RHAPSODY IN FRACTIONAL

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Abstract

This paper studies several topics related with the concept of "fractional" that are not directly related with Fractional Calculus, but can help the reader in pursuit new research directions. We introduce the concept of non-integer positional number systems, fractional sums, fractional powers of a square matrix, tolerant computing and FracSets, negative probabilities, fractional delay discrete-time linear systems, and fractional Fourier transform.

Key Words and Phrases

Positional number systems, fractional sums, matrix power, matrix root, tolerant computing, negative probability, frac- tional delay, difference equations, fractional Fourier transform

1. Introduction

Fractional Calculus has been receiving a considerable attention during the last years. In fact, the concepts of "fractional" embedded in the integro-differential operator allow a remarkable and fruitful generalization of the operators of classical Calculus. The success of this "new" tool in applied sciences somehow outshines other possible mathematical generalizations involving the concept of "fractional". The *leitmotif* of this paper is to highlight several topics that may be useful for researchers, not only in the scope of each area, but also as possible avenues for future progress.

Bearing these ideas in mind, the manuscript is organized as follows. Section **2** focuses the concept of non-integer positional number systems. Section **3** studies fractional sums. Section **4** discusses the fractional powers of a square matrix. Section **5** draws attention to tolerant computing and FracSets. Section **6** introduces the concept of negative probabilities. Section **7** addresses the case of fractional delay discrete-time linear systems. Finally, section **8** analyzes the fundamentals of the fractional Fourier transform.

2. Non-integer Positional Number Systems

A positional number system (PNS) is a method of representing numbers where the same symbol, or digit, can be associated to different orders of magnitude (i.e., it can assume different "weighs"). The Hindu-Arabic *base* 10 PNS is nowadays the most widely used system. In a PNS the base, or radix, usually corresponds to the number of unique symbols (including the zero digit) that are adopted to represent the numbers. For example, in *base* 10 we use the symbols $\{0, 1, 2, \ldots, 9\}$. To represent fractions and the numeric expansions of real numbers, the PNS notation is extended by the use of a radix decimal point. The first known PNS was the Babylonian *base* 60, which is still used for representing time and angles. The *base* 2 system was introduced with the advent of computers and machine-based calculations, while other earlier PNS, as *base* 20 or *base* 12, have nowadays small relevance.

All mentioned PNS have a common characteristic, meaning that they use a positive integer base, b. In general, numbers in base b are expressed as:

$$(a_n a_{n-1} \ldots a_1 a_0 . a_{-1} a_{-2} a_{-3} \ldots)_b = \sum_{k=0}^{n} a_k b^k + \sum_{k=-\infty}^{-1} a_k b^k, \qquad (2.1)$$

where b^k are the weights of the digits and a_k are non-negative integers less than b. The position k is the logarithm of the corresponding weight, which is given by $k = \log_b b^k$.

Positive integer base PNS have been commonly used, but other bases are possible for representing numbers, namely negative integer [27, 40], improper fractional [34], irrational [9, 3, 61], transcendental [27] and complex bases. Negative integer bases have the advantage that no minus sign is needed to represent negative numbers. Negative bases were first considered by V. Grünwald in 1885 [25] and later rediscovered by A.J. Kempner in 1936 [34] and Z. Pawlak & A. Wakulicz in 1959 [52]. Improper fractional bases were first addressed by A.J. Kempner in 1936 [34]. In this PNS most integer numbers have an infinite representation. An irrational base PNS

was proposed by G. Bergman in 1957 [9], known as the *τ− system*, and based on the "golden ratio". A generalization was proposed by A. Stakhov [61] by means of the "golden" *p− proportions*. In transcendental bases, integers greater than the base have infinite digits in its representation, while it has been shown that base *e* (Napier's constant) is theoretically the most efficient base [27]. Different complex bases for PNS were proposed by D. Knuth in 1955 [36], S. Khmelnik in 1964 and W.F. Penney in 1965 [55].

Similarly to expression (2.1), in a non-integer base PNS, denoting the base by $\beta > 1$, we have:

$$(a_n a_{n-1} \ldots a_1 a_0 . a_{-1} a_{-2} \ldots a_{-m})_\beta = \sum_{k=0}^{n} a_k \beta^k + \sum_{k=-m}^{-1} a_k \beta^k, \qquad (2.2)$$

where β^k are the weights of the digits and a_k are non-negative integers less than β. It is worth noting that equation (2.2) is a *β − expansion* [59, 51] and every real number, *x*, has at least one (possibly infinite) *β−expansion*.

Usually, a greedy algorithm is used to choose the canonical *β−expansion* of *x* [24]. First we denote by $\lfloor x \rfloor$ the floor operator (i.e., the greatest integer less than or equal to *x*) and by $\{x\} = x - \lfloor x \rfloor$ the fractional part of *x*. Second, as it exists an integer *k* such that $\beta^k \leq \beta < \beta^{k+1}$, we make $a_k = \left\lfloor \frac{x}{\beta^k} \right\rfloor$ and $r_k = \left\{ \frac{x}{\beta^k} \right\}$. Third, for $k - 1 \geq j > -\infty$, we choose $a_j = \lfloor \beta r_{j+1} \rfloor$ and $r_j = \{\beta r_{j+1}\}$. This means that the canonical *β−expansion* of *x* is obtained starting by choosing the largest a_k such that $\beta^k a_k \leq x$, then choosing the largest a_{k-1} such that $\beta^k a_k + \beta^{k-1} a_{k-1} \leq x$, and adopting the same scheme for the remaining indices.

In the sequel we describe in more detail the Bergman's *τ − system* [9, 61]. This PNS uses the irrational base $\tau = \frac{1+\sqrt{5}}{2}$ which is known as 'golden ratio", "golden proportion" or "golden mean". A given real number, *x*, is represented by:

$$x = \sum_i a_i \tau^i, \qquad (2.3)$$

where $a_i \in \{0, 1\}$ is the i^{th} binary digit ($i = 0, \pm 1, \pm 2, \pm 3, \ldots$) and *τ* represents the base or radix of the PNS.

