Riccati Equations in Optimal Filtering of Nonstabilizable Systems Having Singular State Transition Matrices

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Abstract—**Until recently, it was believed that a necessary and sufficient condition for convergence of the Riccati difference equation of optimal filtering was that the system be both detectable and stabilizable. Recently, it has been shown that the stabilizability condition can be removed but convergence has only established under restrictive assumptions including the requirement that the state transition matrix be nonsingular. The present paper generalizes these results in several directions. First, properties of the algebraic Riccati equation are established for the case of singular state transition matrix. Second, several assumptions previously imposed in establishing convergence of the Riccati difference equation for systems with unreachable modes on the unit circle are relaxed including replacing observability by detectability, weakening the conditions on the initial covariance, and allowing the state transition matrix to be singular. Third, results on the convergence and properties of the Riccati equations are expressed as both necessary and sufficient conditions, whereas previous results were only sufficient. These extensions mean that the results have wider applicability, including fixed-lag smoothing problems and filtering for systems with time delays. The implications of the results in the dual problem of optimal control are also studied.**

I. Introduction

Over the past two decades there has been a great deal of work on the dual problems of optimal control and optimal filtering. Surprisingly, however, the question of optimal filtering of nonstabilizable systems has not been satisfactorily resolved.

Kalman and Bucy [1]–[3], the originators of the problem, established convergence and properties of the solutions of the matrix Riccati equation under the assumption of controllability and observability. Later, these conditions were weakened by Wonham [4], Caines and Mayne [5], and Anderson and Moore [6] to detectability and stabilizability. However, this excludes a wide class of important problems as, for example, the optimal filtering problem of unstable systems without process noise. Results for continuous-time nonstabilizable systems have been presented in [7]–[10]. However, these results are incomplete. For example, they do not allow for deterministic disturbances such as sinusoids which give rise to uncontrollable modes on the imaginary axis (unit circle in discrete time).

In a recent paper Chan, Goodwin, and Sin [11] have presented results in this direction for discrete-time systems.

The present paper generalizes the results of [11] in three directions. One is to establish results on the algebraic Riccati equation (ARE) under weaker assumptions including both necessary and sufficient conditions. Another is to relax the assumptions of the main theorems of [11] on the convergence of the Riccati difference equation (RDE): in particular, the requirements on the initial condition of the RDE are relaxed, and observability is replaced by the weaker requirement of detectability. Finally, all these results are now established for the case of singular state transition matrices. This was explicitly excluded in [11]. These extensions mean that the results have wider applicability, including fixed-lag smoothing problems and systems with time delays, which give rise to singular state transition matrices.

To handle systems having singular state transition matrices we shall use a different proof technique to that employed in [11]. We shall use a generalized eigenvector approach introduced by Pappas, Laub and Sandell [12] and Emami-Naeini and Franklin [13], [14] and used in [15], [16] in the numerical solution of ARE for the restrictive detectable and stabilizable case. Another important contribution of the current paper is to remove the stabilizability requirement from that approach.

The organization of the paper is as follows. In Section II the statement of the problem is given. In Section III the solutions of the algebraic Riccati equation are analyzed from an algebraic point of view using the eigenvectors of an associated generalized eigenvalue problem. We also investigate the necessary and sufficient conditions for existence and uniqueness of *strong solutions* to the ARE, i.e., real symmetric nonnegative definite solutions which give rise to a filter with roots on or inside the unit circle. In Section IV we present new results concerning the convergence of the solution of the Riccati difference equation to the strong solution of the algebraic Riccati equation. In Section V we discuss some consequences of our results for the dual optimal control problem.

II. Problem Statement

Consider the following discrete-time linear system of dimension n, having m outputs:

$$x(t+1)=Ax(t)+v_1(t) \tag{2.1}$$

$$y(t)=Cx(t)+v_2(t); \qquad t\geq 0 \tag{2.2}$$

where A and C are constant matrices of appropriate dimensions and $\{v_1(t)\}$, $\{v_2(t)\}$ are uncorrelated zero mean "white" sequences having covariance matrices Q and R, respectively, with $Q \geq 0$ and $R > 0$. We also assume that the initial state $x(0)$ is a random variable uncorrelated with $\{v_1(t)\}$ and $\{v_2(t)\}$, having mean $\bar{x}_0$ and covariance matrix Σ_0. Note that the case when $\{v_1(t)\}$ and $\{v_2(t)\}$ are correlated can be handled by identical means using a preliminary transformation (see [11], [18]).

It is well known that the best linear estimate $\hat{x}(t)$ of $x(t)$ given data up to time $t-1$ satisfies

$$\hat{x}(t+1)=\bar{A}(t)\hat{x}(t)+K(t)y(t) \tag{2.3}$$

$$\hat{x}(0)=\bar{x}_0 \tag{2.4}$$

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where

$$\bar{A}(t) \triangleq A - K(t)C \quad (2.5)$$

$$K(t) \triangleq A\Sigma(t)C^T(C\Sigma(t)C^T + R)^{-1} \quad (2.6)$$

and $\Sigma(t)$ satisfies the following matrix *Riccati difference equation:*

$$\Sigma(t+1) = A\Sigma(t)A^T - A\Sigma(t)\bar{C}^T[\bar{C}\Sigma(t)\bar{C}^T + I]^{-1}\bar{C}\Sigma(t)A^T + DD^T \quad (2.7)$$

$$\Sigma(0) = \Sigma_0 \geq 0 \quad (2.8)$$

where

$$\bar{C} = R^{-1/2}C \quad (2.9)$$

with D and $R^{1/2}$ being matrices such that

$$Q = DD^T; \; R = (R^{1/2})(R^{1/2})^T \quad (2.10)$$

and $R^{1/2}$ nonsingular. The factorizations above exist due to the fact that $Q \geq 0$, $R > 0$ and can be easily obtained using standard results.

