New York, Berlin, Heidelberg, 1988, xvi + 500 pp., \$64.00. ISBN 0-387-96638-2

The Navier-Stokes equations define an infinite dimensional dynamical system describing the flow of a viscous fluid. Nonetheless, one would like to study the dynamical behavior of fluid flows using techniques and insight gained from studies of finite dimensional dynamical systems. This is a common situation in the theory of partial differential equations; indeed it is implicit in most numerical computations of the asymptotic behavior (in time) of solutions to partial differential equations. This book gives a comprehensive account of a general set of techniques that have been developed to rigorously justify the use of finite dimensional techniques for the study of the dynamics of a large variety of nonlinear partial differential equations. The relevant PDE's include such examples as the Navier-Stokes equations in two dimensions, reaction-diffusion equations, nonlinear wave equations such as the sine-Gordon and pattern formation equations such as the Kuramoto-Sivashinsky equation. Although there is a general point of view that applies to all of the examples, many of the equations have their individual quirks and personalities. Thus separate discussion of each example is appropriate and included in the book.

In the process of reducing an infinite dimensional system to a finite dimensional one, the first question that arises is the classical one of existence and uniqueness of solutions with specified initial conditions. One wants to show that the long time behavior of solutions is characterized by a finite dimensional subset of the phase space, and this requires that the solutions exist for all (forward) time. The usual method to obtain results concerning the existence of solutions for all time is to prove that there is a bounded region of the phase space that is “absorbing.” This means that solutions never leave this bounded region. Note that even finite dimensional vector fields may fail to have absorbing regions and may have solutions that blow up in finite time. Perhaps the simplest example of this phenomenon is the equation $\dot{x} = x^2$. Thus each example requires consideration of why the norm of solutions decreases when this norm is large. These considerations are often based upon a physical understanding of the origins of the equations. For example, an important class of equations come from conservative physical systems to which forcing and damping has been added. The damping is typically in the form of a frictional term that decreases the norm of a solution at a rate proportional to its magnitude while the forcing is bounded. In these situations, the existence of an absorbing region is readily deduced from the form of the equation.

The second step in reducing an infinite dimensional system to a finite dimensional one involves compactness. In the language used in the book, the *universal attractor* is the intersection of the images of the absorbing set as it flows for longer and longer times. When the relevant operators obey a compactness property, general topological considerations lead to compactness of the universal attractor. The compactness results are often closely tied to the proof that the problems being studied are well posed. Intuitively, they can be understood sometimes in the following terms. If the phase space of the dynamical system is a Hilbert space and solutions are expanded in an orthogonal basis of this Hilbert space, then one would like to know that the higher modes in this basis decay strongly. For example, in a one dimensional reaction diffusion system the phase space can be decomposed into Fourier modes. Diffusion then leads to rapid decay of the higher Fourier modes in a manner analogous to the heat equation.

The third step in the reduction to finite dimensions addresses the question of estimating Hausdorff or fractal dimension of the universal attractor. The quantitative estimates are based upon linearizing the system along its trajectories and computing a Lyapunov spectrum. Under very mild hypotheses, most trajectories have a Lyapunov spectrum that yields the asymptotic rates of exponential growth or decay of deviations from the reference trajectory. A rough estimate of part of the Lyapunov spectrum can be obtained from looking at the growth rates of d -dimensional volumes in the linearized flow. If there is a d for which all d -dimensional volumes are decreased along the flow, then d is an upper bound for the dimension of the universal attractor. In situations like that of the Navier-Stokes equations, these estimates can be obtained by finding a length scale for which the effect of viscosity in the equations has a smoothing effect on the fluid velocity that overwhelms the effects of the forcing. The resulting

estimates of the dimension are crude but they do give some idea of how much of the phase space is occupied by the attractors of a fluid flow with a given Reynolds number.

The fourth and final step in the reduction to finite dimensional systems is less widely successful than the theory described thus far. One can hope that not only will the attractors of an infinite dimensional system be finite but that there will be a smooth finite dimensional subsystem that is invariant under the flow and contains the universal attractor. Such a subsystem is called an *inertial manifold*. The existence of inertial manifolds is a more delicate matter than the existence of universal attractors. One can see this already in the theory of invariant manifolds for finite dimensional systems. The question that one asks is when a smooth invariant submanifold in a dynamical system will persist under perturbation. General conditions for this to happen require that one know that separation of trajectories within the invariant manifold are less extreme than in the normal directions. Temam presents these conditions in terms of invariant cone fields that describe how tangent vectors are drawn into a neighborhood of the tangent space of the inertial manifold that one seeks. Within the infinite dimensional setting of Temam, the construction of these cone fields is usually based upon the existence of spectral gaps for the linearization of the partial differential equations. The existence of a cone field is closely related to normal hyperbolicity conditions: an attracting invariant manifold persistent under perturbations must have more extreme Lyapunov exponents in its normal directions than in its tangential directions. If the partial differential equations being studied have large gaps in their spectra, then these can be used to look for invariant manifolds that lie close to the linear space spanned by the modes whose eigenvalues lie to the right of a gap in the complex plane. There are many examples of such spectral gaps. For a reaction-diffusion equation in one dimension, the eigenvalues of the Laplacian give eigenvalues that grow in magnitude like n^2 and this leads to the existence of the appropriate gap conditions.

The book provides an up to date and comprehensive treatment of these topics. The demands on the reader are great, however. Background discussions are cursory and references to the literature are simply pointers to standard sources. For example, at the end of a single introductory paragraph defining Hausdorff dimension, "the reader is referred to H. Federer (Geometric Measure Theory) for the properties of Hausdorff measure and dimension." There is more discussion of Sobolev spaces and inequalities, but the treatment is not self-contained. Throughout, the style is to incorporate the minimum amount of supporting material and to refer the reader to original papers for details of proofs. There is little actual discussion of the dynamical behavior found in the universal attractors and invariant manifolds of the different examples that are discussed repeatedly throughout the book. The emphasis is primarily upon the process of proving the existence of attractors and invariant manifolds for as broad a class of systems as possible.

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