

RICCI CURVATURE OF INTEGRAL SUBMANIFOLDS OF AN \mathcal{S} -SPACE FORM

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ABSTRACT. Involving the Ricci curvature and the squared mean curvature, we obtain a basic inequality for an integral submanifold of an \mathcal{S} -space form. By polarization, we get a basic inequality for Ricci tensor also. Equality cases are also discussed. By giving a very simple proof we show that if an integral submanifold of maximum dimension of an \mathcal{S} -space form satisfies the equality case, then it must be minimal. These results are applied to get corresponding results for C -totally real submanifolds of a Sasakian space form and for totally real submanifolds of a complex space form.

1. Introduction

One of the most fundamental problems in submanifold theory is the following: Establish simple relationships between the main extrinsic invariants and the main intrinsic invariants of a submanifold. In [7], B.-Y. Chen established a sharp relationship between the Ricci curvature and the squared mean curvature for a submanifold in a Riemannian space form with arbitrary codimension. In [8], he gave the corresponding version of this inequality for totally real submanifolds in a complex space form. We find corresponding results for C -totally real submanifolds of a Sasakian space form in [10], [11] and [12].

The concept of framed metric structure unifies the concepts of almost Hermitian and almost contact metric structures. In particular, an \mathcal{S} -structure generalizes Kaehler and Sasakian structure. In [1], D. Blair discusses principal toroidal bundles and generalizes the Hopf fibration to give a canonical example of an \mathcal{S} -manifold playing the role of complex projective space in Kaehler geometry and the odd-dimensional sphere in Sasakian geometry. An \mathcal{S} -manifold

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of constant f -sectional curvature c is called an \mathcal{S} -space form $\widetilde{M}(c)$ [5], which generalizes the complex space form and Sasakian space form.

Motivated by the result of Chen in [7], recently in [9], a general basic inequality involving the Ricci curvature and the squared mean curvature of a submanifold in any Riemannian manifold is established and its several applications are presented. Using this inequality, in the present paper, we find a basic inequality for integral submanifolds of an \mathcal{S} -space form $\widetilde{M}(c)$ and apply this to recover the already known inequalities for totally real submanifolds in complex space forms and C -totally real submanifolds in Sasakian space forms. The paper is organized as follows. In section 2, we recall a brief account of Ricci curvature, k -Ricci curvature, scalar curvature in a Riemannian manifold and basic formulas and definitions for a submanifold. Then, we recall the result of [9] giving a general basic inequality involving the Ricci curvature and the squared mean curvature of a submanifold in any Riemannian manifold. Section 3 presents a brief account of framed metric manifold leading to \mathcal{S} -space forms. In section 4, we give a very simple way to present a basic inequality for integral submanifolds of an \mathcal{S} -space form $\widetilde{M}(c)$. Then, the already known inequalities for totally real submanifolds in complex space forms and C -totally real submanifolds in Sasakian space forms become direct consequences. In section 5, we mainly prove that an integral submanifold of maximum dimension of an \mathcal{S} -space form $\widetilde{M}(c)$ satisfying the equality case becomes minimal. Then, we derive the same conclusion for Lagrangian submanifold of a complex space form and C -totally real submanifold of maximum dimension of a Sasakian space form.

2. Ricci curvature of submanifolds

Let M be an n -dimensional Riemannian manifold. Let $\{e_1, \dots, e_k\}$, $2 \leq k \leq n$, be an orthonormal basis of a k -plane section Π_k of T_pM . If $k = n$ then $\Pi_n = T_pM$; and if $k = 2$ then Π_2 is a plane section of T_pM . For a fixed $i \in \{1, \dots, k\}$, a k -Ricci curvature of Π_k at e_i , denoted $\text{Ric}_{\Pi_k}(e_i)$, is defined by [7]

$$(1) \quad \text{Ric}_{\Pi_k}(e_i) = \sum_{j \neq i}^k K_{ij},$$

where K_{ij} is the sectional curvature of the plane section spanned by e_i and e_j . An n -Ricci curvature $\text{Ric}_{T_pM}(e_i)$ is the usual Ricci curvature of e_i , denoted $\text{Ric}(e_i)$. Thus for any orthonormal basis $\{e_1, \dots, e_n\}$ for T_pM and for a fixed $i \in \{1, \dots, n\}$, we have