In the sequel we shall be interested in the question of the convergence of the filter, i.e., existence of a limiting solution Σ of the RDE. If this solution exists, then it satisfies the following algebraic Riccati equation:

$$\Sigma - A\Sigma A^T + A\Sigma\bar{C}^T(\bar{C}\Sigma\bar{C}^T + I)^{-1}\bar{C}\Sigma A^T - DD^T = 0 \quad (2.11)$$

and the dynamics of the asymptotic filter will be characterized by the following steady-state transition matrix:

$$\bar{A} = A - A\Sigma\bar{C}^T(\bar{C}\Sigma\bar{C}^T + I)^{-1}\bar{C}. \quad (2.12)$$

In the next section we will study the solutions of (2.11). Since $\Sigma(t)$ is real symmetric nonnegative definite, we shall restrict attention to the solutions of the ARE which retain these properties. We shall also isolate those solutions which give rise to a filter having roots inside or on the unit circle. Following the notation first introduced in [11], we call these solutions the *strong solutions*. If the asymptotic filter is exponentially stable, then we shall call Σ the *stabilizing solution.*

III. Solution of the Algebraic Riccati Equation

To construct the solutions of the ARE we shall use the generalized eigenproblem approach introduced in [12] and [13]. The key idea of this approach is to construct the solutions of the ARE from a computationally stable basis for the eigenspace of the following symplectic matrix pencil (see Appendix A for definition and properties of symplectic pencils).

$$P(\lambda) = M - \lambda L \quad (3.1)$$

$$M \triangleq \begin{bmatrix} A^T & 0 \\ -DD^T & I \end{bmatrix}; \; L \triangleq \begin{bmatrix} I & \bar{C}^T\bar{C} \\ 0 & A \end{bmatrix}. \quad (3.2)$$

To simplify the analysis, a basis consisting of the generalized eigenvectors of $P(\lambda)$ was adopted.

Note that when A is nonsingular, the generalized eigenvalue problem for (3.1) reduces to the standard eigenvalue problem for $\hat{M} - \lambda I$ where $\hat{M} = L^{-1}M$ is the well-known Hamiltonian matrix [17].

In the sequel we assume that the pencil (3.1) is regular, i.e., det $(M - \lambda L)$ is not identically zero for all λ.

We now summarize the key properties of the solutions of the ARE. Parts A)–E) are well-known results [12], whereas parts F)–I) are extensions of similar results of [11, Lemma 3.1] to the case where A is singular.

Throughout the paper, the notation x^* will be used to denote the transpose of the complex conjugate of x.

Lemma 3.1: Let each generalized principal vector z_i of the pencil $(M - \lambda L)$ be decomposed into two n-vectors

$$z_i = (x_i^T, y_i^T)^T, \qquad i = 1, 2, \cdots, 2n$$

and let

$$U \triangleq \begin{bmatrix} X \\ Y \end{bmatrix} \triangleq \begin{bmatrix} x_1, x_2, \cdots, x_n \\ y_1, y_2, \cdots, y_n \end{bmatrix} \quad (3.3)$$

be a selection of n such vectors such that if a generalized principal vector of rank k is a column of U, then the generalized principal vectors of rank inferior to k are also columns of U. Then we have the following results.

A) For each solution Σ of the ARE there exists a choice of U such that X^{-1} exists and $\Sigma = YX^{-1}$.

B) For any choice of U such that X^{-1} exists, then $\Sigma = YX^{-1}$ is a solution of the ARE provided $(I + \bar{C}\Sigma\bar{C}^T)$ is nonsingular.

C) Let J be the $n \times n$ Jordan canonical form corresponding to a choice of U. If $\Sigma = YX^{-1}$ is a solution of the ARE, then J is also the Jordan canonical form of $\bar{A}^T$ and X is the corresponding matrix of generalized eigenvectors of $\bar{A}^T$.

D) If A is nonsingular, all the eigenvalues of the pencil $M - \lambda L$ are nonzero. When A is singular, then $M - \lambda L$ has at least one eigenvalue equal to zero.

E) If $\lambda_i \neq 0$ is an eigenvalue of $M - \lambda L$ with multiplicity p_i, then $1/\lambda_i^*$ is also an eigenvalue of $M - \lambda L$ with the same multiplicity. When $\lambda = 0$ is an eigenvalue of $M - \lambda L$ with multiplicity l, then there are only $2n - l$ finite eigenvalues. The other l eigenvalues are "infinite" eigenvalues (or reciprocals of 0).

F) λ is an unreachable mode of (A, D) having multiplicity p such that

$$(A^T - \lambda I)x_i = x_{i-1} \qquad i = 0, 1, \cdots, p-1 \quad (3.4)$$

$$D^Tx_i = 0 \qquad x_{-1} = 0 \quad (3.5)$$

if and only if λ is an eigenvalue of multiplicity p of the pencil $M - \lambda L$ such that

$$(M - \lambda L)\begin{bmatrix} x_i \\ 0 \end{bmatrix} = L\begin{bmatrix} x_{i-1} \\ 0 \end{bmatrix}, \qquad \begin{matrix} i = 0, 1, \cdots, p-1 \\ x_{-1} = 0. \end{matrix} \quad (3.6)$$

G) $\lambda \neq 0$ is an unobservable mode of (C, A) having multiplicity p such that

$$(A - \lambda I)y_i = y_{i-1} \qquad i = 0, 1, \cdots, p-1 \quad (3.7)$$

$$Cy_i = 0 \qquad y_{-1} = 0 \quad (3.8)$$

if and only if λ^{-1} is an eigenvalue of multiplicity p of the pencil $M - \lambda L$ such that

$$(M - \lambda^{-1}L)\begin{bmatrix} 0 \\ \omega_i \end{bmatrix} = L\begin{bmatrix} 0 \\ \omega_{i-1} \end{bmatrix}, \qquad \begin{matrix} i = 0, 1, \cdots, p-1 \\ \omega_{-1} = 0. \end{matrix} \quad (3.9)$$

H) The pencil $M - \lambda L$ has an eigenvalue λ of multiplicity $2p$ on the unit circle if and only if:

1) there is an unreachable mode β of multiplicity p of (A, D) such that $|\beta| = 1$ and

$$(A^T - \beta I)x_i = x_{i-1} \qquad i = 0, 1, \cdots, p-1 \quad (3.10)$$

$$D^Tx_i = 0 \qquad x_{-1} = 0 \quad (3.11)$$

or

2) there is an unobservable mode γ of multiplicity p of (C, A) such that $|\gamma| = 1$ and

$$(A - \gamma I)y_i = y_{i-1} \qquad i = 0, 1, \cdots, p-1 \quad (3.12)$$

$$Cy_i = 0 \qquad y_{-1} = 0. \tag{3.13}$$

I) The matrix (X^*Y) is Hermitian if and only if:

1) for an eigenvalue α_i of multiplicity p_i of the pencil $P(\lambda)$ such that $\alpha_i = 0$, then the associated p_i generalized principal vectors are used in the construction of U, and

2) for each pair of eigenvalues β_j, γ_j of multiplicity p_j of the pencil $P(\lambda)$ such that $\beta_j \neq 0$, $|\beta_j| \neq 1$, and $\gamma_j = (1/\beta_j)^*$, p_j of the total of $2p_j$ principal vectors associated with β_j and γ_j are used in the construction of U, and

3) for each eigenvalue δ_k of multiplicity $2p_k$ of the pencil $P(\lambda)$ such that $|\delta_k| = 1$, exactly p_k of the total of $2p_k$ associated principal vectors are used in the construction of U.

If X is nonsingular, (1)–(3) are necessary and sufficient conditions for the solution $\Sigma = YX^{-1}$ to be a Hermitian solution of the ARE.

Proof: For parts A)–E), see [12]. Parts F)–I) can be easily proved by extending the proof of similar results in [7], [8], [11], and [17] for the case of a generalized eigenproblem. See [26] for further details. $\nabla\nabla\nabla$

The next theorem gives conditions under which X^*Y is nonnegative definite. Parts A) and B) are extensions of results in [17] for the case of a symplectic pencil and a general A matrix.

Theorem 3.1: Let

$$M\begin{bmatrix} X \\ Y \end{bmatrix} = L\begin{bmatrix} X \\ Y \end{bmatrix} J \tag{3.14}$$

where J is the $(n \times n)$ Jordan canonical form corresponding to a choice of n eigenvalues λ_i, $i = 1, 2, \cdots, n$ and $[X^T, Y^T]^T$ is the matrix of the associated principal vectors of the pencil $M - \lambda L$.

A) If $|\lambda_i| \leq 1$, $i = 1, 2, \cdots, n$, then X^*Y is Hermitian nonnegative definite.

B) Provided (C, A) is detectable and $|\lambda_i| \leq 1$, $i = 1, 2, \cdots, n$, then X is nonsingular.

C) Provided X is nonsingular and X^*Y is Hermitian nonnegative definite, then $|\lambda_i| \leq 1$, $i = 1, 2, \cdots, n$ if (A, D) has no unreachable mode outside the unit circle.

Proof:

A) To prove the nonnegative definitiveness of X^*Y we initially restrict the analysis to the case where $|\lambda_i| < 1$, $i = 1, \cdots, n$. Using (3.2) we can rewrite (3.14) as

$$A^TX = XJ + \bar{C}^T\bar{C}YJ \tag{3.15}$$

$$-DD^TX + Y = AYJ. \tag{3.16}$$

Premultiplying (3.15) and (3.16) by, respectively, $(YJ)^*$ and X^*, we obtain

$$(YJ)^*A^TX = J^*(Y^*X)J + (YJ)^*\bar{C}^T\bar{C}(YJ) \tag{3.17}$$

$$-X^*DD^TX + X^*Y = X^*AYJ. \tag{3.18}$$

Adding (3.17) to the conjugate transpose of (3.18) gives

$$X^*Y - J^*(X^*Y)J = X^*DD^TX + (YJ)^*\bar{C}^T\bar{C}(YJ) \tag{3.19}$$

and the result follows immediately from the discrete lemma of Lyapunov since J is stable and the RHS of (3.19) is nonnegative definite.

We shall now analyze the case where $|\lambda_i| \leq 1$, $i = 1, \cdots, n$. Let

$$\begin{bmatrix} X \\ Y \end{bmatrix} \triangleq \left[\begin{array}{c|c} X_0 & X_1 \\ Y_0 & Y_1 \end{array}\right]$$

where

$$\begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}, \quad \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix}$$

are the matrices of generalized principal vectors of $M - \lambda L$ associated, respectively, with the eigenvalues on and inside the unit circle. From Lemma 3.1 parts F), G), and H) we know that $X_0 = 0$ or $Y_0 = 0$. We shall assume $Y_0 = 0$. The case $X_0 = 0$ can be handled by similar means. Thus, we have

$$X^*Y = \left[\begin{array}{c|c} 0 & X_0^*Y_1 \\ 0 & X_1^*Y_1 \end{array}\right] = \left[\begin{array}{c|c} 0 & 0 \\ 0 & X_1^*Y_1 \end{array}\right]$$

where the second equality is obtained using Lemmas A2 and A3 together with the fact that $X_1^*Y_0 = 0$. Finally, the result follows immediately because the first part of this proof implies $X_1^*Y_1 \geq 0$ and Hermitian.

B) This can be easily proved by extending the proof of the "if" part of [17, Theorem 6] to the case of a generalized eigenproblem to cope with singular state transition matrices. See [26] for further details.

C) The proof will be carried out by contradiction. Suppose that there exists an eigenvalue λ of J such that $|\lambda| > 1$ and let z be the associated eigenvector. Premultiplying (3.19) by z^* and postmultiplying by z, we have

$$(1 - |\lambda|^2)z^*(X^*Y)z = \|D^TXz\|^2 + \|\bar{C}YJz\|^2. \tag{3.20}$$

Since $|\lambda| > 1$ and $X^*Y \geq 0$, then the LHS of (3.20) is nonpositive while the RHS is nonnegative. Thus, this implies the following:

$$D^TXz = 0 \tag{3.21}$$

$$\bar{C}YJz = 0. \tag{3.22}$$

Postmultiplying (3.15) by z and using (3.22) we obtain

$$A^T(Xz) = XJz = \lambda(Xz). \tag{3.23}$$

Since X is nonsingular, $Xz \neq 0$, then (3.21) and (3.23) imply that λ is an unreachable mode of (A, D) which is a contradiction because by assumption the modes of (A, D) outside the unit circle are reachable. Hence, $|\lambda_i| \leq 1$, $i = 1, 2, \cdots, n$. $\nabla\nabla\nabla$

The previous theorem gives a sufficient condition for the solution $\Sigma = YX^{-1}$ of the ARE to be nonnegative definite, since provided X is nonsingular, $\Sigma = YX^{-1} = (X^*)^{-1}(X^*Y)X^{-1}$.

