PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 122, Number 2, October 1994

RICCI CURVATURE AND A CRITERION FOR SIMPLE-CONNECTIVITY ON THE SPHERE

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(Communicated by Christopher Croke)

ABSTRACT. From the recent work of Osgood and Stowe on the Schwarzian derivative for conformal maps between Riemannian manifolds we derive a sharp sufficient condition for a domain on the sphere to be simply-connected. We show further that a less restrictive form of the condition yields a uniform lower bound for the length of closed geodesics.

INTRODUCTION

Osgood and Stowe have recently defined a notion of Schwarzian derivative for conformal mappings of Riemannian manifolds which generalizes the classical operator for analytic functions in the plane [O-S1]. As in complex analysis, where the Schwarzian derivative has been central as a means of characterizing conditions for global univalence, these authors establish in [O-S2] an injectivity criterion for conformal local diffeomorphisms ψ of a Riemannian n-manifold (M, g) to the standard sphere S^n. The univalence of ψ follows from a bound on the norm of the Schwarzian derivative by geometric quantities of M (Theorem 1.1). This result allows a unified approach to a vast class of injectivity theorems in the plane, as different criteria can be derived from it on a given domain just by changing the metric g conformally. Indeed, in [O-S2] the authors obtain as corollaries, with M the unit disc in the plane and g alternately the euclidean and hyperbolic metric, two classical conditions of Nehari. Most of the known criteria, including a recent injectivity result of Epstein [Ep], and some new conditions on the unit disc and simply-connected domains are derived in [Ch1] from Theorem 1.1.

We shall show in this paper that a local diffeomorphism ψ as before satisfying a particular form of the criterion in [O-S2] forces the manifold M to be simply-connected (Theorem 2.1). Our main result, a sharp criterion for simple-connectivity for domains in S^n, will appear as a reformulation of Theorem 2.1 when using conformal invariance. This allows one to translate the existence of ψ to that of a conformal metric on a domain $\Omega \subset S^n$ with the property that

Received by the editors October 9, 1991 and, in revised form, January 8, 1993.
1991 *Mathematics Subject Classification*. Primary 53A30; Secondary 34C10.
Key words and phrases. Schwarzian derivative, univalence, conformal metric, simple-connectivity, closed geodesic, trace-free Ricci, scalar curvature.

the norm of the trace-free Ricci tensor is bounded above by a dimensional constant multiple c_n of the scalar curvature. The effect of changing the constant c_n to c is, in our opinion, quite remarkable. With $c < c_n$ one can construct a reflection $\Lambda: S^n \to S^n$ which maps Ω to $S^n - \overline{\Omega}$ and which fixes pointwise $\partial\Omega$ [Ch2]. The mapping Λ is quasiconformal in the sense of Ahlfors, i.e., the ratio of the largest and smallest eigenvalue of the symmetrized differential $(D\Lambda)(D\Lambda)^t$ is uniformly bounded. Here we show that for $c > c_n$ the criterion yields a uniform lower bound for the length of closed geodesics in Ω.

1. Preliminaries

We shall present in this section enough of the work in [O-S1] so that we can state the injectivity criterion in [O-S2]. We will omit proofs and refer the reader to the sources for more details.

Let M be an n-dimensional Riemannian manifold with metric g. When $M = R^n$, we will denote by g_0 the euclidean metric and g_1 will stand for the standard metric on the sphere S^n.

Given a conformal metric $\hat{g} = e^{2\varphi}g$ on M, Osgood and Stowe define the Schwarzian tensor of $\hat{g}$ with respect to g as the symmetric, trace-free $(0, 2)$-tensor

$$B_g(\varphi) = \mathrm{Hess}(\varphi) - d\varphi \otimes d\varphi - \frac{1}{n}(\Delta\varphi - |\operatorname{grad}\varphi|^2)g,$$

where the metric dependent quantities on the right-hand side are computed with respect to the metric g. We mention here that the tensor $B_g(\varphi)$ appears as the term by which the trace-free part of the Ricci tensor changes under the conformal change of metric g to $e^{2\varphi}g$. When ψ is a conformal local diffeomorphism of (M, g) to another Riemannian manifold (N, g'), then $\psi^*(g') = e^{2\varphi}g$ with $\varphi = \log|D\psi|$. The Schwarzian derivative of ψ is defined by