$$\text{Ric}_{T_pM}(e_i) \equiv \text{Ric}(e_i) = \sum_{j \neq i}^n K_{ij}.$$

The scalar curvature $\tau(\Pi_k)$ of the k -plane section Π_k is given by

$$(2) \quad \tau(\Pi_k) = \sum_{1 \leq i < j \leq k} K_{ij}.$$

Geometrically, $\tau(\Pi_k)$ is the scalar curvature of the image $\exp_p(\Pi_k)$ of Π_k at p under the exponential map at p . The scalar curvature $\tau(p)$ of M at p is identical with the scalar curvature of the tangent space T_pM of M at p , that is, $\tau(p) = \tau(T_pM)$.

Let M be an n -dimensional submanifold of an m -dimensional Riemannian manifold \widetilde{M} equipped with a Riemannian metric \widetilde{g} . We use the inner product notation $\langle \cdot, \cdot \rangle$ for both the metrics \widetilde{g} of \widetilde{M} and the induced metric g on the submanifold M . The Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y) \quad \text{and} \quad \widetilde{\nabla}_X N = -A_N X + \nabla_X^\perp N$$

for all $X, Y \in TM$ and $N \in T^\perp M$, where $\widetilde{\nabla}$, ∇ and ∇^\perp are respectively the Riemannian, induced Riemannian and induced normal connections in \widetilde{M} , M and the normal bundle $T^\perp M$ of M respectively, and σ is the second fundamental form related to the shape operator A by $\langle \sigma(X, Y), N \rangle = \langle A_N X, Y \rangle$. The equation of Gauss is given by

$$(3) \quad R(X, Y, Z, W) = \widetilde{R}(X, Y, Z, W) + \langle \sigma(X, W), \sigma(Y, Z) \rangle - \langle \sigma(X, Z), \sigma(Y, W) \rangle$$

for all $X, Y, Z, W \in TM$, where \widetilde{R} and R are the Riemann curvature tensors of \widetilde{M} and M respectively. The curvature tensor R^\perp of the normal bundle of M is defined by

$$R^\perp(X, Y)N = \nabla_X^\perp \nabla_Y^\perp N - \nabla_Y^\perp \nabla_X^\perp N - \nabla_{[X, Y]}^\perp N$$

for all $X, Y \in TM$ and $N \in T^\perp M$. If $R^\perp = 0$, then the normal connection ∇^\perp of M is said to be *flat*.

The mean curvature vector H is given by $H = \frac{1}{n} \text{trace}(\sigma)$. The submanifold M is *totally geodesic* in \widetilde{M} if $\sigma = 0$, and *minimal* if $H = 0$. If $\sigma(X, Y) = g(X, Y)H$ for all $X, Y \in TM$, then M is *totally umbilical*.

The *relative null space* of M at p is defined by [7]

$$\mathcal{N}_p = \{X \in T_pM \mid \sigma(X, Y) = 0 \text{ for all } Y \in T_pM\},$$

which is also known as the *kernel of the second fundamental form* at p [8].

Now, let $\{e_1, \dots, e_n\}$ be an orthonormal basis of the tangent space T_pM and e_r belongs to an orthonormal basis $\{e_{n+1}, \dots, e_m\}$ of the normal space $T_p^\perp M$. We put

$$\sigma_{ij}^r = \langle \sigma(e_i, e_j), e_r \rangle \quad \text{and} \quad \|\sigma\|^2 = \sum_{i, j=1}^n \langle \sigma(e_i, e_j), \sigma(e_i, e_j) \rangle.$$

Let K_{ij} and \widetilde{K}_{ij} denote the sectional curvature of the plane section spanned by e_i and e_j at p in the submanifold M and in the ambient manifold \widetilde{M} respectively. Thus, we can say that K_{ij} and \widetilde{K}_{ij} are the “intrinsic” and “extrinsic” sectional curvature of the $\text{Span}\{e_i, e_j\}$ at p . In view of (3), we get

$$(4) \quad K_{ij} = \widetilde{K}_{ij} + \sum_{r=n+1}^m (\sigma_{ii}^r \sigma_{jj}^r - (\sigma_{ij}^r)^2).$$