The "golden ratio" is the positive root of the algebraic equation:

$$x^2 = x + 1 \qquad (2.4)$$

which is known as "golden section problem" and from which the following identity results:

$$\tau^n = \tau^{n-1} + \tau^{n-2} = \tau \times \tau^{n-1}. \qquad (2.5)$$

Using expression (2.3) we have the "golden" representation of number x. This means that x is expressed in a binary code consisting of two parts separated by a decimal point. The first part $a_n a_{n-1} \ldots a_1 a_0$ corresponds to the weights with positive powers τ^n, τ^{n-1}, ..., τ^1, τ^0 and the second part $a_{-1} a_{-2} \ldots a_{-m}$ corresponds to the weights with negative powers τ^{-1}, τ^{-2}, ..., τ^{-m}. The weights τ^i ($i = 0, \pm 1, \pm 2, \pm 3, \ldots$) are given by (2.5).

The $\tau -$ *system* has the following properties [61]:

(1) Certain irrational numbers (e.g., the powers of the golden ratio τ^i and their sums) can be represented by a finite numbers of digits, which is not possible in classical number systems (e.g., decimal, binary);
(2) The "golden" representations of natural numbers are in fact fractional numbers, consisting of a finite number of digits;
(3) All non-zero real numbers have various "golden" representations, which can be obtained from another by means of two transformations: "devolution" and "convolution". These transformations are based on equation (2.5). The "devolution" applies to any three neighboring digits of the initial "golden" representation and corresponds to the transformation $100 \rightarrow 011$. The "convolution" is the back transformation and is given by $011 \rightarrow 100$.

If we perform all possible "devolution" operations on a given "golden" representation of a number, x, then we get the "maximal form" of the number. On the contrary, performing all possible "convolutions" we obtain the "minimal form" representation. The "minimal" and "maximal" forms are the extreme "golden" representations of x. In the "minimal form" there are no contiguous 1's and in the "maximal" form there are no contiguous 0's in the binary code.

Let now consider some important properties of natural numbers [61]. If N is a natural number, then in the τ *_system* we can write the τ *_code* of N:

$$N = \sum_i a_i \tau^i. \tag{2.6}$$

Applying Binet's formula [67]:

$$\tau^i = \frac{L_i + F_i\sqrt{5}}{2}, \; i = 0, \pm 1, \pm 2, \pm 3, \ldots, \tag{2.7}$$

where, F_i and L_i represent the Fibonacci and Lucas numbers, respectively, we get:

$$2N = \sum_i a_i L_i + \sqrt{5} \sum_i a_i F_i. \tag{2.8}$$

Equation (2.8) shows that the natural even number $2N$ is given by the sum of an integer and the product of other integer by the irrational number $\sqrt{5}$. On the other hand, it is worth noting that the identity (2.8) is true for every natural number, N, if the following condition is met:

$$\sum_i a_i F_i = 0, \tag{2.9}$$

taking into account equation (2.9) we can observe that the sum of Lucas numbers in (2.8) is always even.

Properties of natural numbers [61]:

(1)

(2) $Z-$ *property* – for any natural number, N, expressed using the $\tau-$ *code*, if we replace all powers of the golden ratio τ^i (i = 0, ±1, ±2, ±3, … with the corresponding Fibonacci numbers F_i, then the sum obtained equals to 0;

(3) $D-$ *property* – for any natural number, N, expressed using the $\tau-$ *code*, if we replace all powers of the golden ratio τ^i (i = 0, ±1, ±2, ±3, … with the corresponding Lucas numbers L_i, then the sum obtained is an even number equal to $2N$.

Taking into account the $Z-$ *property*, equation (2.8) yields:

$$2N = \sum_i a_i L_i + \sum_i a_i F_i, \qquad (2.10)$$

$$N = \sum_i a_i \frac{L_i + F_i}{2} = \sum_i a_i F_{i+1}, \qquad (2.11)$$

where $F_{i+1} = \frac{L_i + F_i}{2}$. Equation (2.11) represents the $F-$ *code* of N. The binary digits in the $\tau-$ *code* and $F-$ *code* coincide, which means that the $F-$ *code* can be obtained from the $\tau-$ *code* by substituting the golden ratio powers τ^i for the Fibonacci numbers F_{i+1} ($i = 0, \pm1, \pm2, \pm3, \ldots$).

If we rewrite equations (2.10) and (2.11), we can obtain:

$$N = \sum_i a_i F_{i+1} + 2\sum_i a_i F_i = \sum_i a_i L_{i+1}, \qquad (2.12)$$

this means that we can also obtain a $L-$ *code* by substituting the golden ratio powers for the Lucas numbers L_{i+1}.

About Bergman's mathematical discovery, Alexey Stakhov [62] said that "... can be considered as the major mathematical discovery in the field of number systems (following the Babylonian discovery of the positional principle of number representation and also decimal and binary systems)".

3. Fractional Sums

Everybody knows the meaning of the expression $\sum_{k=1}^{n} f(k)$, where n is a positive integer or ∞. But what meaning can we assign to $\sum_{\nu=1}^{x} f(\nu)$ where $x \in \mathbb{R}$?

According to Müller and Schleicher [44] such theme dates back to Euler and Ramanujan. In fact Euler presented the first example: $\sum_{\nu=1}^{-1/2} \frac{1}{\nu} = -2\ln 2$. However the formulation and solution of the problem was done by Müller and Schleicher. Among several examples they found the interesting case: $\sum_{\nu=1}^{x} \ln \nu = \ln \Gamma(x+1)$.

We are going to do a brief description of the theory involved. We will follow closely the presentation in [44] based on 6 axioms.

3.1. The axioms

Assume that we are working in the context of complex functions defined in $\mathbb{C}$.

(1) Continued Summation

$$\sum_{\nu=x}^{y} f(\nu) + \sum_{\nu=y+1}^{z} f(\nu) = \sum_{\nu=x}^{z} f(\nu)$$

This is exactly what happens with the normal integer order summation.

(2) Translation Invariance

$$\sum_{\nu=x+z}^{y+z} f(\nu) = \sum_{\nu=x}^{y} f(\nu+z)$$

(3) Linearity

For arbitrary constants $\alpha, \beta \in \mathbb{C}$,

$$\sum_{\nu=x}^{y} [\alpha f(\nu) + \beta g(\nu)] = \alpha \sum_{\nu=x}^{y} f(\nu) + \beta \sum_{\nu=x}^{y} g(\nu)$$

(4) Consistency with Classical Definition

$$\sum_{\nu=1}^{1} f(\nu) = f(1)$$

(5) Monomials

For every $n \in \mathbb{N}$, the mapping

$$z \mapsto \sum_{\nu=1}^{z} \nu^n$$

is holomorphic on $\mathbb{C}$. This axiom covers the classic cases: sums of integer powers.