$$S_g(\psi) = B_g(\varphi).$$

For an analytic map ψ in the plane, with $g = g' = g_0$, $\varphi = \log|\psi'|$. Computing in standard coordinates one gets

$$S_g(\psi) = \begin{pmatrix} \mathrm{Re}\{\psi, z\} & -\mathrm{Im}\{\psi, z\} \\ -\mathrm{Im}\{\psi, z\} & -\mathrm{Re}\{\psi, z\} \end{pmatrix},$$

where $\{\psi, z\} = (\frac{\psi''}{\psi'})' - \frac{1}{2}(\frac{\psi''}{\psi'})^2$ is the classical Schwarzian derivative.

On M, the conformal metric $\hat{g} = e^{2\varphi}g$ is called Möbius with respect to g if $B_g(\varphi) = 0$, and so a conformal local diffeomorphism ψ is said to be Möbius if $S_g(\psi) = 0$. If φ and σ are smooth functions on M, then there is an important identity:

$$B_g(\varphi + \sigma) = B_g(\varphi) + B_{\hat{g}}(\sigma) \tag{1.1}$$

where $\hat{g} = e^{2\varphi}g$. In a chain of conformal local diffeomorphisms $\psi_1 : (M, g) \to N_1, g')$ and $\psi_2 : (N_1, g') \to (N_2, g'')$, equation (1.1) can be formulated as

$$S_g(\psi_2 \circ \psi_1) = S_g(\psi_1) + \psi_1^*(S_{g'}(\psi_2)). \tag{1.2}$$

This recovers the classical formula of the Schwarzian derivative of a composition of analytic maps in the plane.

By $\|B_g(\varphi)\|$ we mean the norm of the Schwarzian tensor $B_g(\varphi)$ with respect to g, as a bilinear form on each tangent space, that is,

$$\|B_g(\varphi)\| = \max\{|B_g(\varphi)(X, Y)| : |X| = |Y| = 1\}.$$

In some cases, we will need to consider the norm of $B_g(\varphi)$ in a metric $\hat{g} = e^{2\sigma} g$ conformal to g. Then

$$\|B_g(\varphi)\|_{\hat{g}} = e^{-2\sigma}\|B_g(\varphi)\|.$$

With this, we present the theorem in [O-S2].

Theorem 1.1. *Let (M, g) be a Riemannian manifold of dimension $n \geq 2$ and $\psi: (M, g) \to (S^n, g_1)$ a conformal local diffeomorphism. Suppose that the scalar curvature of M is bounded above by $n(n-1)K$ for some $K \in R$, and that any two points in M can be joined by a geodesic of length $< \delta$ for some $0 < \delta \leq \infty$. If*

$$\|S_g(\psi)\| \leq \frac{2\pi^2}{\delta^2} - \frac{1}{2}K$$

then ψ is injective.

With M the unit disc in the plane and g alternately the euclidean and hyperbolic metric, Osgood and Stowe derive from this theorem the classical criteria of Nehari; namely, that

$$|\{\psi, z\}| \leq \frac{\pi^2}{2} \quad \text{or} \quad |\{\psi, z\}| \leq \frac{1}{(1-|z|^2)^2}, \quad \text{all } |z| < 1,$$

imply that ψ is injective.

We point out that in Theorem 1.1, the target (S^n, g_1) can be replaced by the standard hyperbolic space H^n or (R^n, g_0). This follows from the transformation law (1.1) and the fact that both g_1 and the hyperbolic metric are Möbius with respect to the euclidean metric. Finally, let $\mathrm{scal}(g)$ be the scalar curvature of g. It is easy to see that the proof given by Osgood and Stowe works equally well when assuming that at each point in M the norm of the Schwarzian derivative of ψ is bounded above by

$$\frac{2\pi^2}{\delta^2} - \frac{\mathrm{scal}(g)}{2n(n-1)}.$$

2. A criterion for simple-connectivity

To begin with, we note the following consequence of Theorem 1.1.

Theorem 2.1. *Let (M, g) be a complete Riemannian manifold of dimension $n \geq 2$ and $\psi: (M, g) \to (S^n, g_1)$ a conformal local diffeomorphism. If*

$$\|S_g(\psi)\| \leq -\frac{\mathrm{scal}(g)}{2n(n-1)}$$

then M is simply-connected.

