



Ricci Almost Solitons And Gradient Ricci Almost Solitons In (k, μ) -Paracontact Geometry

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ABSTRACT: The purpose of this paper is to study Ricci almost soliton and gradient Ricci almost soliton in (k, μ) -paracontact metric manifolds. We prove the non-existence of Ricci almost soliton in a (k, μ) -paracontact metric manifold M with $k < -1$ or $k > -1$ and whose potential vector field is the Reeb vector field ξ . Further, if the metric g of a (k, μ) -paracontact metric manifold M^{2n+1} with $k \neq -1$ is a gradient Ricci almost soliton, then we prove that either the manifold is locally isometric to a product of a flat $(n + 1)$ -dimensional manifold and an n -dimensional manifold of negative constant curvature equal to -4 , or, M^{2n+1} is an Einstein manifold. Finally, an illustrative example is given.

Key Words: (k, μ) -paracontact manifold, Ricci almost soliton, Gradient Ricci almost soliton, Einstein manifold.

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1. Introduction

A Ricci soliton is a natural generalization of an Einstein metric [2]. In a pseudo-Riemannian manifold (M, g) a Ricci soliton is defined by

$$\mathcal{L}_V g + 2S - 2\lambda g = 0, \tag{1.1}$$

where $\mathcal{L}_V g$ is the Lie derivative of g along a vector field V (called the potential vector field), S the Ricci tensor of type $(0, 2)$ and λ a constant. Naturally, a Ricci soliton with V zero or Killing is an Einstein metric. A Ricci soliton on a compact manifold is a gradient Ricci soliton [18]. For details, we refer to Chow and Knopf

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[10], Bejan and Crasmareanu [1] about Ricci solitons and gradient Ricci solitons. Ricci solitons and gradient Ricci solitons on several types of (almost) contact metric manifolds were studied by several authors such as Cho [8,9], De and Matsuyama [11], Deshmukh [12], Hamilton [14], Turan et al [21], Wang [22], Wang et al [23], Yildiz et al [24] and many others.

Recently, Calvaruso and Perrone [4] studied Ricci solitons in three-dimensional paracontact geometry. Also in [5], Calvaruso and Perrone proved that a paracontact metric manifold is H -paracontact if and only if the Reeb vector field is a Ricci eigenvector.

On the other hand, Pigola et al [19] introduced the notion of Ricci almost soliton in the study of Riemannian manifolds, defined by the same equation (1.1) in which λ is a smooth function. The Ricci almost soliton is said to be shrinking, steady or expanding according as λ is positive, zero or negative, respectively. If the complete vector field V is the gradient of a potential function f , then g is said to be a gradient Ricci almost soliton and equation (1.1) takes the form

$$\text{Hess}f + S = \lambda g, \quad (1.2)$$

where $\text{Hess}f$ denotes the Hessian of a smooth function f on M and defined by $\text{Hess}f = \nabla\nabla f$.

Sharma [20] studied Ricci almost solitons in K -contact geometry. Also Ghosh [13] studied Ricci almost solitons and gradient Ricci almost solitons in (k, μ) -contact geometry.

Motivated by these circumstances in this paper we study Ricci almost solitons and gradient Ricci almost solitons in (k, μ) -paracontact metric manifolds. The present paper is organized as follows: Section 2 contains some preliminary results of (k, μ) -paracontact metric manifolds. In Section 3, we prove the non-existence of Ricci almost soliton in a (k, μ) -paracontact metric manifold M with $k < -1$ or $k > -1$ and whose potential vector field is the Reeb vector field ξ . In the next section, we study gradient Ricci almost soliton in a (k, μ) -paracontact metric manifold M^{2n+1} with $k \neq -1$ and prove that if the metric g of M^{2n+1} is a gradient Ricci almost soliton, then either the manifold is locally isometric to a product of a flat $(n+1)$ -dimensional manifold and an n -dimensional manifold of negative constant curvature equal to -4 , or, M^{2n+1} is an Einstein manifold. Finally, an illustrative example is given.