(6) Right Shift Continuity

If $\lim_{n \to \infty} f(z+n) = 0$ pointwise for every $z \in \mathbb{C}$, then

$$\lim_{n \to \infty} \sum_{\nu=x}^{y} f(\nu + n) = 0.$$

In [44] a more general format of this theorem is presented by considering the uniform approximation of $f(x)$ by a sequence of polynomials.

In some applications it may be interesting to consider the **Left Shift Contiuity** that can be stated as:

If $\lim_{n \to \infty} f(z - n) = 0$ pointwise for every $z \in \mathbb{C}$, then

$$\lim_{n \to \infty} \sum_{\nu=x}^{y} f(\nu - n) = 0.$$

Consider the case $\sum_1^n K = nK$ and assume that n is odd such that $n/2$ is fractional. We have $\sum_1^{n/2} K + \sum_{n/2+1}^{n} K = nK$. By the axiom (2) the second summation is equal to the first. We have then $2\sum_1^{n/2} K = nK$ and $\sum_1^{n/2} K = nK/2$. In particular if $n = 1$, it gives $\sum_1^{1/2} K = K/2$.

To go ahead consider a polynomial $P(z)$ defined in $\mathbb{C}$ such that $P(0) = 0$ and its differenced polynomial is $p(z) = P(z) - P(z-1)$. Then

$$\sum_{\nu=x}^{y} p(\nu) = P(y) - P(x-1). \tag{3.1}$$

It is not difficult to see that this definition is conform with the above axioms [44]. In the computation of the fractional sums, we must be aware of the computational direction as we will see. This means that we must modify the summation symbol to include the direction. In [42, 43, 44] an arrow overstrikes the summation symbol. Any way in most applications the direction of summation is clear. So, we will avoid such change of notation.

3.2. The summation formula

In agreement with the above subsection we are going to obtain a general summation formula using the axioms. Let x, y be any complex numbers and n a positive integer; also assume that $f(x)$ is a function that verifies Axiom (6). We have successively:

- The Axiom (1) gives

$$\sum_{\nu=x}^{y+n} f(\nu) = \sum_{\nu=x}^{y} f(\nu) + \sum_{\nu=y+1}^{y+n} f(\nu),$$

and also

$$\sum_{\nu=x}^{y+n} f(\nu) = \sum_{\nu=x}^{x+n-1} f(\nu) + \sum_{\nu=x}^{y+n} f(\nu).$$

Then

$$\sum_{\nu=x}^{y} f(\nu) + \sum_{\nu=y+1}^{y+n} f(\nu) = \sum_{\nu=x}^{x+n-1} f(\nu) + \sum_{\nu=x}^{y+n} f(\nu).$$

- Also

$$\sum_{\nu=x}^{y} f(\nu) = \sum_{\nu=x}^{x+n-1} f(\nu) - \sum_{\nu=y+1}^{y+n} f(\nu) + \sum_{\nu=x+n}^{y+n} f(\nu)$$

- By Axiom (2)

$$\sum_{\nu=x}^{y} f(\nu) = \sum_{\nu=1}^{n} [f(\nu+x-1) - f(\nu+y)] + \sum_{\nu=x}^{y} f(\nu+n).$$

The first sum on the right-hand side involves an integer number of terms that can be evaluated classically. Now let n go to infinity.

$$\sum_{\nu=x}^{y} f(\nu) = \sum_{\nu=1}^{\infty} [f(\nu+x-1) - f(\nu+y)] + \lim_{n\to\infty} \sum_{\nu=x}^{y} f(\nu+n).$$

- Using Axiom (6) we obtain

$$\sum_{\nu=x}^{y} f(\nu) = \sum_{\nu=1}^{\infty} [f(\nu+x-1) - f(\nu+y)], \qquad (3.2)$$

that is the fundamental summation formula. Although it is not valid with enough generality the procedure above described can be used to treat other cases.

3.3. Examples

- The geometric sequence summation.

Let $f(x) = r^x$ with $|r| < 1$. We have:

$$\sum_{\nu=0}^{x} r^{\nu} = \sum_{\nu=1}^{\infty} [r^{\nu-1} - r^{\nu+x}] = (1 - r^{x+1}) \frac{1}{1-r}.$$

- The binomial series

$$\sum_{\nu=0}^{x}\binom{\alpha}{\nu}z^{\nu}=\sum_{\nu=1}^{\infty}\left[\binom{\alpha}{\nu-1}z^{\nu-1}-\binom{\alpha}{\nu+x}z^{\nu+x}\right].$$

This general case is not easy to deal. Let $x=\alpha$. We obtain for the second term on the right: $\binom{\alpha}{\nu+\alpha}z^{\nu+\alpha}=\frac{\Gamma(\alpha+1)}{\Gamma(-\nu+1)\Gamma(\nu+\alpha+1)}z^{\nu+\alpha}$ that is null for positive ν. Then

$$\sum_{\nu=0}^{x}\binom{\alpha}{\nu}z^{\nu}=\sum_{\nu=1}^{\infty}\binom{\alpha}{\nu-1}z^{\nu-1},$$

and so,

$$\sum_{\nu=0}^{\alpha}\binom{\alpha}{\nu}z^{\nu}=(1+z)^{\alpha}.$$

The procedure can be used to other functions even with $\lim_{n\longrightarrow\infty} f(z+n)\neq 0$. Also can be adopted for computing some fractional products: $\prod_{\nu=1}^{x} f(\nu)$, see [42, 43, 44].

4. Fractional Powers of a Square Matrix

The calculation of the p^{th} power of a square $n \times n$ real matrix $\mathbf{A}$, where p is a real or complex value, arises in applications such us Markov chain models in finance and healthcare [16, 32], fractional differential equations, nonlinear matrix equations and in computation [23, 10].