From (4) it follows that

$$(5) \quad 2\tau(p) = 2\widetilde{\tau}(T_p M) + n^2 \|H\|^2 - \|\sigma\|^2,$$

where $\widetilde{\tau}(T_p M)$ denotes the scalar curvature of the n -plane section $T_p M$ in the ambient manifold \widetilde{M} . Thus, we can say that $\tau(p)$ and $\widetilde{\tau}(T_p M)$ are the “intrinsic” and “extrinsic” scalar curvature of the submanifold at p respectively.

We denote the set of unit vectors in $T_p M$ by $T_p^1 M$; thus

$$T_p^1 M = \{X \in T_p M \mid \langle X, X \rangle = 1\}.$$

Now, we recall the following result from [9].

Theorem 2.1. *Let M be an n -dimensional submanifold of a Riemannian manifold \widetilde{M} . Then the following statements are true.*

(a) *For $X \in T_p^1 M$ we have*

$$(6) \quad \text{Ric}(X) \leq \frac{n^2}{4} \|H\|^2 + \widetilde{\text{Ric}}_{(T_p M)}(X),$$

where $\widetilde{\text{Ric}}_{(T_p M)}(X)$ is the n -Ricci curvature of $T_p M$ at $X \in T_p^1 M$ with respect to the ambient manifold \widetilde{M} .

(b) *The equality case of (6) is satisfied by $X \in T_p^1 M$ if and only if*

$$(7) \quad \sigma(X, X) = \frac{n}{2} H(p) \quad \text{and} \quad \sigma(X, Y) = 0$$

for all $Y \in T_p M$ such that $\langle X, Y \rangle = 0$.

(c) *The equality case of (6) holds for all $X \in T_p^1 M$ if and only if either (1) p is a totally geodesic point or (2) $n = 2$ and p is a totally umbilical point.*

From Theorem 2.1, we immediately have the following

Corollary 2.2. *Let M be an n -dimensional submanifold of a Riemannian manifold. For $X \in T_p^1 M$ any two of the following three statements imply the remaining one.*

(a) *The mean curvature vector $H(p)$ vanishes.*

(b) *The unit vector X belongs to the relative null space \mathcal{N}_p .*

(c) *The unit vector X satisfies the equality case of (6), namely*

$$(8) \quad \text{Ric}(X) = \frac{1}{4} n^2 \|H\|^2 + \widetilde{\text{Ric}}_{(T_p M)}(X).$$

3. \mathcal{S} -space forms

Let \widetilde{M} be a $(2m + s)$ -dimensional framed metric manifold [17] (also known as framed f -manifold [13] or almost r -contact metric manifold [15]) with a framed metric structure $(f, \xi_\alpha, \eta^\alpha, \widetilde{g})$, $\alpha \in \{1, \dots, s\}$, that is, f is a $(1, 1)$ tensor field defining an f -structure of rank $2m$; ξ_1, \dots, ξ_s are vector fields; η^1, \dots, η^s are 1-forms and \widetilde{g} is a Riemannian metric on \widetilde{M} such that for all $X, Y \in T\widetilde{M}$ and $\alpha, \beta \in \{1, \dots, s\}$

$$(9) \quad f^2 = -I + \eta^\alpha \otimes \xi_\alpha, \quad \eta^\alpha(\xi_\beta) = \delta_\beta^\alpha, \quad f(\xi_\alpha) = 0, \quad \eta^\alpha \circ f = 0,$$

$$(10) \quad \langle fX, fY \rangle = \langle X, Y \rangle - \sum_\alpha \eta^\alpha(X)\eta^\alpha(Y),$$

$$(11) \quad \Omega(X, Y) \equiv \langle X, fY \rangle = -\Omega(Y, X), \quad \langle X, \xi_\alpha \rangle = \eta^\alpha(X),$$

where \langle, \rangle denotes the inner product of the metric \widetilde{g} . A framed metric structure is an \mathcal{S} -structure [1] if the Nijenhuis tensor of f equals $-2d\eta^\alpha \otimes \xi_\alpha$ and $\Omega = d\eta^\alpha$ for all $\alpha \in \{1, \dots, s\}$.