2. Preliminaries on (k, μ) -paracontact Metric Manifolds

A $(2n+1)$ -dimensional smooth manifold M is said to be an almost paracontact manifold if it admits an almost paracontact structure (ϕ, ξ, η) , where ϕ is a $(1, 1)$ -tensor field, ξ a vector field and its dual 1-form η and for any vector field X on M satisfying [15]

- (i) $\phi^2 X = X - \eta(X)\xi$,
- (ii) $\phi(\xi) = 0$, $\eta \circ \phi = 0$, $\eta(\xi) = 1$,
- (iii) the tensor field ϕ induces an almost paracomplex structure on each fibre of $\mathcal{D} = \ker(\eta)$, that is, the eigendistributions \mathcal{D}_ϕ^+ and \mathcal{D}_ϕ^- of ϕ corresponding to the eigenvalues 1 and -1 , respectively, have same dimension n .

An almost paracontact structure is said to be normal [25] if and only if the $(1, 2)$ -type torsion tensor $N_\phi = [\phi, \phi] - 2d\eta \otimes \xi$ vanishes identically, where $[\phi, \phi]$ denotes the Nijenhuis tensor of ϕ . If an almost paracontact manifold M equipped with a pseudo-Riemannian metric g of signature $(n + 1, n)$ such that

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{2.1}$$

for all $X, Y \in \chi(M)$, where $\chi(M)$ is the Lie algebra of all smooth vector fields on the manifold M , then (M, g) is called an almost paracontact metric manifold. An almost paracontact structure is said to be a paracontact structure if $g(X, \phi Y) = d\eta(X, Y)$ with the associated metric g [25]. For any almost paracontact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ admits (at least, locally) a ϕ -basis [25], that is, a pseudo-orthonormal basis of vector fields of the form $\{\xi, E_1, E_2, \dots, E_n, \phi E_1, \phi E_2, \dots, \phi E_n\}$, where $\xi, E_1, E_2, \dots, E_n$ are space-like vector fields and then, by (2.1) vector fields $\phi E_1, \phi E_2, \dots, \phi E_n$ are time-like. In a paracontact metric manifold there exists a symmetric, trace-free $(1, 1)$ -tensor $h = \frac{1}{2}\mathcal{L}_\xi\phi$ satisfying [25]

$$\phi h + h\phi = 0, \quad h\xi = 0, \tag{2.2}$$

$$\nabla_X\xi = -\phi X + \phi hX, \tag{2.3}$$

where ∇ is Levi-Civita connection of the pseudo-Riemannian manifold and for all $X \in \chi(M)$. It is clear that the tensor h satisfies $h = 0$ if and only if ξ is a Killing vector field and then (ϕ, ξ, η, g) is said to be a K -paracontact manifold. An almost paracontact manifold is said to be para-Sasakian if and only if the following condition holds [25]

$$(\nabla_X\phi)Y = -g(X, Y)\xi + \eta(Y)X, \tag{2.4}$$

for any $X, Y \in \chi(M)$. A normal paracontact metric manifold is para-Sasakian and satisfies

$$R(X, Y)\xi = -(\eta(Y)X - \eta(X)Y), \tag{2.5}$$

for any $X, Y \in \chi(M)$, but unlike contact metric geometry the relation (2.5) does not imply that the paracontact manifold is para-Sasakian manifold. Every para-Sasakian manifold is a K -paracontact manifold, but the converse is not always true, as it is shown in three dimensional case [3]. Paracontact metric manifolds have been studied by Cappelletti-Montano et al [6,7], Martin-Molina [16,17] and many others.

According to Cappelletti-Montano et al [6] we have the following definition.

Definition 2.1. *A paracontact metric manifold is said to be (k, μ) -paracontact manifold if the curvature tensor R satisfies*

$$R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY), \tag{2.6}$$

for all vector fields $X, Y \in \chi(M)$ and k, μ are real constants.