Theorem 3.1 together with Lemma 3.1 allows us to establish the existence and uniqueness of the strong solution of the ARE subject only to a detectability assumption. This represents a weakening of the assumptions of previous work dealing with singular state transition matrix [12]–[16] which assumed both detectability and stabilizability.

We consider in the following the existence conditions and properties of the strong solution of the ARE. The theorem discussed below is an extension of the results of [11] to the case of singular transition matrices and strengthens the results to include both necessary and sufficient conditions.

Theorem 3.2:

A) The strong solution of the ARE exists and is unique if and only if (C, A) is detectable.

B) The strong solution is the only nonnegative definite solution of the ARE if and only if (C, A) is detectable and (A, D) has no unreachable mode outside the unit circle.

C) The strong solution coincides with the stabilizing solution if and only if (C, A) is detectable and (A, D) has no unreachable modes on the unit circle.

D) The stabilizing solution is positive definite if and only if (C, A) is detectable and (A, D) has no unreachable mode inside, or on the unit circle.

Proof: Part A): The proof of the "if" part parallels the proof of a similar result in [11] once Lemma 3.1 and Theorem 3.1 are established.

To prove the converse, let Σ be the strong solution of the ARE. From Lemma 3.1 A) there exists a choice $\left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$ of principal vectors of $M - \lambda L$ such that X^{-1} exists and $\Sigma = YX^{-1}$. As Σ is a strong solution, from Lemma 3.1 C) and I) it follows that the choice of principal vectors is unique.

The proof of the detectability of (C, A) will be made by contradiction. Assume that (C, A) has an unobservable mode λ_i of multiplicity p_i such that $|\lambda_i| \geq 1$. From Lemma 3.1 G) it follows that $\beta_i = 1/\lambda_i$ is an eigenvalue of $M - \lambda L$ with multiplicity p_i and the associated principal vectors z_i, $i = 0, \cdots, p_i - 1$ have the form $z_i = \left[\begin{smallmatrix}0\\ w_i\end{smallmatrix}\right]$ where w_i is an n-vector. Since $|\beta_i| \leq 1$, then the uniqueness of the choice of $\left[\begin{smallmatrix}X\\ Y\end{smallmatrix}\right]$ implies that z_i, $i = 0, p_i - 1$ must be used in the construction of $\left[\begin{smallmatrix}X\\ Y\end{smallmatrix}\right]$ and so X is singular, contradicting the fact that X^{-1} exists. Hence, (C, A) is detectable.

Part B):

Sufficiency: From part A) the strong solution Σ_s exists and is unique. Let P be another nonnegative definite solution of the ARE. From Lemma 3.1 A) it follows that there exists a choice of principal vectors of $M - \lambda L$ such that $P = YX^{-1}$. Therefore, Theorem 3.1 C) and Lemma 3.1 C) imply that P is also a strong solution. Since the strong solution is unique, it follows that $P = \Sigma_s$.

Necessity: This can be proved in an identical manner as used for the "only if" part of [17, Theorem 7].

Part C):

The "if" part of the proof follows immediately from part A) of this theorem and part H) of Lemma 3.1.

To prove the converse, let Σ_s be the stabilizing solution of the ARE. Using Lemma 3.1 part I) this implies $|\lambda_i| \neq 1$, $i = 1, \cdots, 2n$ where λ_i are the eigenvalues of $M - \lambda L$. Otherwise, by Lemma 3.1 parts I) and C), the eigenvalues λ_j with $|\lambda_j| = 1$ were also eigenvalues of the steady-state transition matrix $\bar{A}$ of the filter, which is a contradiction. The nonexistence of unreachable modes of (A, D) on the unit circle follows from Lemma 3.1 H) and the detectability of (C, A) follows from part A) of this theorem.

Part D):

For the "if" part, the existence of a unique stabilizing solution Σ_s follows immediately from the detectability of (C, A) and from the fact that (A, D) has no unreachable modes on the unit circle. Using the ARE we have

$$\Sigma_s - \bar{A}_s \Sigma_s \bar{A}_s^T = KK^T + DD^T \tag{3.28}$$

where

$$K = A\Sigma_s \bar{C}^T(\bar{C}\Sigma_s\bar{C}^T + I)^{-1} \tag{3.29}$$

and $\bar{A}_s$ is the steady-state filter transition matrix corresponding to Σ_s, i.e., (3.30)

$$\bar{A}_s = A - K\bar{C}. \tag{3.31}$$

Since $|\lambda_i(\bar{A}_s)| < 1$, it follows from (3.28)

$$\Sigma_s = \sum_{k=0}^{\infty} \bar{A}_s^k(FF^T)(\bar{A}_s^T)^k \tag{3.32}$$

where

$$FF^T = KK^T + DD^T. \tag{3.33}$$

To prove that $\Sigma_s > 0$, initially note that if λ is an unreachable mode of $(\bar{A}_s, F)$, then λ is also an unreachable mode of (A, D). This implies that the modes of $(\bar{A}_s, F)$ inside or on the unit circle are reachable. Otherwise, (A, D) has an unreachable mode inside or on the unit circle, which is a contradiction. Since the modes of $\bar{A}_s$ are inside the unit circle, it follows that $(\bar{A}_s, F)$ is completely reachable and thus $\Sigma_s > 0$.