When $s = 1$, a framed metric structure is an almost contact metric structure, while an \mathcal{S} -structure is a Sasakian structure. When $s = 0$, a framed metric structure is an almost Hermitian structure, while an \mathcal{S} -structure is a Kaehler structure. If a framed metric structure on \widetilde{M} is an \mathcal{S} -structure then it is known [1] that

$$(12) \quad (\widetilde{\nabla}_X f)Y = \sum_\alpha (\langle fX, fY \rangle \xi_\alpha + \eta^\alpha(Y)f^2X),$$

$$(13) \quad \widetilde{\nabla} \xi_\alpha = -f, \quad \alpha \in \{1, \dots, s\}.$$

The converse may also be proved. In case of Sasakian structure (that is, $s = 1$), (12) implies (13). In Kaehler case (that is, $s = 0$), we get $\widetilde{\nabla} f = 0$. For $s > 1$, examples of \mathcal{S} -structures are given in [1], [2] and [4]. Thus, the bundle space of a principal toroidal bundles over a Kaehler manifold with certain conditions is an \mathcal{S} -manifold. Thus, a generalization of the Hopf fibration $\pi' : S^{2m+1} \rightarrow PC^m$ is a canonical example of an \mathcal{S} -manifold playing the role of complex projective space in Kaehler geometry and the odd-dimensional sphere in Sasakian geometry.

A plane section in $T_p\widetilde{M}$ is a f -section if there exists a vector $X \in T_p\widetilde{M}$ orthogonal to ξ_1, \dots, ξ_s such that $\{X, fX\}$ span the section. The sectional curvature of a f -section is called a f -sectional curvature. It is known that [5]

in an \mathcal{S} -manifold of constant f -sectional curvature c

$$\begin{aligned}
 (14) \quad & \widetilde{R}(X, Y)Z \\
 &= \sum_{\alpha, \beta} \{ \eta^\alpha(X) \eta^\beta(Z) f^2 Y - \eta^\alpha(Y) \eta^\beta(Z) f^2 X \\
 &\quad - \langle fX, fZ \rangle \eta^\alpha(Y) \xi_\beta + \langle fY, fZ \rangle \eta^\alpha(X) \xi_\beta \} \\
 &\quad + \frac{c+3s}{4} \{ - \langle fY, fZ \rangle f^2 X + \langle fX, fZ \rangle f^2 Y \} \\
 &\quad + \frac{c-s}{4} \{ \langle X, fZ \rangle fY - \langle Y, fZ \rangle fX + 2 \langle X, fY \rangle fZ \}
 \end{aligned}$$

for all $X, Y, Z \in T\widetilde{M}$, where \widetilde{R} is the curvature tensor of \widetilde{M} . An \mathcal{S} -manifold of constant f -sectional curvature c is called an \mathcal{S} -space form $\widetilde{M}(c)$.

When $s = 1$, an \mathcal{S} -space form $\widetilde{M}(c)$ reduces to a Sasakian space form $\widetilde{M}(c)$ [3] and (14) reduces to

$$\begin{aligned}
 \widetilde{R}(X, Y)Z &= \frac{c+3}{4} \{ \langle Y, Z \rangle X - \langle X, Z \rangle Y \} \\
 &\quad + \frac{c-1}{4} \{ \langle X, fZ \rangle fY - \langle Y, fZ \rangle fX + 2 \langle X, fY \rangle fZ \\
 &\quad \quad + \eta(X) \eta(Z) Y - \eta(Y) \eta(Z) X \\
 &\quad \quad + \langle X, Z \rangle \eta(Y) \xi - \langle Y, Z \rangle \eta(X) \xi \},
 \end{aligned}$$

where $\xi_1 \equiv \xi$ and $\eta^1 \equiv \eta$. When $s = 0$, an \mathcal{S} -space form $\widetilde{M}(c)$ becomes a complex space form and (14) moves to

$$\begin{aligned}
 4\widetilde{R}(X, Y)Z &= c \{ \langle Y, Z \rangle X - \langle X, Z \rangle Y \\
 &\quad + \langle X, fZ \rangle fY - \langle Y, fZ \rangle fX + 2 \langle X, fY \rangle fZ \}.
 \end{aligned}$$