Even though one can give an independent proof of this theorem, it will follow from Theorem 2.2. Let us consider the image $\Omega = \psi(M)$. Under the hypotheses of Theorem 2.1, ψ is injective and we let $g_2 = e^{2\rho} g_1 = \phi^*(g)$, where $\phi = \psi^{-1}$. The addition formula (1.2) implies

$$S_g(\psi) = -\psi^*(S_{g_1}(\phi)) = -\psi^*(B_{g_1}(\rho))$$

and therefore

$$\|S_g(\psi)\| = \|B_{g_1}(\rho)\|_{g_2}.$$

Our main result is

Theorem 2.2. *Let $\Omega \subset S^n$ be a domain with a complete metric $g_2 = e^{2\rho} g_1$. If*

$$\|B_{g_1}(\rho)\|_{g_2} \leq -\frac{\operatorname{scal}(g_2)}{2n(n-1)}$$

then Ω is simply-connected.

Proof. Let $\tilde{\Omega}$ be the universal cover of Ω with covering map π and metric $\tilde{g} = \pi^*(g_2)$. We consider π as a conformal map from $(\tilde{\Omega}, \tilde{g})$ into (S^n, g_1). We shall show that

$$\|S_{\tilde{g}}(\pi)\| = \|B_{g_1}(\rho)\|_{g_2}, \tag{2.1}$$

which by Theorem 1.1 implies the univalence of π and, consequently, the theorem.

We have

$$\pi^*(g_1) = \pi^*(e^{-2\rho} g_2) = e^{-2(\rho\circ\pi)}\tilde{g};$$

hence

$$S_{\tilde{g}}(\pi) = B_{\tilde{g}}(-\rho\circ\pi) = \pi^*(B_{g_2}(-\rho)) = -\pi^*(B_{g_1}(\rho)).$$

This proves (2.1).

Remarks. (1) Theorem 2.2 can be stated as well for domains in R^n with g_0 as the background metric. The hypotheses of the theorem implicitly require that $\operatorname{scal}(g_2) \leq 0$. Moreover one can show that g_2 has nonpositive curvature. Indeed, since g_2 is conformally flat, its Weyl tensor vanishes, and now a classical decomposition of the Riemann curvature tensor allows us to compute sectional curvatures solely in terms of $\operatorname{scal}(g_2)$ and the trace-free part of the Ricci tensor. If X, Y are orthonormal tangent vectors in the metric g_2, then the sectional curvature $K(X, Y)$ of g_2 is given by

$$K(X, Y) = \frac{\operatorname{scal}(g_2)}{n(n-1)} + B_{g_1}(\rho)(X, X) + B_{g_1}(\rho)(Y, Y).$$

Therefore $K(X, Y) \leq 0$ and so Ω as in the theorem is actually diffeomorphic to R^n.

(2) The condition that g_2 be complete can be relaxed; what one really needs is that any two points in $\tilde{\Omega}$ can be joined by some geodesic in the metric $\tilde{g}$. Also, the theorem can be stated slightly more generally: without the assumption of completeness, if

$$\|B_{g_1}(\rho)\|_{g_2} \leq \frac{2\pi^2}{\delta^2} - \frac{\operatorname{scal}(g)}{2n(n-1)}$$

then there are no closed (not even nonsmoothly closing) geodesics in the metric g_2 of length $< \delta$.

(3) Theorem 2.2 is sharp. For $n = 2$ this can be verified by taking in the plane the ring $R_1 < |z| < R_2$ with its Poincaré metric $g = e^{2\varphi} g_0$. This metric satisfies the inequality

$$\|B_{g_0}(\varphi)\|_g \leq -(1+\varepsilon)\frac{\operatorname{scal}(g)}{4}$$

where $\varepsilon(\log(R_2/R_1)/\pi)^2$ can be made arbitrarily small.