For the "only if" part the detectability of (C, A) follows immediately from part A) of this theorem, whereas the proof that the modes of (A, D) inside, or on the unit circle are reachable can be carried out by contradiction using a discrete-time version of [7, Theorem 9]. $\nabla\nabla\nabla$

Part D) of the above theorem represents an improvement over previous results since only the reachability of the modes of (A, D) inside or on the unit circle is required instead of the complete reachability of (A, D) as in [4]. Also it is worth noting that Theorem 3.2 provides a theoretical foundation for the numerical solution given in [24] for the ARE's of nonstabilizing systems (in the filtering context) which were explicitly excluded in [12]–[16].

IV. Convergence of the Riccati Difference Equation

In this section we present new results on the convergence of the solution of the RDE to the strong solution of the ARE for nonstabilizing systems (in the filtering context). They include both necessary and sufficient conditions and can handle systems with singular state transition matrices.

Theorem 4.1: Subject to $\Sigma_0 > 0$, then the detectability of (C, A) and the nonexistence of unreachable modes of (A, D) on the unit circle are necessary and sufficient conditions for

$$\lim_{t\to\infty} \Sigma(t) = \Sigma_s \text{ (exponentially fast),}$$

where $\Sigma(t)$ is the solution of the RDE with initial condition Σ_0 and Σ_s is the unique stabilizing solution of the ARE.

Proof: The "if" part parallels the proof of Theorem 4.2 of [11] once the properties of the strong solution of the ARE are established as in Section III.

The "only if" part follows from Theorem 3.2 part C). $\nabla\nabla\nabla$

Theorem 4.2: Subject to $(\Sigma_0 - \Sigma_s) \geq 0$, then $\lim_{t\to\infty} \Sigma(t) = \Sigma_s$ if and only if (C, A) is detectable, where $\Sigma(t)$ is the solution of the RDE with initial condition Σ_0 and Σ_s is the unique strong solution of the ARE.

Proof: (The proof in the initial version of this paper was extremely long. We give here a new and much shorter proof based on an idea suggested to us by D. J. Clements.)

We first recall a device in [19] and inspired by Nishimura [20]. Let $\Sigma(t+1) = f(\Sigma(t), Q)$ represent the mapping (2.7) for fixed A and $\bar{C}$, and with $Q \triangleq DD^T$ and let $\hat{\Sigma}(t+1) = f(\hat{\Sigma}(t), \hat{Q})$ represent the same mapping (i.e., same A and $\bar{C}$) for different $\hat{\Sigma}(t)$ and $\hat{Q}$. Then (see [19] or [21]),

$$\hat{\Sigma}(t+1) \geq \Sigma(t+1) \text{ if } \hat{\Sigma}(t) \geq \Sigma(t) \text{ and } \hat{Q} \geq Q. \tag{4.1}$$

Now assume that (C, A) is detectable and consider the family of RDE's

$$\Sigma^{(k)}(t+1) = A\Sigma^{(k)}(t)A^T - A\Sigma^{(k)}(t)\bar{C}^T[\bar{C}\Sigma^{(k)}(t)\bar{C}^T + I]^{-1}\bar{C}\Sigma^{(k)}(t)A^T + Q_k \tag{4.2a}$$

$$\Sigma^{(k)}(0) = \Sigma_0 \geq \Sigma_s \qquad k = 1, 2, \cdots \tag{4.2b}$$

$$Q_k = DD^T + \frac{1}{k}I \qquad k = 1, 2, \cdots. \tag{4.2c}$$

Then, with Σ_s the unique strong solution of (2.11) and $\Sigma(t)$ the solution of (2.7), we have

$$\Sigma_s \leq \Sigma(t) \leq \Sigma^{(k+1)}(t) \leq \Sigma^{(k)}(t) \quad t \geq 0, \; k = 1, 2, \cdots \tag{4.3}$$

where all three inequalities follow from the device (4.1).

Since $Q_k > 0$, taking limits as $t \to \infty$, (4.3) implies that for any fixed $x \in \mathbb{R}^n$

$$x^T\Sigma_s x \leq \lim_{t\to\infty} \sup x^T\Sigma(t)x \leq x^T\Sigma^{(k-1)}x \leq x^T\Sigma^{(k)}x \tag{4.4}$$

where $\Sigma^{(k)}$ satisfies the ARE

$$\Sigma^{(k)} = A\Sigma^{(k)}A^T - A\Sigma^{(k)}\bar{C}^T(\bar{C}\Sigma^{(k)}\bar{C}^T + I)^{-1}\bar{C}\Sigma^{(k)}A^T + Q_k. \tag{4.5}$$

Since $Q_k > 0$ for all k, $(A, Q_k^{1/2})$ is stabilizable, and since $(\bar{C}, A)$ is detectable, it follows by Theorem 3.2 that

$$|\lambda_i\{A - A\Sigma^{(k)}\bar{C}^T(\bar{C}\Sigma^{(k)}\bar{C}^T + I)^{-1}\bar{C}\}| < 1 \qquad i = 1, \cdots, n;\; k = 1, 2, \cdots. \tag{4.6}$$

We now take the limits in (4.4) for $k \to \infty$. Since $\Sigma^{(k)}$ is monotonically nonincreasing (in the sense that $\Sigma^{(k+1)} \le \Sigma^{(k)}$) and bounded below by Σ_s, it converges to a constant $\bar{\Sigma}$ (see [22, Theorem 1, p. 169]). Its limit $\bar{\Sigma}$ satisfies the ARE. In addition, since $\Sigma^{(k)}$ is an analytic function of A, $\bar{C}$ and Q_k, [27] it follows by (4.6) that

$$|\lambda_i\{A - A\bar{\Sigma}\bar{C}^T(\bar{C}\bar{\Sigma}\bar{C}^T + I)^{-1}\bar{C}\}| \le 1 \qquad i = 1, \cdots, n. \tag{4.7}$$

Therefore, $\bar{\Sigma} = \Sigma_s$ by the uniqueness of the strong solution. Hence,

$$x^T\Sigma_s x \le \lim_{t\to\infty} \sup\, x^T\Sigma(t)x \le x^T\Sigma_s x \qquad \forall x. \tag{4.8}$$

The same argument holds if $\lim_{t\to\infty}$ sup is replaced by $\lim_{t\to\infty}$ inf in (4.4) up to (4.8). Therefore, $\lim_{t\to\infty} x^T\Sigma(t)x$ exists for all x

$$\lim_{t\to\infty} x^T\Sigma(t)x = x^T\Sigma_s x \qquad \forall\, x. \tag{4.9}$$

It follows easily that $\lim_{t\to\infty} \Sigma(t) = \Sigma_s$ (Take unit vectors for x, then sums of two unit vectors.) This concludes the sufficiency part. Necessity follows from Theorem 3.2 A). $\nabla\nabla\nabla$

Combining the results of Theorems 4.1 and 4.2 also yields the following new result.