In a (k, μ) -paracontact manifold $(M^{2n+1}, \phi, \xi, \eta, g), n > 1$, the following relations hold [6]:

$$h^2 = (k + 1)\phi^2, \quad (2.7)$$

$$(\nabla_X \phi)Y = -g(X - hX, Y)\xi + \eta(Y)(X - hX), \quad \text{for } k \neq -1, \quad (2.8)$$

$$\begin{aligned} QY &= [2(1 - n) + n\mu]Y + [2(n - 1) + \mu]hY \\ &\quad + [2(n - 1) + n(2k - \mu)]\eta(Y)\xi, \quad \text{for } k \neq -1, \end{aligned} \quad (2.9)$$

$$S(X, \xi) = 2nk\eta(X), \quad (2.10)$$

$$Q\xi = 2nk\xi, \quad (2.11)$$

$$\begin{aligned} (\nabla_X h)Y &= -[(1 + k)g(X, \phi Y) + g(X, \phi hY)]\xi \\ &\quad + \eta(Y)\phi h(hX - X) - \mu\eta(X)\phi hY, \quad \text{for } k \neq -1, \end{aligned} \quad (2.12)$$

$$Q\phi - \phi Q = 2[2(n - 1) + \mu]h\phi, \quad (2.13)$$

for any vector fields $X, Y \in \chi(M)$, where Q is the Ricci operator defined by $g(QX, Y) = S(X, Y)$. Making use of (2.3) we have

$$(\nabla_X \eta)Y = g(X, \phi Y) + g(\phi hX, Y), \quad (2.14)$$

for all vector fields $X, Y \in \chi(M)$.

First of all we recall some useful results:

Lemma 2.2. ([26], Theorem 3.3) *Let $M^{2n+1}, n > 1$, be a paracontact metric manifold satisfies $R(X, Y)\xi = 0$, for all $X, Y \in \chi(M)$. Then M^{2n+1} is locally isometric to a product of a flat $(n + 1)$ -dimensional manifold and an n -dimensional manifold of negative constant curvature equal to -4 .*

Lemma 2.3. ([6], p.683, 687) *Let (M, ϕ, ξ, η, g) be a (k, μ) -paracontact metric manifold. Then for any vector fields $X, Y \in \chi(M)$ we have*

$$\begin{aligned} (\nabla_X \phi h)Y &= g(h^2 X - hX, Y)\xi + \eta(Y)(h^2 X - hX) \\ &\quad - \mu\eta(X)hY, \quad \text{for } k > -1 \end{aligned} \quad (2.15)$$

$$\begin{aligned} (\nabla_X \phi h)Y &= (1 + k)g(X, Y)\xi - g(hX, Y)\xi + \eta(Y)(h^2 X - hX) \\ &\quad - \mu\eta(X)hY, \quad \text{for } k < -1. \end{aligned} \quad (2.16)$$

3. Ricci Almost Solitons in (k, μ) -paracontact Metric Manifolds

In this section we discuss about Ricci almost solitons in (k, μ) -paracontact manifolds. We prove the following:

Theorem 3.1. *There does not exist Ricci almost soliton in a (k, μ) -paracontact metric manifold M^{2n+1} ($n > 1$) whose potential vector field is the Reeb vector field ξ with $k < -1$ or $k > -1$.*

Proof: Suppose a (k, μ) -paracontact metric manifold admits a Ricci almost soliton (g, ξ) . Then we have from (1.1)

$$(\mathcal{L}_\xi g)(Y, Z) + 2S(Y, Z) - 2\lambda g(X, Y) = 0. \tag{3.1}$$

This is equivalent to

$$g(\nabla_Y \xi, Z) + g(\nabla_Z \xi, Y) + 2S(Y, Z) - 2\lambda g(Y, Z) = 0. \tag{3.2}$$

Using (2.3) in the above equation implies

$$g(\phi hY, Z) + S(Y, Z) - \lambda g(Y, Z) = 0, \tag{3.3}$$

that is,

$$\phi hY + QY - \lambda Y = 0. \tag{3.4}$$

Taking covariant differentiation of (3.4) with respect to an arbitrary vector field X , we have

$$(\nabla_X \phi h)Y + (\nabla_X Q)Y - (X\lambda)Y = 0. \tag{3.5}$$

Now we split our discussions into two cases:

Case 1. Let $k > -1$. Applying (2.9) and (2.15) in (3.5) we have

$$\begin{aligned} & (1+k)g(X, Y)\xi - g(hX, Y)\xi + (k+1)\eta(Y)X - (k+1)\eta(X)\eta(Y)\xi \\ & - \eta(Y)hX - \mu\eta(X)hY + \{2(n-1) + \mu\}(\nabla_X h)Y \\ & + \{2(n-1) + n(2k - \mu)\}\{(\nabla_X \eta)(Y)\xi + \eta(Y)\nabla_X \xi\} - (X\lambda)Y = 0. \end{aligned} \tag{3.6}$$

With the help of (2.3), (2.12) and (2.14) we obtain from the above equation

$$\begin{aligned} & (1+k)g(X, Y)\xi - g(hX, Y)\xi + (k+1)\eta(Y)X - (k+1)\eta(X)\eta(Y)\xi \\ & - \eta(Y)hX - \mu\eta(X)hY - \{2(n-1) + \mu\}\{(k+1)g(X, \phi Y)\xi + g(X, \phi hY)\xi \\ & - (k+1)\eta(Y)\phi X + \eta(Y)\phi hX + \mu\eta(X)\phi hY\} + \{2(n-1) + n(2k - \mu)\} \\ & \{g(X, \phi Y)\xi + g(\phi hX, Y)\xi - \eta(Y)\phi X + \eta(Y)\phi hX\} - (X\lambda)Y = 0. \end{aligned} \tag{3.7}$$

Contracting X in (3.7) we get

$$(2n+1)(k+1)\eta(Y) = Y\lambda. \tag{3.8}$$

Also putting $Y = Z = \xi$ in (3.3) yields $\lambda = 2nk$, which is a constant. Applying this in (3.8) we have $k = -1$, which is a contradiction as we consider $k > -1$.

Case 2. Let $k < -1$. Making use of (2.9) and (2.16) in (3.5) we have

$$\begin{aligned} & g((k+1)\phi^2 X, Y)\xi - g(hX, Y)\xi + (k+1)\eta(Y)\phi^2 X \\ & - \eta(Y)hX - \mu\eta(X)hY + \{2(n-1) + \mu\}(\nabla_X h)Y \\ & + \{2(n-1) + n(2k - \mu)\}\{(\nabla_X \eta)(Y)\xi + \eta(Y)\nabla_X \xi\} - (X\lambda)Y = 0. \end{aligned} \tag{3.9}$$

Using (2.3), (2.12) and (2.14) in the above equation gives

$$\begin{aligned} & g((k+1)\phi^2 X, Y)\xi - g(hX, Y)\xi + (k+1)\eta(Y)\phi^2 X \\ & - \eta(Y)hX - \mu\eta(X)hY - \{2(n-1) + \mu\}\{(k+1)g(X, \phi Y)\xi + g(X, \phi hY)\xi \\ & - (k+1)\eta(Y)\phi X + \eta(Y)\phi hX + \mu\eta(X)\phi hY\} + \{2(n-1) + n(2k - \mu)\} \\ & \{g(X, \phi Y)\xi + g(\phi hX, Y)\xi - \eta(Y)\phi X + \eta(Y)\phi hX\} - (X\lambda)Y = 0. \end{aligned} \quad (3.10)$$

Contracting X in (3.10) we have

$$2n(k+1)\eta(Y) = Y\lambda. \quad (3.11)$$

Again substituting $Y = Z = \xi$ in (3.3) gives $\lambda = 2nk$, which is a constant. Using this in (3.11) we obtain $k = -1$, which is a contradiction as we consider $k < -1$. Combining the two cases our theorem follows. \square

Next, we prove the following:

Theorem 3.2. *If a (k, μ) -paracontact metric manifold M^{2n+1} ($n > 1$) admits a Ricci almost soliton for $k = -1$ whose potential vector field is the Reeb vector field ξ , then the Ricci almost soliton is expanding with $Q\xi = -2n\xi$.*