Many authors have investigated methods for computing the p^{th} power of matrices. In [29] are presented the Schur, Newton and Inverse Newton methods. The Schur-Newton and Schur-Padé algorithms are also discussed in [30]. Some of these methods impose additional conditions for matrix $\mathbf{A}$.

In this section a reliable method for computing $\mathbf{A}^p$, $p \in$, based on the eigenvalue decomposition, is presented.

Given a real square $n \times n$ matrix $\mathbf{A}$, with the eigenvalues λ_i and the corresponding eigenvectors $\mathbf{v}_i$, $1 \leq i \leq n$, $\mathbf{v}_i \neq \mathbf{o}$, there is the well know relation

$$\mathbf{A}\mathbf{v}_i = \lambda_i\mathbf{v}_i. \tag{4.1}$$

Is easily show that

$$\mathbf{A}^n\mathbf{v}_i = \lambda_i^n\mathbf{v}_i \tag{4.2}$$

for all positive integers n. Thus, we have one economical substitution of a power of the scalar λ_i for the more complex computed power $\mathbf{A}^n$ of the given matrix.

It is normal to consider the generalization of (4.2) to *non-integer* values of the n. For a nonsingular matrix $\mathbf{A}$ if we have $n = -1$ it produces the multiplicative inverse matrix $\mathbf{A}^{-1}$ by a somewhat circuitous way of computing it. All other n negative-integer values are, of course, powers of

$\mathbf{A}^{-1}$, [13, 14, 6].

Let us now considerer the case $n = \frac{1}{p}$, $p \in \mathbb{Z}$, $p \neq 0$. So, (4.2), yields:

$$\mathbf{A}^{\frac{1}{p}}\mathbf{v}_i = \lambda_i^{\frac{1}{p}}\mathbf{v}_i. \tag{4.3}$$

The p root of the matrix $\mathbf{A}$ should be defined as a matrix $\mathbf{B}$ such that $\mathbf{B}^p = \mathbf{A}$, therefore $\mathbf{B} = \mathbf{A}^{\frac{1}{p}} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ satisfies (4.3) for any eigenvalue and corresponding eigenvectors [5, 68, 4, 10] and expression (4.3) is

$$\mathbf{B}\mathbf{v}_i = \lambda_i^{\frac{1}{p}}\mathbf{v}_i. \tag{4.4}$$

Take, for example, the 3×3 following matrix $\mathbf{A}$ and $p = 3$,

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 1 \\ -1 & 1 & -3 \\ 1 & 0 & 4 \end{bmatrix}.$$

The *spectrum* of $\mathbf{A}$, is $\lambda_1 = 3$, $\lambda_2 = 2$, $\lambda_3 = 1$. The set of eigenvectors is $\mathbf{v}_1 = [-x_1, -x_1, x_1]$, $\mathbf{v}_2 = [-2x_2, -x_2, x_2]$, $\mathbf{v}_3 = [-9x_3, -x_3, 3x_3]$, respectively. Considering, for example, $x_1 = x_2 = x_3 = 1$ the set of eigenvectors results $\mathbf{v}_1 = [-1, -1, 1]$, $\mathbf{v}_2 = [-2, -1, 1]$, $\mathbf{v}_3 = [-9, -1, 3]$.

To calculate the desired p root of $\mathbf{A}$ as $\mathbf{B} = \mathbf{A}^{\frac{1}{p}}$ we need to find the real or complex elements $a, b, c, d, e, f, g, h, i$, we have

$$\mathbf{B}\mathbf{v}_i = \lambda_i^{\frac{1}{3}}\mathbf{v}_i \Leftrightarrow \begin{cases} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = 3^{\frac{1}{3}} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \\ \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} = 2^{\frac{1}{3}} \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \\ \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} -9 \\ -1 \\ 3 \end{bmatrix} = 1^{\frac{1}{3}} \begin{bmatrix} -9 \\ -1 \\ 3 \end{bmatrix} \end{cases}. \tag{4.5}$$

$$\text{Solving the system (4.5), gives } \begin{cases} a = 1.0776 \\ b = 0.8962 \\ c = 0.5315 \\ d = -0.1823 \\ e = 1.1164 \\ f = -0.5082 \\ g = 0.1823 \\ h = -0.1164 \\ i = 1.5082 \end{cases},$$

so the desired cubic root matrix is

$$\mathbf{B} = \mathbf{A}^{\frac{1}{3}} = \sqrt[3]{\mathbf{A}} = \begin{bmatrix} 1.0776 & 0.8962 & 0.5315 \\ -0.1823 & 1.1164 & -0.5082 \\ 0.1823 & -0.1164 & 1.5082 \end{bmatrix}$$

as a check, since we should have $\mathbf{B}^3 = \mathbf{A}$.

Obviously, different sort of eigenvalues may produce distinct results. In the example we analyzed a case with *distinct* positive eigenvalues. If we have negative eigenvalues, the p root matrix have complex entries.

Naturally the same process applies to larger order square matrices. The main limitation is to calculate the matrix's spectrum.

To compute noninteger powers in general, as they only involve raising an eigenvalue to the appropriated power and following the procedure presented.

Using the identity $\lambda_i^j = e^{j \ln \lambda_i} = \cos(\ln \lambda_i) + j \sin(\ln \lambda_i)$, $j = \sqrt{-1}$ it is possible to extend the determination of $\mathbf{A}^z$ to complex powers, $z \in \mathbb{C}$.

5. On Tolerant Computing and FracSets

Tolerant computing is the ability to compute coping with missing data and it is becoming increasingly important in today's computer dependent world. Hence there are plenty of strategies for incorporating missing data in a computing process. A recent practical list can be found in [33]. By a *tolerant operation* we mean an operation still defined and closed (in the abstract algebra sense) when some operand value is missing. Following Allouche and Shallit we consider the notion of symbol well known and do not define it further [1]. As it is customary in telecommunications a *symbol* is the communicational atomic token. In order to perform a computation symbols must be received by the computing apparatus and sometimes it receives nothing. For a general purpose computing system designed to deal with any symbol missing data is a no-symbol.

When there is the need to refer just to the symbol itself without any other meaning, just to its suitable glyph, it will be written between ' (single

straight quotes). As an example we can refer the Greek letter 'π'. When used without the surrounding straight quotes this letter can stand for a number, for a plane, for a partition or for any other meaning deemed appropriate by an author. Another example stems from the usual convention of representing the blank by '␣'. By this convention, abc xyz and abc␣ xyz represent the same.