Corollary 4.1: Subject to either $\Sigma_0 > 0$ or $\Sigma_0 \ge \Sigma_s$, then the detectability of (C, A) and the nonexistence of unreachable modes of (A, D) on the unit circle are necessary and sufficient conditions for

$$\lim_{t\to\infty} \Sigma(t) = \Sigma_s$$

where $\Sigma(t)$ is the solution of the RDE with initial condition Σ_0 and Σ_s is the unique stabilizing solution of the ARE.

Proof: The result for $\Sigma_0 > 0$ is established in Theorem 4.1. For $\Sigma_0 \ge \Sigma_s$, $\Sigma(t)$ converges to the unique strong solution of the ARE by Theorem 4.2. Since (A, D) has no unreachable modes on the unit circle, that solution is stabilizing by Theorem 3.2. The converse follows from Theorem 3.2 C). $\nabla\nabla\nabla$

Theorem 4.2 represents an improvement over [11, Theorem 4.3], which required the observability of (C, A) and $\Sigma_0 > \Sigma_s$. The requirement on the initial condition of the RDE was relaxed to $\Sigma_0 \ge \Sigma_s$ and observability was replaced by detectability. Also notice that Corollary 4.1 relaxes the assumptions of Theorem 4.1 because if (A, D) has unreachable modes inside the unit circle, then Σ_s is singular by Theorem 3.2. In that case, Σ_0 need not be positive definite.

V. Optimal Control Interpretations

Although our results were originally motivated by filtering applications, they imply a number of new results for the dual optimal control problem, and these have some interesting consequences.

We consider the following system:

$$x(t+1) = Fx(t) + Gu(t), \qquad x(t_0) = x_0 \tag{5.1}$$

with the cost function

$$J(x_0, N) = x^T(N)\Sigma_f x(N) + \sum_{t=t_0}^{N-1} [x^T(t)Qx(t) + u^T(t)Ru(t)] \tag{5.2}$$

where $\Sigma_f \ge 0$, $Q \ge 0$ and $R > 0$.

Factorizing $Q \triangleq H^TH$ and defining

$$y(t) = Hx(t) \tag{5.3}$$

corresponds to replacing $x^T(t)Qx(t)$ by $y^T(t)y(t)$ in (5.2).

Minimizing $J(x_0, N)$ with respect to $u(t_0), \cdots, u(N-1)$ yields the optimal control law

$$u(t) = -K(t)x(t) \tag{5.4}$$

$$K(t) = (G^T\Sigma(t+1)G + R)^{-1}G^T\Sigma(t+1)F \tag{5.5}$$

where $\Sigma(t)$ obeys a reversed-time RDE

$$\Sigma(t) = F^T\Sigma(t+1)F - F^T\Sigma(t+1)\bar{G} \cdot [\bar{G}^T\Sigma(t+1)\bar{G} + I]^{-1}\bar{G}^T\Sigma(t+1)F + H^TH \tag{5.6}$$

$$\Sigma(N) = \Sigma_f$$

where R has been factorized as in (2.10) and where $\bar{G} \triangleq G(R^{-1/2})^T$; compare to (2.9).

The resulting closed-loop system is

$$x(t+1) = [F - GK(t)]x(t), \qquad x(t_0) = x_0. \tag{5.8}$$

The optimal cost is

$$\min_{u_{t_0}, \cdots, u_{N-1}} J(x_0, N) \triangleq J^*(x_0, N) = x_0^T\Sigma(t_0)x_0. \tag{5.9}$$

The RDE (5.6) is identical to (2.7) provided the following duality transformations are made:

$$A \leftrightarrow F^T,\; \bar{C} \leftrightarrow \bar{G}^T,\; D \leftrightarrow H^T,\; \Sigma_0 \leftrightarrow \Sigma_f. \tag{5.10}$$

The corresponding ARE is

$$\Sigma = F^T\Sigma F - F^T\Sigma\bar{G}[\bar{G}^T\Sigma\bar{G} + I]^{-1}\bar{G}\Sigma F + H^TH. \tag{5.11}$$

Therefore, all the results of Theorems 3.2, 4.1, 4.2, and Corollary 4.1 can be rewritten for this optimal control problem provided the following duality changes are made:

$$(C, A) \text{ detectable} \to (F, G) \text{ stabilizable} \tag{5.12a}$$

$$(A, D) \text{ stabilizable} \to (H, F) \text{ detectable} \tag{5.12b}$$

$$\left.\begin{array}{l}(A, D) \text{ has an unreachable}\\ \text{mode on the unit circle}\end{array}\right\} \to \begin{array}{l}(H, F) \text{ has an unobservable}\\ \text{mode on the unit circle}\end{array} \tag{5.12c}$$

$$\Sigma_0 > 0 \to \Sigma_f > 0 \tag{5.12d}$$

$$\Sigma_0 \ge \Sigma_s \to \Sigma_f \ge \Sigma_s \tag{5.12e}$$

$$t \to \infty \to t \to -\infty. \tag{5.12f}$$

The filtering interpretations of the results are straightforward since the RDE can be iterated simultaneously with the filter state transition equation, but the optimal control interpretations are a little more subtle since the RDE (5.6) is solved backwards in time, while the control gain sequence is applied forward in time [see (5.4)].

For the infinite-time optimal control problem, one may wonder whether it makes any sense to apply a penalty on the final state (i.e., $\Sigma_f \ge 0$) since this state is reached only after an infinite time period. For this reason, most authors assume $\Sigma_f = 0$ for the infinite-time optimal control problem; see, e.g., [23]. Yet our Theorems 4.1 and 4.2 tell us that the asymptotic value of $\Sigma(t)$, and hence the asymptotic cost $\lim J^*_{t_0\to-\infty}(x_0, N)$, may depend on Σ_f. In fact, we show below that in some cases it is important to penalize the final cost in infinite-time optimal control problems.