4. Ricci curvature of integral submanifolds

Let \widetilde{M} be an \mathcal{S} -manifold equipped with an \mathcal{S} -structure $(f, \xi_\alpha, \eta^\alpha, \widetilde{g})$. A submanifold M of \widetilde{M} is an *integral submanifold* if $\eta_\alpha(X) = 0, \alpha = 1, \dots, s$, for every tangent vector X . A submanifold M of \widetilde{M} is an *anti-invariant submanifold* if $f(TM) \subseteq T^\perp M$. An integral submanifold is identical with an anti-invariant submanifold normal to the structure vector fields ξ_1, \dots, ξ_s . In particular case of $s = 1$, an integral submanifold M of a Sasakian manifold is a *C-totally real submanifold* [16]. It is known that [6] an n -dimensional integral submanifold M , of an \mathcal{S} -manifold \widetilde{M} of dimension $(2n + s)$, is of constant curvature s if and only if the normal connection is flat.

First, we give the following Lemma.

Lemma 4.1. *Let M be an n -dimensional integral submanifold of an \mathcal{S} -space form $\widetilde{M}(c)$. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of the tangent space $T_p M$.*

Then

$$(15) \quad \tilde{K}_{ij} = \frac{1}{4}(c + 3s),$$

$$(16) \quad \widetilde{\text{Ric}}_{(T_p M)}(e_i) = \frac{1}{4}(n - 1)(c + 3s),$$

$$(17) \quad \tilde{\tau}(T_p M) = \frac{1}{8}n(n - 1)(c + 3s).$$

Proof. Equation (15) follows from (14). Using $\widetilde{\text{Ric}}_{(T_p M)}(e_i) = \sum_{j \neq i}^n \tilde{K}_{ij}$ in (15), we get (16). Next, using $2\tilde{\tau}(T_p M) = \sum_{i=1}^n \widetilde{\text{Ric}}_{(T_p M)}(e_i)$, from (16) we get (17). \square

Now, we have the following Theorem.

Theorem 4.2. *If M is an n -dimensional integral submanifold of an S -space form $\tilde{M}(c)$, then the following statements are true.*

(a) *For $X \in T_p^1 M$, it follows that*

$$(18) \quad \text{Ric}(X) \leq \frac{1}{4} \{n^2 \|H\|^2 + (n - 1)(c + 3s)\}.$$

(b) *The equality case of (18) is satisfied by $X \in T_p^1 M$ if and only if (7) is true. If $H(p) = 0$, $X \in T_p^1 M$ satisfies equality in (18) if and only if $X \in \mathcal{N}_p$.*

(c) *The equality case of (18) holds for all $X \in T_p^1 M$ if and only if either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.*

Proof. Using (16) in (6), we find the inequality (18). Rest of the proof is straightforward. \square

By polarization, from Theorem 4.2, we derive

Theorem 4.3. *Let M be an n -dimensional integral submanifold of an S -space form $\tilde{M}(c)$. Then the Ricci tensor S satisfies*

$$(19) \quad S \leq \frac{1}{4} \{n^2 \|H\|^2 + (n - 1)(c + 3s)\} g,$$

where g is the induced Riemannian metric on M . The equality case of (19) is true if and only if either M is a totally geodesic submanifold or M is a totally umbilical surface.

When $s = 0$, we have the following two results.

Theorem 4.4. *If M is an n -dimensional totally real submanifold (or isotropic submanifold) of a complex space form $\tilde{M}(c)$, then the following statements are true.*

(a) *It follows that*

$$(20) \quad \text{Ric}(X) \leq \frac{1}{4} \{n^2\|H\|^2 + (n-1)c\}, \quad X \in T_p^1M.$$

(b) *The equality case of (20) is satisfied by $X \in T_p^1M$ if and only if (7) is true. If $H(p) = 0$, $X \in T_p^1M$ satisfies equality in (20) if and only if $X \in \mathcal{N}_p$.*

(c) *The equality case of (20) holds for all $X \in T_p^1M$ if and only if either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.*

Theorem 4.5. *If M is an n -dimensional totally real submanifold (or isotropic submanifold) of a complex space form $\widetilde{M}(c)$, then the following statements are true.*

(a) *It follows that*

$$(21) \quad S \leq \frac{1}{4} \{n^2\|H\|^2 + (n-1)c\}g.$$

(b) *The equality case of (21) holds identically if and only if either M is totally geodesic submanifold or M is a totally umbilical surface.*

For $s = 1$, we again have the following two results.