In higher dimensions we consider a similar example: a hyperbolic solid torus. Let $n \geq 2$ and let Ω be the domain in R^{n+1} given by

$$\{(x_1\cos\theta, x_1\sin\theta, x_2, \ldots, x_n) : (x_1-a)^2 + x_2^2 + \cdots + x_n^2 < 1\},$$

where $a > 1$. To simplify notation, we write $r^2 = (x_1-a)^2 + x_2^2 + \cdots + x_n^2$. Let $g = e^{2\varphi} g_0$ with $\varphi = -\log(1-r^2)$. This metric is complete and we will show that given $\varepsilon > 0$, the inequality

$$\|B_{g_0}(\varphi)\|_g \leq -(1+\varepsilon)\frac{\operatorname{scal}(g)}{2n(n+1)} \tag{2.2}$$

will hold throughout Ω provided the constant a is sufficiently large. With respect to the coordinates $x_1, \ldots, x_n, \theta$ the tensor $B_{g_0}(\varphi)$ is a diagonal matrix with eigenvalues $\lambda = \lambda_1 = \cdots = \lambda_n$ and λ_{n+1} given by

$$\lambda = \frac{2a}{(n+1)(1-r^2)x_1}$$

and

$$\lambda_{n+1} = \frac{-2nax_1}{(n+1)(1-r^2)}.$$

A standard formula gives

$$\begin{aligned} -e^{2\varphi}\frac{\operatorname{scal}(g)}{n(n+1)} &= \frac{2}{n+1}\Delta\varphi + \frac{n-1}{n+1}|\operatorname{grad}\varphi|^2 \\ &= \frac{4}{(1-r^2)^2} - \frac{4a}{(n+1)(1-r^2)x_1}. \end{aligned}$$

The vector fields $\frac{\partial}{\partial x_i}$ have euclidean length 1 while $\frac{\partial}{\partial\theta}$ has length x_1. It follows that $\|B_{g_0}(\varphi)\|_g = e^{-2\varphi}x_1^{-2}\lambda_{n+1}$. If $c = 1+\varepsilon$ then the desired inequality is

$$\frac{2na}{(n+1)(1-r^2)x_1} \leq (1+\varepsilon)\left\{\frac{2}{(1-r^2)^2} - \frac{2a}{(n+1)(1-r^2)x_1}\right\}$$

which simplifies to

$$\frac{(n+1+\varepsilon)a}{(n+1)x_1} \leq \frac{1+\varepsilon}{1-r^2}.$$

Since $a - 1 < x_1$ the last inequality will hold if $a \geq \frac{(n+1)(1+\varepsilon)}{n\varepsilon}$.

3. Short geodesics

The proof of Theorem 1.1 relies on translating the given inequality on ψ to a differential inequality along geodesics of a suitably chosen test function w. To be more precise, let γ be a geodesic joining two given points x, $y \in M$.

The function w is nonnegative and constructed so that it vanishes at p if and only if $\psi(x) = \psi(p)$ [O-S2]. Along γ

$$w'' \geq -w\left(\|S_g(\psi)\| + \frac{\operatorname{scal}(g)}{2n(n-1)}\right) + \frac{(w')^2}{2w}$$

whenever $w > 0$. The estimate on $\|S_g(\psi)\|$ as in Theorem 2.2 implies that $(w^{1/2})'' \geq 0$ and therefore $\psi(x) \neq \psi(y)$.

We assume now that g is complete and consider the case when ψ satisfies the estimate

$$\|S_g(\psi)\| \leq -c\frac{\operatorname{scal}(g)}{2n(n-1)}$$

for some $c > 1$. Suppose also that $\frac{\operatorname{scal}(g)}{n(n-1)} \geq -s > -\infty$. Then

$$(w^{1/2})'' \geq -\frac{(c-1)s}{4}w^{1/2}.$$

A standard Sturm comparison theorem guarantees that w cannot vanish again before time

$$d = \frac{2\pi}{\sqrt{s(c-1)}}.$$

In other words, if $\psi(x) = \psi(y)$ for $x \neq y$, then the distance between these two points is at least equal to d. We reformulate this as:

Theorem 3.1. *Let $\Omega \subset S^n$ be a domain with a complete metric $g_2 = e^{2\varphi}g_1$. Assume that*

$$-\infty < -s \leq \frac{\operatorname{scal}(g)}{n(n-1)} \leq 0.$$

If for some $c > 1$

$$\|B_{g_1}(\varphi)\|_{g_2} \leq -c\frac{\operatorname{scal}(g)}{2n(n-1)} \tag{3.1}$$

then any closed geodesic in Ω has length at least d.

The ring domain $R_1 < |z| < R_2$ with its hyperbolic metric shows that (3.1) is sharp.

Acknowledgments

I would like to thank the referee for pointing out some mistakes in the original computations in the main example, as well as for other thoughtful comments.

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