Proof: Replacing Y by ξ in (3.4) we get $Q\xi = \lambda\xi$. On the other hand from (2.11) and $k = -1$ we have $Q\xi = -2n\xi$. Thus we obtain $\lambda = -2n$. This shows that the Ricci almost soliton is expanding. \square

Remark 3.3. *Since $\lambda = \text{constant}$, the Ricci almost soliton in a (k, μ) -paracontact metric manifold reduces to a Ricci soliton.*

4. Gradient Ricci Almost Solitons in (k, μ) -paracontact Metric Manifolds

This section is devoted to study gradient Ricci almost soliton in (k, μ) -paracontact manifolds. We prove the following:

Theorem 4.1. *Let (M, g) be a $(2n+1)$ -dimensional ($n > 1$) (k, μ) -paracontact metric manifold with $k \neq -1$. If g is a gradient Ricci almost soliton, then either the manifold is locally isometric to a product of a flat $(n+1)$ -dimensional manifold and an n -dimensional manifold of negative constant curvature equal to -4 , or, M^{2n+1} is an Einstein manifold.*

Proof: Let (M, g) be a $(2n+1)$ -dimensional (k, μ) -paracontact metric manifold and g a gradient Ricci almost soliton. Then (1.2) reduces to

$$\nabla_Y Df = -QY + \lambda Y, \quad (4.1)$$

for any $Y \in \chi(M)$, where D denotes the gradient operator of g . From (4.1) it follows that

$$R(X, Y)Df = (\nabla_Y Q)X - (\nabla_X Q)Y - (Y\lambda)X + (X\lambda)Y. \quad (4.2)$$

Taking covariant differentiation of (2.9) along arbitrary vector field X and using (2.14) we have

$$(\nabla_X Q)Y = \{2(n-1) + n(2k - \mu)\}[g(X, \phi Y)\xi + g(\phi hX, Y)\xi - \eta(Y)\phi X + \eta(Y)\phi hX] + \{2(n-1) + \mu\}(\nabla_X h)Y. \quad (4.3)$$

Applying (2.12) in (4.3) gives

$$\begin{aligned} (\nabla_X Q)Y - (\nabla_Y Q)X &= \{2(n-1) + \mu\}[-(k+1)\{2g(X, \phi Y)\xi + \eta(X)\phi Y \\ &\quad - \eta(Y)\phi X\} + (1-\mu)\{\eta(X)\phi hY - \eta(Y)\phi hX\}] \\ &\quad + \{2(n-1) + n(2k - \mu)\}[2g(X, \phi Y)\xi + \eta(X)\phi Y \\ &\quad - \eta(Y)\phi X + \eta(Y)\phi hX - \eta(X)\phi hY]. \end{aligned} \quad (4.4)$$

Making use of (4.4) we have from (4.2)

$$\begin{aligned} R(X, Y)Df &= \{2(n-1) + \mu\}[(k+1)\{2g(X, \phi Y)\xi + \eta(X)\phi Y \\ &\quad - \eta(Y)\phi X\} + (\mu-1)\{\eta(X)\phi hY - \eta(Y)\phi hX\}] \\ &\quad - \{2(n-1) + n(2k - \mu)\}[2g(X, \phi Y)\xi + \eta(X)\phi Y \\ &\quad - \eta(Y)\phi X + \eta(Y)\phi hX - \eta(X)\phi hY] - (Y\lambda)X + (X\lambda)Y. \end{aligned} \quad (4.5)$$

Taking inner product of (4.5) with ξ we obtain

$$\begin{aligned} g(R(X, Y)Df, \xi) &= 2(\mu - 2k + \mu k + n\mu)g(X, \phi Y) \\ &\quad - (Y\lambda)\eta(X) + (X\lambda)\eta(Y). \end{aligned} \quad (4.6)$$

Substituting $X = \xi$ in (4.6) yields

$$g(R(\xi, Y)Df, \xi) = (\xi\lambda)\eta(Y) - (Y\lambda). \quad (4.7)$$

Also from (2.6) it follows that

$$R(\xi, Y)X = k\{g(X, Y)\xi - \eta(X)Y\} + \mu\{g(hX, Y)\xi - \eta(X)hY\}. \quad (4.8)$$