Theorem 4.6. *If M is an n -dimensional C -totally real submanifold of a Sasakian space form $\widetilde{M}(c)$, then the following statements are true.*

(a) *It follows that*

$$(22) \quad \text{Ric}(X) \leq \frac{1}{4} \{n^2\|H\|^2 + (n-1)(c+3)\}, \quad X \in T_p^1M.$$

(b) *The equality case of (22) is satisfied by $X \in T_p^1M$ if and only if (7) is true. If $H(p) = 0$, $X \in T_p^1M$ satisfies equality in (22) if and only if $X \in \mathcal{N}_p$.*

(c) *The equality case of (22) holds for all $X \in T_p^1M$ if and only if either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.*

(d) *The equality case of (23) holds identically if and only if either M is totally geodesic submanifold or M is a totally umbilical surface.*

Theorem 4.7. *If M is an n -dimensional C -totally real submanifold of a Sasakian space form $\widetilde{M}(c)$, then the following statements are true.*

(a) *It follows that*

$$(23) \quad S \leq \frac{1}{4} \{n^2\|H\|^2 + (n-1)(c+3)\}g.$$

(b) *The equality case of (23) holds identically if and only if either M is totally geodesic submanifold or M is a totally umbilical surface.*

It is known that (Theorem 4, [14]) if M is an n -dimensional compact minimal C -totally real submanifold of a Sasakian space form $M^{2n+1}(c)$, $c > -3$, such that M has positive sectional curvature, then M is totally geodesic. Therefore, in view of Theorem 4.7, we have the following

Theorem 4.8. *An n -dimensional compact minimal C -totally real submanifold of a Sasakian space form $M^{2n+1}(c)$, $c > -3$ with positive sectional curvature is an Einstein manifold and satisfies $4S = (n - 1)(c + 3)g$.*

The inequality (22) is the inequality (2.1) in Theorem 2.1 of [12]. The inequality (23) is the inequality (9) in Theorem 3.1 of [10]. The inequality (21) is the inequality (2.1) in Theorem 1 of [8]. Here, we find the proofs very much simplified.

5. Minimality of integral submanifolds of maximum dimension

We already know the following result [6]. If M is an n -dimensional integral submanifold of any $(2m + s)$ -dimensional \mathcal{S} -space form $\widetilde{M}(c)$, then the following four statements are equivalent: (i) M is totally geodesic. (ii) M is of constant curvature $\frac{1}{4}(c + 3s)$. (iii) The Ricci tensor is $\frac{1}{4}(n - 1)(c + 3s)g$. (iv) The scalar curvature is $\frac{1}{4}n(n - 1)(c + 3s)$. In Theorem 5.2, we find a condition for minimality.

Now, we begin with the following

Theorem 5.1. *Let M be an n -dimensional integral submanifold of a $(2n + s)$ -dimensional \mathcal{S} -space form $\widetilde{M}(c)$. If a unit vector of T_pM satisfies the equality case of (18), then $H(p) = 0$.*

Proof. Choose an orthonormal basis $\{e_1, \dots, e_n\}$ of T_pM such that e_1 satisfies the equality case of (18). Then, $\{e_{n+1}, \dots, e_{2n}, e_{2n+1} = \xi_1, \dots, e_{2n+s} = \xi_s\}$ is an orthonormal basis of $T_p^\perp M$ such that $e_{n+j} = fe_j$, $j \in \{1, \dots, n\}$. We then have $A_{\xi_\alpha} = 0$ for all $\alpha \in \{1, \dots, s\}$ and $A_{fX}Y = A_{fY}X$ for $X, Y \in TM$. Using these two facts alongwith (7), for any $Y = \sum_{j=1}^n a_j e_{n+j} + \sum_{\alpha=1}^s a_\alpha \xi_\alpha \in T_p^\perp M$, we have