For the no-symbol we need a similar convention. The *no-symbol* is defined as symbol absence and some glyphed representation is needed for it. To stand for symbol absence, we introduce the glyph '$\varnothing$'. When used without the surrounding straight quotes this symbol sole meaning is "here there is no symbol". We equate $\varnothing$ with the empty bunch in the Hehner sense [28]. As an example of its use we can write the empty set as $\{\}$ or as $\{\varnothing\}$.

On the notation:

We define the comparison equal as = . The production equal as =: . The equivalent equal as $\equiv$. The set of the finite natural numbers with zero as $\mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$. The bilogic value of an expression *expr* as (: *expr* :). The trilogic value of an expression *expr* as $\cdot$: *expr* :$\cdot$). The tolerant version (. .

of an entity θ as θ. TT as the abbreviation of "tolerant". NTT as the abbreviation of "non-tolerant".

On the appropriate logic for the tolerant equal :

The first question. As in Codd [17], we start by asking: What is the logic value of $\varnothing = \varnothing$? This question can formally be rephrased as: what is the value produced by (:$\varnothing$ = $\varnothing$:)? As stated in the introduction, this is the same as asking for the logic value of an isolated comparison equal sign (: = :).

Assertion 1. *The logic value of an equal sign,* (: = :)*, is not* F *or* T.

P r o o f. As (: (:$\varnothing$ = $\varnothing$:) =: F :) =: F and (: (:$\varnothing$ = $\varnothing$:) =: T :) =: F ✷

This leads us into the need for a logic with more than just two values.

A review of symbolic logic: This section is intended to provide a brief overview of symbolic logic and to introduce some terminology and notation since we assume that the reader is familiar with its fundamentals. Mind that in order to fully deal with tolerant operations a 3-valued (tri-valued) symbolic logic framework is needed, see [56].

Two-valued logic: Classical logic, or *bilogic*, has only two values. These values are $\ddot{\mathrm{T}}$ and $\ddot{\mathrm{F}}$. Usually, as bilogic is assumed to be the only logic in use, these values are written in a simplified way as T and F.

Tri-valued logic: Tri-valued logic, or trilogic, has three independent logic values wich are $\ddot{0}$, $\ddot{1}$, and $\ddot{2}$ with no order whatsoever defined between any pair of them, albeit shown by their preferred presentation order. These values have neither any connection with the integers 0, 1, 2, nor with the bilogic values $\ddot{\mathrm{T}}$ and $\ddot{\mathrm{F}}$.

Semantics of symbolic logic: We use $\ddot{\mathrm{T}}$ and $\ddot{\mathrm{F}}$ for bilogic True and False. We use $\dddot{\mathrm{T}}$ and $\dddot{\mathrm{F}}$ for trilogic True and False. No semantic connection is set up for now between $\dddot{\mathrm{T}}$ and $\ddot{\mathrm{T}}$ and between $\dddot{\mathrm{F}}$ and $\ddot{\mathrm{F}}$. We equate $\ddot{\ddot{\mathrm{T}}}$ with $\dddot{2}$ and $\dddot{\mathrm{F}}$ with $\dddot{0}$. We postulate the remaining trilogic value to be $\varnothing$.

Introducing the tolerant equal: As the expression $(:\varnothing = \varnothing :)$ produces neither a $\ddot{\mathrm{F}}$ nor a $\ddot{\mathrm{T}}$ we define

$$(: \widehat{\varnothing} = \widehat{\varnothing} :) =: \widehat{\varnothing}. \qquad (5.1)$$

This is most inelegant, since this forces $(: \mathit{expr} :)$ not to be closed in the sense of always producing a valid bilogic value. Closure in the above sense is natural in a trilogic framework where $(:\varnothing =: \varnothing :) =: \varnothing$. For this reason from now on when working in a TT environment we will do so in a trilogic framework.

In order to be able to compare $\varnothing$ we define the tolerant equal, with symbol $\eqsim$ with the following characteristic property:

$$(: \widehat{\varnothing} \eqsim \widehat{\varnothing} :) =: \ddot{\mathrm{T}}. \qquad (5.2)$$

A TT variable can assume the $\varnothing$value. A NTT variable can never assume the $\varnothing$value.

5.1. Tolerant operations and FracSets

A tolerant operation is a generalization of a non tolerant one. So, the definition of the two-set tolerant Cartesian product must be made upfront. The formal definition of the Cartesian product stems from the definition of ordered pair. The most common definition of ordered pairs, the Kuratowski definition, is $(x, y) \equiv \{\{x\}\{x\ y\}\}$. This definition is not suitable for our purposes as

$$(x, \widehat{\varnothing}) \equiv \{\{x\}\ \{x\ \widehat{\varnothing}\}\} \equiv \{\{x\}\ \{x\}\} \equiv \{\{x\}\}$$

and

$$(x, x) \equiv \{\{x\}\ \{x\ x\}\} \equiv \{\{x\}\ \{x\}\} \equiv \{\{x\}\}.$$

Minding that an urelement is an isolated set element that is not a set in itself, an elementary bunch, we define the *urset* as a finite set that when written in extension all symbols are either braces or stand for urelements. From this definition we can conclude that the empty set is an urset. For ursets these are the tolerant ordered pairs [57]:

$\{\{a\}b\}$	the strict dituple
$\{\{a\}a\}$	the dituple with both components equal
$\{\{\}a\}$	the dituple with null first component
$\{\{a\}\}$	the dituple with null second component
$\{\{\}\}$	the null dituple

With these dituples it is possible to define the tolerant Cartesian product or *urcartesian* product of two ursets as the urset of all possible dituples.

The urcartesian symbol is $\times$ A tolerant operation is defined by its urcartesian. This definition is strictly for ursets and an urset is a finite set. In this text we don't address the question of tolerant operations on any set.

5.1.1. *FracSets.* Using tolerance it is possible to define a certain kind of set, neither fuzzy nor crisp, the FracSet.