We illustrate this by two situations which show that our convergence results for the RDE provide interesting insights into the infinite-time regulator problem.

Application 1: Stabilization of an unstable undetectable system. Suppose, to simplify understanding, that the system (5.1) has all its poles strictly outside the unit circle (i.e., $|\lambda_i(F)| > 1$, $i = 1, \cdots, n$) and that it is stabilizable but completely undetectable, i.e., $H = 0$ in (5.3) or, correspondingly $Q = 0$ in (5.2). The optimal control problem corresponds to minimizing

$$J(x_0, N) = x^T(N)\Sigma_f x(N) + \sum_{t=t_0}^{N-1} u^T(t)Ru(t). \quad (5.13)$$

Suppose we want to solve the infinite-time optimal control problem

$$\lim_{t_0 \to -\infty} \min_{u(\cdot)} J(x_0, N). \quad (5.14)$$

With $\Sigma_f = 0$ and F unstable, the states of the system will grow unbounded since the optimal control is clearly $u^*(t) = 0$, $t \geq t_0$. However, Theorem 4.1 shows that by taking $\Sigma_f > 0$, we have

$$\lim_{t \to -\infty} \Sigma(t) = \Sigma_s \quad (5.15)$$

where Σ_s is the strong and in this case the stabilizing solution of the ARE (5.11).

Hence, the corresponding closed-loop state transition matrix is stable, and the optimal cost is $x_0^T\Sigma_s x_0$. The intuitive interpretation is that, because there is a penalty on the final state, then even though this final state is reached infinitely far into the future and although there is no penalty along the way, the controller cannot allow the states to blow up. Hence, the optimal steady state controller is to apply

$$u(t) = -K_s x(t), \ t \geq t_0 \quad (5.16)$$

where

$$K_s = (G^T\Sigma_s G + R)^{-1} G\Sigma_s F. \quad (5.17)$$

In fact, this strategy has the following interesting property.

Lemma 5.1: Consider the system (5.1) with $x_0 \neq 0$ and (F, G) stabilizable, and assume that F has no eigenvalues on the unit circle. If the objective is to obtain $x(N) = 0$ via the infinite-time regulator problem: $\lim_{t_0 \to -\infty} \min_{u(\cdot)} J(x_0, N)$, where

$$J(x_0, N) = \sum_{t=t_0}^{N-1} u^T(t)Ru(t) \quad (5.19)$$

for some $R > 0$, then the optimal control is given by (5.16), (5.17), where Σ_s is the solution of the ARE (5.11).

Proof: If we solve the optimal control problem (5.13), (5.14) with $\Sigma_f \geq \Sigma_s$, we know by the preceding discussion that (5.16), (5.17) is the optimal solution. Since Σ_s is stabilizing by Theorem 3.2, $x(t) \to 0$. Therefore,

$$\lim_{t_0 \to -\infty} \min_{u(\cdot)} J(x_0, N) = \lim_{t_0 \to -\infty} \left[\sum_{t=t_0}^{N-1} u^{*T}(t)Ru^*(t)\right] \quad (5.19)$$

where $u^*(t)$ is given by (5.16). $\nabla\nabla\nabla$

Note that if our objective is to achieve $\lim x(t) = 0$ while minimizing the control energy, then the solution of any other optimal control problem (such as taking $Q > 0$) would be suboptimal.

The result of Lemma 5.1 was obtained in a different formulation in an excellent paper of Willems and Callier [25] (see Theorem 2), which was brought to our attention by a reviewer. That paper examines the relationship between four different infinite horizon LQ-optimal control problems for continuous-time systems. It assumes throughout that (H, F) has no unobservable modes on the $j\omega$-axis. We now conclude with a simple application where (H, F) has an unobservable mode on the unit circle, a case not covered by [25].

Application 2: Consider a very simple scalar control problem

$$x(t+1) = x(t) + u(t), \ x(t_0) = 1. \quad (5.20)$$

Suppose we want to minimize $\lim_{t_0 \to -\infty} J(x_0, N)$, where

$$J(x_0, N) = x^2(N) + \sum_{t=t_0}^{N-1} u^2(t). \quad (5.21)$$

This is an optimal control problem where (F, G) is stabilizable, but (H, F) has an unobservable mode on the unit circle. It appears from (5.20), (5.21) that it should not be possible to obtain

$$\lim_{t_0 \to -\infty} J(x_0, N) = 0 \quad (5.22)$$

since applying no control yields $x(N) = 1$ and applying any control will contribute a positive quantity to the cost function. Yet the optimal cost is $x_0^T\Sigma(t_0)x_0$ and

$$\lim_{t_0 \to -\infty} \Sigma(t_0) = \Sigma_s = 0 \quad (5.23)$$

in accordance with Theorems 4.2 and 3.2. This appears paradoxical.

However, an examination of the solution to the finite-time optimal control problem shows that these conclusions are indeed correct. The finite time optimal control problem gives

$$u^*(t) = -K(t)x(t) = -\frac{1}{\bar{N}+1} = \text{constant} \quad (5.24)$$

$$\Sigma(t) = \frac{1}{N-t+1} = K(t) \quad (5.25)$$

$$x(t) = \frac{N-t+1}{\bar{N}+1}; \ x(N) = \frac{1}{\bar{N}+1} \quad (5.26)$$

$$J^*(x_0, N) = \left(\frac{1}{\bar{N}+1}\right)^2 + \bar{N}\left(\frac{1}{\bar{N}+1}\right)^2 = \frac{1}{\bar{N}+1} \quad (5.27)$$

where $\bar{N} = N - t_0$.