Taking inner product of (4.8) with ξ gives

$$g(R(\xi, Y)Df, \xi) = kg(Y, Df - (\xi f)\xi) + \mu g(hY, Df). \quad (4.9)$$

In view of (4.7) and (4.9) we have

$$kg(Y, Df - (\xi f)\xi) + \mu g(hY, Df) - (\xi\lambda)\eta(Y) + (Y\lambda) = 0, \quad (4.10)$$

from which we obtain

$$kDf - k(\xi f)\xi + \mu hDf + D\lambda - (\xi\lambda)\xi = 0. \quad (4.11)$$

Contracting X in (4.2) and noticing that the scalar curvature of the manifold is constant, we have $QDf = -2nD\lambda$. Applying this in (4.11) gives

$$2nkDf + 2n\mu hDf = QDf + 2n(k(\xi f) + (\xi\lambda))\xi. \quad (4.12)$$

Taking inner product of (4.12) with ξ and since $Q\xi = 2nk\xi$ it follows that $k(\xi f) + (\xi\lambda) = 0$. Using this in the above equation one gets

$$2nkDf + 2n\mu hDf = QDf. \quad (4.13)$$

Putting $X = \xi$ in (4.1) we obtain

$$\nabla_\xi Df = (\lambda - 2nk)\xi. \quad (4.14)$$

Replacing X by ϕX and Y by ϕY in (4.6) and (2.6), respectively, then comparing the right hand sides we have

$$(\mu - 2k + \mu k + n\mu)g(\phi X, Y) = 0. \quad (4.15)$$

Since $d\eta \neq 0$, it follows from the above equation

$$k = \frac{\mu(n+1)}{2-\mu}. \quad (4.16)$$

Differentiating (4.13) along ξ implies

$$2nk\nabla_\xi Df + 2n\mu(\nabla_\xi h)Df + 2n\mu h(\nabla_\xi Df) = (\nabla_\xi Q)Df + Q(\nabla_\xi Df). \quad (4.17)$$

Making use of (2.12), (4.3) and (4.14) in (4.17) we obtain

$$\mu\{\mu(2n-1) - 2(n-1)\}h\phi Df = 0. \quad (4.18)$$

Operating h on (4.18) and since $k \neq -1$ one obtains

$$\mu\{\mu(2n-1) - 2(n-1)\}\phi Df = 0. \quad (4.19)$$

Thus we consider the following cases:

Case1. If $\mu = 0$, then from (4.16) it follows that $k = 0$. Consequently (2.6) gives $R(X, Y)\xi = 0$. Therefore, using Lemma 2.2 we can state M^{2n+1} is locally isometric to a product of a flat $(n+1)$ -dimensional manifold and an n -dimensional manifold of negative constant curvature equal to -4 .

Case2. If $\phi Df = 0$. Applying ϕ on both sides we obtain

$$Df = (\xi f)\xi. \quad (4.20)$$

Taking differentiation of (4.20) along any arbitrary vector field X , we have $\nabla_X Df = X(\xi f)\xi + (\xi f)(-\phi X + \phi hX)$. Replacing X by ϕX and taking inner product with ϕY we have

$$g(\nabla_{\phi X} Df, \phi Y) = -(\xi f)\{g(X, \phi Y) + g(hX, \phi Y)\}. \quad (4.21)$$

Interchanging X and Y in the above equation yields

$$g(\nabla_{\phi Y} Df, \phi X) = -(\xi f)\{g(Y, \phi X) + g(hY, \phi X)\}. \quad (4.22)$$

Applying Poincaré's lemma: On a contractible manifold, all closed forms are exact. Therefore $d^2 f(X, Y) = 0$, for all $X, Y \in \chi(M)$. From which we have

$$XY(f) - YX(f) - [X, Y]f = 0,$$

that is,

$$Xg(\text{grad}f, Y) - Yg(\text{grad}f, X) - g(\text{grad}f, [X, Y]) = 0.$$

This is equivalent to

$$\nabla_X g(\text{grad}f, Y) - g(\text{grad}f, \nabla_X Y) - \nabla_Y g(\text{grad}f, X) + g(\text{grad}f, \nabla_Y X) = 0.$$