Tolerant belonging: Let us consider the "belongs to", $\in$, symbol. It is used in expressions like $a \in \{a\ b\ c\}$. We introduce now the *tolerant belonging* with symbol $\widehat{\in}$. The empty set, $\varnothing$, can be written as $\{\}$ or equivalently as $\{\varnothing\}$. We cannot say that $\varnothing \in \{\ \}$ since $(\cdot.\ \varnothing \in \{\ \}\ .\cdot) =: \dddot{\mathrm{F}}$ but we can say that $\varnothing\ \widehat{\in}\ \{\}$ since we define

$$(:\widehat{\varnothing}\,\widehat{\in}\,\{\}:) =: \dddot{\mathrm{T}}. \tag{5.3}$$

5.1.2. *The stuff set operation.* The *stuff* set operation, with symbol $\widehat{\in}^{++}$ can be defined as

$$x\,\widehat{\in}^{++}A =: \{x\} \cup A, \tag{5.4}$$

where A is an UrSet and x is an optional urelement or UrSet. As an illustrative example we have $a\,\widehat{\in}^{++}\{\} =: \{a\}$ We have $\widehat{\varnothing}$ as the neutral stuff argument, since $\widehat{\varnothing}\widehat{\in}^{++}A =: \{\} \cup A =: A$.

Before we can deal with the symbol $\widetilde{\in}^{\sim}$, we need to develop the FracSet definition.

5.1.2. *FracSets and FracOmegas.* An UrSet is written in *ket notation* if every of its elements and $\varnothing$ are depicted as ket labels. As an illustrative example the expression $\{|\varnothing\bullet\,|a\bullet\,|b\bullet\}$ depicts the UrSet $\{a\,b\}$ in ket notation. A *FracSet* is an UrSet where each element and $\varnothing$ is associated with a $z \in \mathbb{C}$. Let us use a ket notation for this association. Without loss of generality we will depict FracSet elements as an ordered sequence of not necessarily contiguous integers with the zs themselves as the ket labels. An UrSet with $N \in \mathbb{N}_0$ elements will have $N+1$ zs. The zero index is reserved for the z associated with the $|\varnothing\bullet$ ket. As an illustrative example the UrSet $\{1\,3\}$ originates the FracSet $\{z_0\bullet\, z_1|\,\bullet\, z_3|\cdot\,\bullet\}$

The zs can be function of parameters, like t or τ. In this situation we will have $\{|z_0(t)\bullet\,|z_1(t)\bullet\,|z_3(t)\bullet\}$.

A *FracOmega* is FracSet, where

$$\sum_{All\ i} z_i(t)z_i^*(t)|_{\forall t} = 1. \tag{5.5}$$

So we have $z_i(t)z_i^*(t) = |z_i(t)|^2 = \langle z_i(t)|z_i(t)\rangle$.

A rule for mixing the zs when joining or intersecting two FracSets: Let A and B be not necessarily distinct FracSets. When joining or intersecting A with B the common zs modules (phases) must be added and the result divided by two.

5.2. Potentialities: Tolerant probabilities

We call Potentialities to tolerant probabilities. Let Ω be a FracOmega.

5.2.1. *Classic axioms of probabilities.* The classic axioms are [58]: $P(A) \geq 0$, $\forall A \subseteq \Omega$; $P(\Omega) = 1$ and $A1 \cap A2 = \varnothing \Rightarrow P(A_1 \cup A_2) = P(A_1) + P(A_2)$. In a FracOmega where $z_0 = 0$ the $(z|z\bullet$s can be seen as the probabilities.

5.2.2. *Axioms of potentialities.*

$$\widehat{P}(A) \geq 0, \qquad \forall A \subseteq \Omega, \tag{5.6}$$

$$\widehat{P}(\Omega) = 1, \tag{5.7}$$

$$A_1 \cap A_2 = \varnothing \Rightarrow \widehat{P}(A_1 \cup A_2) = \widehat{P}(A_1) + \widehat{P}(A_2) - \widehat{P}(\varnothing). \tag{5.8}$$

In a FracOmega the $(z|z\bullet$s can be seen as the potentialities. When $P(\varnothing) = 0$ potentialities are equal to probabilities.

5.2.3. *The probe set operation.* The probe set operation, with symbol $\in^{00}$ can be applied to UrSets. It shows at most a member of the probed UrSet in a non deterministic fashion. If the original UrSet is a FracOmega, this is related to potentialities by

$$\widehat{P}(\widehat{\in}^{00}\{a\,b\,c\} =: a) \equiv |z_a|^2. \tag{5.9}$$

5.2.4. *The unstuff set operation.* The unstuff set operation, with symbol $\in^{--}$ can be applied to FracSets. It extracts, in the sense that shows like the probe but also reduces its z to zero, at most a member of the original FracSet in a non deterministic fashion. If the original FracSet is a FracOmega, this is is related to potentialities because the element extraction obeys

$$\widehat{P}(\widehat{\in}^{--}\{a\,b\,c\} =: a) \equiv \langle z_a|z_a\rangle. \tag{5.10}$$

These are the tools to work with elements and FracSets.

6. Negative Probabilities

The concept of probability emerged in 1654 with the correspondence between Fermat and Pascal and the modern theory paradigm is usually credited to Kolmogorov (1931). The first axiom says that the probability of an event X is a non-negative real number, that is, $P(X) \in \mathbb{R}$ and $P(X) \geq 0$. Therefore, the concept of Negative probability (NP), or of some other value outside the interval between zero and one, is excluded.

Paul Dirac [19], in the scope of quantum mechanics, introduced the concepts of negative energies and NP. He wrote "Negative energies and probabilities should not be considered as nonsense since they are well-defined concepts mathematically, like a negative of money". Later Richard Feynman [21, 22] discussed also NP. He observed that we adopt negative numbers in calculations, but that "minus three apples" is not a valid concept in the real world. Nevertheless, the first efforts towards a formal definition of NP should be credited to M. Bartlett [7]. More recently, Gábor Székely [63] introduced the concept of "half-coins" as objects related to NP. He considered a fair coin with two sides "0" and "1" and probabilities $\frac{1}{2}$. It is well known that for a discrete random variable X with probability mass function $P(X = k)$, the probability generating function (pgf) is defined as $G_X(z) = E(z^X) = \sum_{k=0}^{\infty} P(X = k)\, z^k$, $z \in \mathcal{C}$. Moreover, the addition of independent random variables corresponds to the multiplication of their pgf. Therefore, the pgf of one fair coin is $G_X(z) = \frac{1}{2}(1 + z)$ and the pgf of the sum of n fair coins is $G_X(z) = \left[\frac{1}{2}(1 + z)\right]^n$. Székely generalized the pgf and defined "half-coin" as the object having pgf:

$$G_X(z) = \left[\frac{1}{2}(1+z)\right]^{\frac{1}{2}}. \tag{6.1}$$

If we flip two half coins it yields a complete coin, since then the sum of the outcomes is either "0" or "1", having probability $\frac{1}{2}$, similarly to what happens when flipping a fair coin. The Taylor expansion of (6.1) reveals that "half-coins" have an infinite number of sides and that some exhibit NP. In fact, it results:

$$\begin{aligned} P(X=0) &= \frac{1}{\sqrt{2}} \\ P(X=1) &= \frac{1}{2\sqrt{2}} \\ P(X=2) &= -\frac{1}{8\sqrt{2}} \\ P(X=3) &= \frac{1}{16\sqrt{2}} \\ P(X=4) &= -\frac{5}{128\sqrt{2}} \\ &\cdots \end{aligned} \tag{6.2}$$

A formulation for NP relaxes the first axiom of standard probabilities. NP are defined in the scope of quasiprobability distributions, that share some of the common features of probabilities, but that violate the first and third axioms of classical probability theory.

NP have been discussed and applied in physics [8, 41, 38, 35, 31, 54], but the topic emerged only recently in finance, economy [26, 66, 12]. In the scope of control the concepts were also extended to "anti-flipping half coins", [39].

7. Fractional Delay Discrete-Time Linear Systems

Consider a discrete-time signal, x_n and its discrete-time Fourier transform

$$X(e^{i\omega}) = \sum_{-\infty}^{+\infty} x_n e^{-i\omega n}, \tag{7.1}$$

where $\omega = 2\pi f$, f being the frequency, $|f| \leq 1/2$. Let $\alpha \in \mathrm{R}$. A fractionally delayed signal is a signal with Fourier transform $e^{i\omega\alpha}X(e^{i\omega})$ [46, 48]. Such signal is obtained from

$$x_{n-\alpha} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\omega\alpha}X(e^{i\omega})e^{i\omega n}\mathrm{d}\omega. \tag{7.2}$$

Using the expression of the Fourier transform, we obtain the interesting relation:

$$x_{n-\alpha} = h_n * x_n = \sum_{-\infty}^{+\infty} h_k x_{n-k}, \qquad (7.3)$$

where the symbol $*$ means discrete convolution. So, there is a linear system with impulse response, h_n, that produces a fractional delay. It is not very difficult to conclude that [37, 46]

$$h_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\omega\alpha}e^{i\omega n}\mathrm{d}\omega = \frac{\sin[\pi(n-\alpha)]}{\pi(n-\alpha)}. \qquad (7.4)$$

In the last years some effort has been done to find FIR (finite impulse response) implementations for h_n [65, 20] With this delay we are led to consider systems with the general format

$$\sum_{k=0}^{N} a_k D^{\alpha_k} y_n = \sum_{k=0}^{M} b_k D^{\beta_k} x_n \quad n \in \mathbb{Z} \qquad (7.5)$$

with $a_N = 1$. The orders N and M are any given positive integers and the parameters a_k and b_k are real numbers. We represented the delay operator by a D in similarity with the differential systems [47]. The delays α_k, $\beta_k, k \in$ are real numbers that can be considered as having absolute value less than or equal to one: $|\alpha_k| \leq 1$, $|\beta_k| \leq 1$. This means that we made $D^{\alpha_k} y_n = y_\alpha$. It can be shown [46] that, contrarily to the current (integer) delay operator, the relation $D^{\alpha_k} z^n = z^{-\alpha_k} z^n$ is only valid for $|z| = 1$. This means that we cannot use the Z-transform in studying this kind of systems. For $z = e^{j\omega}$ we obtain the frequency response of the system,

$$G(z) = \frac{\sum_{k=0}^{M} b_k z^{-\beta_k}}{\sum_{k=0}^{N} a_k z^{-\alpha_k}}. \qquad (7.6)$$

As we cannot use directly the Z-transform, we have to use other tools to get the transfer functions, corresponding to causal and anti-causal systems; such tools are the Cauchy integrals that are used for projecting $G(z)$ above defined defined on the unit circle to the spaces outside and inside it.

The general case is difficult to deal because it is not easy to find the poles. The simpler commensurate case

$$\sum_{k=0}^{N} a_k D^{\alpha k} y_n = \sum_{k=0}^{M} b_k D^{\alpha k} x_n \quad n \in \mathbb{Z} \qquad (7.7)$$

can be easily treated [46]. If $M < N$ we obtain a system that is very similar to the continuous-time case. In fact we have:

(1) Consider the function $G(w)$, by substitution of w for z^{α}.
(2) The polynomial denominator $G(w)$ is the indicial polynomial or characteristic pseudo-polynomial. Perform the expansion of $G(w)$ into partial fractions like:

$$F(w) = \frac{1}{(1 - pw^{-1})^k},$$

where we represented by p a generic root of the indicial polynomial (pole).
(3) Substitute back z^{α} for w to obtain $G(z)$ expanded as a linear combination of fractions like:

$$F(z) = \frac{1}{(1 - pz^{-\alpha})^k}.$$

(4) Compute the Impulse Responses corresponding to each partial fraction.
(5) Add the Impulse Responses.

The causal impulse response, f_n corresponding to $F(z) = \frac{1}{1-pz^{-\alpha}}$, is given by [46]

$$f_n = \sum_{k=0}^{\infty} p^k \frac{\sin[\pi(n-k-\alpha)]}{\pi(n-k-\alpha)}, \qquad (7.8)$$

that is formally similar to the Mittag-Leffler function.

These systems are interesting because they allow us to relate signals defined on different time scales.

8. Fractional Fourier Transform

The fractional Fourier transform (FrFT) is a generalization of the classical Fourier transform (FT). With the development of the FrFT it was verified that the ordinary frequency domain is merely a special case of a continuum of fractional Fourier domains, that are related to time-frequency (or space-frequency) representations.