$$\begin{aligned} \langle \sigma(e_1, e_1), Y \rangle &= a_1 \langle \sigma(e_1, e_1), fe_1 \rangle \\ &\quad + \sum_{j=2}^n a_j \langle \sigma(e_1, e_1), fe_j \rangle + \sum_{\alpha=1}^s a_\alpha \langle \sigma(e_1, e_1), \xi_\alpha \rangle \\ &= a_1 \left\langle \sum_{j=2}^n \sigma(e_j, e_j), fe_1 \right\rangle + \sum_{j=2}^n a_j \langle \sigma(e_1, e_1), fe_j \rangle + 0 \\ &= a_1 \sum_{j=2}^n \langle \sigma(e_1, e_j), fe_j \rangle + \sum_{j=2}^n a_j \langle \sigma(e_1, e_j), fe_1 \rangle \\ &= 0 + 0 = 0. \end{aligned}$$

Hence in view of (7), $H(p) = 0$. □

The maximum Ricci curvature function ([8]) on a Riemannian manifold M , denoted $\overline{\text{Ric}}$, is defined as

$$\overline{\text{Ric}}(p) = \max \{ \text{Ric}(X) \mid X \in T_p^1 M \}.$$

Now, in view of Theorem 5.1, we immediately have the following

Theorem 5.2. *Let M be an n -dimensional integral submanifold of a $(2n + s)$ -dimensional \mathcal{S} -space form $\widetilde{M}(c)$. Then*

$$(24) \quad \overline{\text{Ric}} \leq \frac{1}{4} \{ n^2 \|H\|^2 + (n - 1)(c + 3s) \}.$$

If M satisfies the equality case of (24) identically, then M is a minimal submanifold and

$$(25) \quad \overline{\text{Ric}} = \frac{1}{4} (n - 1)(c + 3s).$$

When $s = 0$, from Theorem 5.2 we have the following

Theorem 5.3. ([8], Theorem 2) *Let M be a Lagrangian submanifold of a $2n$ -dimensional complex space form $\widetilde{M}(c)$. Then*

$$\overline{\text{Ric}} \leq \frac{1}{4} \{ n^2 \|H\|^2 + (n - 1)c \}.$$

If M satisfies the equality case of (24) identically, then M is a minimal submanifold and

$$\overline{\text{Ric}} = \frac{1}{4} (n - 1)c.$$

When $s = 1$, from Theorem 5.2 we have the following (Theorem 4.1 of [10] or Theorem 3.1 of [11])

Theorem 5.4. ([10], Theorem 4.1 or Theorem 3.1 of [11]) *Let M be an n -dimensional C -totally real submanifold of a $(2n + 1)$ -dimensional Sasakian space form $\widetilde{M}(c)$. Then*

$$\overline{\text{Ric}} \leq \frac{1}{4} \{ n^2 \|H\|^2 + (n - 1)(c + 3) \}.$$

If M satisfies the equality case of (24) identically, then M is a minimal submanifold and

$$\overline{\text{Ric}} = \frac{1}{4} (n - 1)(c + 3).$$

Following the arguments as in [8], we can prove

Theorem 5.5. *Let M be an n -dimensional minimal integral submanifold of a $(2n + s)$ -dimensional \mathcal{S} -space form $\widetilde{M}(c)$. Then the following statements are true.*

- (1) *The submanifold M satisfies the equality case of (24) if and only if $\dim(\mathcal{N}_p) \geq 1$.*

- (2) If $\dim(\mathcal{N}_p)$ is a positive constant d , then \mathcal{N}_p is completely integral distribution and M is d -ruled, that is, for each $p \in M$, M contains a d -dimensional totally geodesic submanifold M' of $\widetilde{M}(c)$ passing through p .
- (3) If the submanifold M is also ruled, then it satisfies the equality case of (24) identically if and only if, for each ruling M' in M , the normal bundle $T^\perp M$ restricted to M' is a parallel normal subbundle of the normal bundle $T^\perp M'$ along M' .

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