Since $\nabla g = 0$, the above equation yields

$$g(\nabla_X \text{grad}f, Y) - g(\nabla_Y \text{grad}f, X) = 0,$$

that is, $g(\nabla_X Df, Y) = g(\nabla_Y Df, X)$. Replacing X by ϕX and Y by ϕY in the foregoing equation we obtain $g(\nabla_{\phi X} Df, \phi Y) = g(\nabla_{\phi Y} Df, \phi X)$. Applying this in (4.21) and (4.22) we have $(\xi f)g(X, \phi Y) = 0$, that is, $(\xi f)d\eta(X, Y) = 0$. Since $d\eta \neq 0$, it follows that $\xi f = 0$. Consequently from (4.20) we obtain $Df = 0$, this implies f is constant. Therefore from (4.1) we have

$$S(X, Y) = \lambda g(X, Y).$$

This shows the manifold is an Einstein manifold.

Case3. If $\mu(2n - 1) - 2(n - 1) = 0$, that is, $\mu = \frac{2(n-1)}{2n-1}$. Using (4.16) we get $k = n - \frac{1}{n}$. From (2.9) and (4.13) we obtain

$$(2(1 - n) + n\mu - 2nk)(Df - (\xi f)\xi) + (2(n - 1) + \mu - 2n\mu)hDf = 0. \quad (4.23)$$

Making use of $\mu = \frac{2(n-1)}{2n-1}$ and $k = n - \frac{1}{n}$ in the above equation and noticing $n > 1$ we have $Df = (\xi f)\xi$. Proceeding in the same way as in Case 2 we obtain the manifold is an Einstein manifold. This completes the proof of our theorem. \square

5. Example of a 5-dimensional (k, μ) -paracontact Metric Manifold

In this section we give an example of a 5-dimensional (k, μ) -paracontact metric manifold such that $k = -2$ and $\mu = 2$. In [6], the authors construct an example of a 5-dimensional (k, μ) -paracontact metric manifold. With the help of that example we construct a new example as follows:

Let \mathfrak{g} be the Lie algebra of a Lie group G admits a basis $\{e_1, e_2, e_3, e_4, e_5\}$ such that [6]

$$\begin{aligned} [e_1, e_5] &= e_1 + e_2, [e_2, e_5] = e_1 + e_2, [e_3, e_5] = -e_3 + e_4, \\ [e_4, e_5] &= e_3 - e_4, [e_1, e_2] = e_1 + e_2, [e_1, e_3] = e_2 + e_4 - 2e_5, \\ [e_1, e_4] &= e_2 + e_3, [e_2, e_3] = e_1 - e_4, [e_2, e_4] = e_1 - e_3 + 2e_5, \\ [e_3, e_4] &= -e_3 + e_4. \end{aligned}$$

We consider the metric such that

$$\begin{aligned} g(e_1, e_1) &= g(e_4, e_4) = g(e_5, e_5) = 1, \\ g(e_2, e_2) &= g(e_3, e_3) = -1 \text{ and } g(e_i, e_j) = 0, \text{ for } i \neq j. \end{aligned}$$

Setting $\xi = e_5$ and denote by η its dual 1-form. We define a tensor ϕ by $\phi e_1 = e_3$, $\phi e_2 = e_4$, $\phi e_3 = e_1$, $\phi e_4 = e_2$, $\phi e_5 = 0$. Therefore we have $\phi^2 X = X - \eta(X)\xi$ and $g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y)$. Thus (ϕ, ξ, η, g) makes G a paracontact metric manifold.