The FrFT has been found to play an important role in the study of optical systems, known as Fourier optics, and with applications in optical information processing, allowing a reformulation of this area in a much more general way. It also generalized the notion of the frequency domain and extended our understanding of the time-frequency plane, two central concepts in signal analysis and signal processing. FrFT is expected to have an impact in the form of deeper understanding or new applications in every area in which the FT plays a significant role, and to take its place

among the standard mathematical tools of physics and engineering, see e.g. [45, 11, 15].

Like in the case of the FT, the FrFT can be applied to problems in different fields. Some gain can be expected in most applications because the advantage of the additional degree of freedom associated with the fractional order parameter (α) of the FrFT. Typical applications of FrFT are the areas of linear partial differential equations of fractional order, signal and image processing, communications and wave propagation.

FrFT, in the form of fractional powers of the Fourier operator, appears in the mathematical literature as early as 1929 [18]. Later on it was used in quantum mechanics and signal processing [49, 64, 2], but it was mainly the optical interpretation and the applications in optics that gave a burst of publications since the nineties that culminated in the publication of the book of Ozaktas et al. [50].

The FT of a function can be considered as a linear differential operator acting on that function, while the FrFT generalizes this differential operator by letting it depend on a continuous parameter α.

Several FrFT definitions are found in the literature. Among them the most commonly used, the α^{th} order of FrFT of function $s(t)$ is a linear operation defined by:

$$FrFT^{\alpha}[s(u)] = \int_{-\infty}^{\infty} K_{\alpha}(u,t)s(t)dt, \tag{8.1}$$

where α indicates the rotation angle in the time-frequency plane, $K_{\alpha}(u, t)$ is the kernel function

$$K_{\alpha}(u,t) = \begin{cases} \sqrt{\frac{1-j\cot(\alpha)}{2\pi}} e^{j\left[\frac{t^2+u^2}{2}\cot(\alpha)-\csc(\alpha)ut\right]}, & \alpha \neq n\pi \\ \delta(t-u) & \alpha = 2n\pi \\ \delta(t+u) & \alpha = 2n\pi \pm \pi \end{cases}, \tag{8.2}$$

where $n \in \mathbb{Z}$ and $j = \sqrt{-1}$, [8].

The FrFT has the following special cases:

$$FrFT^{2n}[s(u)] = s(u), \tag{8.3}$$

$$FrFT^{2n\pi+\frac{\pi}{2}}[s(u)] = FT[s(u)], \tag{8.4}$$

$$FrFT^{2n\pi\pm\pi}[s(u)] = s(-u), \tag{8.5}$$

$$FrFT^{2n\pi-\frac{\pi}{2}}[s(u)] = FT[s(-u)]. \tag{8.6}$$

The time domain consists of the FrFT domain with $\alpha = 2n\pi$, while the frequency domain is the FrFT domain with $\alpha = 2n\pi + \frac{\pi}{2}$. Since the FrFT is periodic with the period of 2π, α can be limited to the interval $[-\pi, \pi]$ [64].

The problem of the existence of the FrFT has been widely investigated [50] and it was concluded that the FrFT of a signal $s(t)$ exists under the similar conditions in which the classical FT exists.

Considering $FrFT^{(\alpha)}$ as an operator corresponding to the FrFT of α angle, some important properties are listed:

(1) *Identity*: $FrFT^{(0)}$ is the identity operator. The $FrFT^{(\alpha)}[s(t)]$ with $\alpha = 0$ is the input signal $s(t)$ itself. The $FrFT^{(\alpha)}[s(t)]$ with $\alpha = 2\pi$ corresponds to the successive application of the classical FT four times, and acts as the identity operator, i.e. $FrFT^{(0)} = FrFT^{(\frac{\pi}{2})^2} = 1$. This property follows from the definition of the kernel (8.2) with $\alpha = 0$.

(2) *Fourier transform operator*: $FrFT^{(\frac{\pi}{2})}$ is the Fourier transform operator. The $FrFT^{(\alpha)}[s(t)]$ with $\alpha = \frac{\pi}{2}$ gives de classical Fourier transform of the input signal, i.e. $FrFT^{(\frac{\pi}{2})}[s(t)] = FT[s(t)]$. This property can be proved by expanding the kernel (8.2) with $\alpha = \frac{\pi}{2}$.

(3) *Successive application*: Successive applications of FrFT are equivalent to a single application of FrFT whose order is the sum of individual orders, i.e. $FrFT^{(\alpha)}\left[FrFT^{(\beta)}[s(t)]\right] = FrFT^{(\alpha+\beta)}[s(t)]$. This property follows from the convolution property of the kernel (8.2).

(4) *Inverse*: The FrFT of order $(-\alpha)$ is the inverse of FrFT of order α since $FrFT^{(-\alpha)}\left[FrFT^{(\alpha)}\right]^{-1} = FrFT^{(-\alpha+\alpha)} = FrFT^{(0)} = 1$. This property follows as a consequence of of properties (2) and (3).

(5) *Parseval's theorem*: The well-know Parseval's theorem for classical FT can be extended to the FrFT by the equation:

$$\int_{-\infty}^{\infty} s(t)r^*(t)dt = \int_{-\infty}^{\infty} FrFT^{(\alpha)}s(u)\left[FrFT^{(\alpha)}\right]^* r(u)du, \qquad (8.7)$$

where $*$ denotes complex conjugation.

One conclusion that can be obtained from these properties is that the signal $s(t)$ and corresponding FrFT of order α ($FrFt^{(\alpha)}$) form a transform pair related by

$$\begin{aligned} FrFT^{\alpha}[s(u)] &= \int_{-\infty}^{\infty} K_{\alpha}(u,t)s(t)dt, \\ s(t) &= \int_{-\infty}^{\infty} FrFT^{\alpha}[s(u)]\,K_{-\alpha}(u,t)du. \end{aligned} \qquad (8.8)$$

The flexibility and efficiency of FrFT permits new solutions to a variety of problems that involve Fourier analysis. In many cases, the resulting algorithms are faster than the conventional methods. We have limited our

presentation to real values of α. Complex ordered transforms also have potential applications, so that it might be of interest to explore the FrFT with a complex order. This transform can be considered as a particular case of the more general linear canonical transformation. There are several attempts to obtain discrete-time versions of this transform.

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