The above equations give a complete explanation of the unexpected results (5.23). If $\bar{N}_1$ is the optimization interval, then the cost increases only from $1/(\bar{N}_1 + 1)$ to $1/(\bar{N}_2 + 1)$ if one takes no action until $\bar{N}_2$ steps from the final time ($\bar{N}_2 < \bar{N}_1$). This example suggests that, for optimal control problems in which F has no repeated roots on the unit circle, and the criterion function penalizes only the control energy and the final state, little additional cost is incurred by indefinitely delaying the initiation of control action. The authors hasten to add, that despite appearances, they have not included this example simply to provide a rationalization for procrastination.

VI. Conclusions

This paper has analyzed the solutions of the discrete Riccati equation for systems which are not necessarily stabilizable. The generalizations relative to previous results are in three directions: a) the results on the algebraic Riccati equation have been established under weaker conditions and include both necessary and sufficient conditions, whereas previously only sufficient conditions were given; b) the convergence of the Riccati difference equation has been studied under weaker conditions, including replacing observability by dectability, weakening the conditions on the initial covariance; and c) all these results have been established for the case of singular state transition matrices. The implications of the results in the dual problem of optimal control have also been studied leading to some interesting conclusions.

Appendix A

Properties of Symplectic Pencils

We give a review of certain definitions and properties of symplectic pencils used in the paper.

Definition A1—Symplectic Pencil [15]: Let M and L be $2n$ real square matrices. The pencil of matrices $P(\lambda) = M - \lambda L$ is called symplectic if $MSM^T = LSL^T$ where

$$S \triangleq \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix} \tag{A.1}$$

and I_n is the identity matrix of dimension n.

Definition A2—Eigenvalue and Generalized Right Principal Vectors: The eigenvalues of a pencil $M - \lambda L$ are the roots of det $(M - \lambda L)$ and the generalized right eigenvector associated with λ is the nonzero vector x satisfying $(M - \lambda L)x = 0$. If λ_i is a root of multiplicity p_i, the generalized right principal vectors x_{i+r}, $r = 0, 1, \cdots, p_i - 1$ with rank $1, 2, \cdots, p_i$ associated with λ_i are the nonzero vectors satisfying

$$(M - \lambda_i L)x_i = 0 \tag{A.2}$$

$$(M - \lambda_i L)x_{i+r} = Lx_{i+r-1}, \qquad r = 1, 2, \cdots, p_i - 1. \tag{A.3}$$

Remark: In the text when there is no possibility of confusion, the terms "right" and "generalized" are dropped.

Definition A3—Generalized Left Principal Vectors: The nonzero row vectors z_{j+s}, $s = 0, 1, \cdots, p_j - 1$ are the generalized left principal vectors with rank $1, 2, \cdots, p_j$ of the pencil $M - \lambda L$ associated with a generalized eigenvalue λ_j of multiplicity p_j if:

$$z_j(M - \lambda_j L) = 0 \tag{A.4}$$

$$z_{j+s}(M - \lambda_j L) = z_{j+s-1}L, \qquad s = 1, 2, \cdots, p_j - 1. \tag{A.5}$$

Lemma A1: If $\lambda_j^* \neq 0$ is an eigenvalue of multiplicity p_j of the symplectic pencil $M - \lambda L$ and the $2n$-row vectors z_{j+s}^*, $s = 0, 1, \cdots, p_j - 1$ are the corresponding generalized left principal vectors, then λ_j^{-1} is also an eigenvalue of the same multiplicity and the associated generalized right principal vectors are given by

$$SL^T z_j; \; (-1)^k \lambda_j^{k+1} SL^T Q_k^j(z), \qquad k = 1, 2, \cdots, p_j - 1 \tag{A.6}$$

where S is as in (A.1) and $Q_k^j(z)$ is a polynomial vector in λ_j of the form

$$Q_k^j(z) = \sum_{l=0}^{k-1} \frac{(k-1)!\lambda_j^{k-l-1}}{(k-1-l)!l!} z_{j+k-l}. \tag{A.7}$$

Proof: By direct substitution. See [26] for further details. $\nabla\nabla\nabla$

Lemma A2: Let z_{j+s}, $s = 0, 1, \cdots, p_j - 1$ and x_{i+r}, $r = 0, 1, \cdots, p_i - 1$ be, respectively, the generalized left and right principal vectors of $M - \lambda L$. Then, without loss of generality, they can be chosen such that

$$z_{j+s}Lx_{i+r} \begin{cases} \neq 0, & i = j, \; s + r = p_i - 1 = p_j - 1 \\ = 0 & \text{otherwise.} \end{cases} \tag{A.8}$$

Proof: It is similar to the proof of the orthogonality between generalized left and right eigenvectors of a matrix. See [26] for further details. $\nabla\nabla\nabla$

Lemma A3: Let z_{i+r}^* and $\hat{x}_{i+r} = \begin{bmatrix} x_{i+r} \\ y_{i+r} \end{bmatrix}$, $r = 0, 1, \cdots, p_i - 1$ be, respectively, the left and right $2n$-principal vectors of the symplectic pencil $M - \lambda L$ associated, respectively, with $(\lambda_i^*)^{-1}$ and λ_i. Let $\omega_j^T = (u_j^T, v_j^T)$, $j = 0, 1, \cdots, q$ be a sequence of vectors decomposed into two n-vectors. If $z_{i+r}^* L\omega_j = 0$ for $r = 0, 1, \cdots, p_i - 1$ and $j = 0, 1, \cdots, q$, then

$$(-y_{i+r}^*, \; x_{i+r}^*) \begin{bmatrix} u_j \\ v_j \end{bmatrix} = 0 \qquad \begin{matrix} r = 0, 1, \cdots, p_i - 1 \\ j = 0, 1, \cdots, q. \end{matrix} \tag{A.9}$$

Proof: The proof can be carried out easily using Lemmas A1 and A2 together with the fact that $S^2 = -I_{2n}$. See [26] for further details. $\nabla\nabla\nabla$

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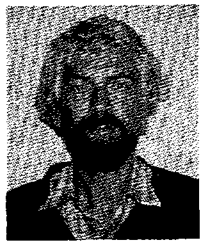

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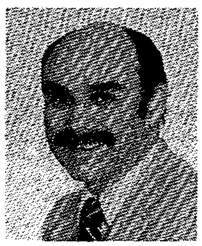

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