Using the well known Koszul's formula we have the following:

$$\begin{aligned} \nabla_{e_1} e_5 &= e_1 - e_3, \nabla_{e_2} e_5 = e_2 - e_4, \nabla_{e_3} e_5 = -e_1 - e_3, \nabla_{e_4} e_5 = -e_2 - e_4, \\ \nabla_{e_5} e_1 &= -e_2 - e_3, \nabla_{e_5} e_2 = -e_1 - e_4, \nabla_{e_5} e_3 = -e_1 - e_4, \nabla_{e_5} e_4 = -e_2 - e_3, \\ \nabla_{e_1} e_1 &= e_2 - e_5, \nabla_{e_1} e_2 = e_1, \nabla_{e_1} e_3 = e_4 - e_5, \nabla_{e_1} e_4 = e_3, \\ \nabla_{e_2} e_1 &= -e_2, \nabla_{e_2} e_2 = -e_1 + e_5, \nabla_{e_2} e_3 = -e_4, \nabla_{e_2} e_4 = -e_3 + e_5, \\ \nabla_{e_3} e_1 &= -e_2 + e_5, \nabla_{e_3} e_2 = -e_1, \nabla_{e_3} e_3 = -e_4 - e_5, \nabla_{e_3} e_4 = -e_3, \\ \nabla_{e_4} e_1 &= -e_2, \nabla_{e_4} e_2 = -e_1 - e_5, \nabla_{e_4} e_3 = -e_4, \nabla_{e_4} e_4 = -e_3 + e_5, \nabla_{e_5} e_5 = 0. \end{aligned}$$

Comparing the above relations with (2.3) we get

$$he_1 = e_3, he_2 = e_4, he_3 = -e_1, he_4 = -e_2, he_5 = 0.$$

Using the formula $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$, we may calculate the following:

$$\begin{aligned} R(e_1, e_2)e_1 &= e_2 - e_4, R(e_1, e_2)e_2 = e_1 - e_3, R(e_1, e_2)e_3 = e_2 - e_4, \\ R(e_1, e_2)e_4 &= e_1 - e_3, R(e_1, e_3)e_1 = -4e_3, R(e_1, e_3)e_2 = -2e_4, \\ R(e_1, e_3)e_3 &= -4e_1, R(e_1, e_3)e_4 = -2e_2, R(e_1, e_4)e_1 = e_2 - e_4, \\ R(e_1, e_4)e_2 &= e_1 + e_3, R(e_1, e_4)e_3 = e_4 - e_2, R(e_1, e_4)e_4 = e_1 + e_3, \\ R(e_1, e_5)e_1 &= 2e_5, R(e_1, e_5)e_3 = 2e_5, R(e_1, e_5)e_5 = 2e_3 - 2e_1, \\ R(e_2, e_3)e_1 &= -e_2 - e_4, R(e_2, e_3)e_2 = e_3 - e_1, R(e_2, e_3)e_3 = -e_2 - e_4, \\ R(e_2, e_3)e_4 &= e_1 - e_3, R(e_2, e_4)e_1 = 2e_3, R(e_2, e_4)e_2 = 4e_4, \\ R(e_2, e_4)e_3 &= 2e_1, R(e_2, e_4)e_4 = 4e_2, R(e_2, e_5)e_2 = -2e_5, \\ R(e_2, e_5)e_4 &= -2e_5, R(e_2, e_5)e_5 = 2e_4 - 2e_2, R(e_3, e_4)e_1 = e_2 + e_4, \\ R(e_3, e_4)e_2 &= e_1 + e_3, R(e_3, e_4)e_3 = -e_2 - e_4, R(e_3, e_4)e_4 = -e_1 - e_3, \\ R(e_3, e_5)e_1 &= 2e_5, R(e_3, e_5)e_3 = -2e_5, R(e_3, e_5)e_5 = -2e_1 - 2e_3, \\ R(e_4, e_5)e_2 &= -2e_5, R(e_4, e_5)e_4 = 2e_5, R(e_4, e_5)e_5 = -2e_2 - 2e_4. \end{aligned}$$

With the help of the expressions of the curvature tensor we conclude that the manifold is a (k, μ) -paracontact metric manifold with $k = -2$ and $\mu = 2$. Also from the above expressions we obtain the following:

$$S(e_1, e_1) = S(e_4, e_4) = -4, S(e_2, e_2) = S(e_3, e_3) = S(e_5, e_5) = 0.$$

Using the above results it can be easily verified that such a manifold does not satisfy the Equation (3.1). Thus Theorem 3.1 is verified.

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