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RICCI DEFORMATION OF THE METRIC ON A RIEMANNIAN MANIFOLD

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The interaction between algebraic properties of the curvature tensor and the global topology and geometry of a Riemannian manifold has been studied extensively. Of particular interest is the question under which conditions on its curvature tensor a Riemannian manifold is homeomorphic or diffeomorphic to a space of constant positive sectional curvature.

A Riemannian manifold M with sectional curvature κ is said to be δ-pinched if $\delta < \kappa \leqslant 1$ holds globally on M. The famous sphere theorem then states that a complete, simply connected $\frac{1}{4}$-pinched manifold is homeomorphic to the standard sphere [1], [6], [7]. It is also known that the homeomorphism theorem can be sharpened to a diffeomorphism theorem, if a more restrictive pinching condition is imposed, [3], [8]. Recently, Ruh [9] was able to show with the help of the Calderon-Zygmund inequalities, that the global pinching condition can be weakened to a local one: If the curvature ratios of a compact Riemannian manifold of positive sectional curvature are close to one, then the manifold is diffeomorphic to a spherical space form.

In this paper we use the heat flow method developed by Hamilton in [4] to give a new proof of Ruh's result and to obtain a more precise pointwise condition for the curvature tensor which ensures the existence of a diffeomorphism to a spherical space form. In [4] Hamilton showed that on a three-dimensional manifold of strictly positive Ricci curvature the metric can be deformed into a metric of constant positive curvature. We show that this heat flow method works for any dimension $n \geqslant 4$, provided the norm of the Weyl conformal curvature tensor and the norm of the traceless Ricci tensor are not too large compared to the scalar curvature at each point.

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1. The result

It is well known that the curvature tensor $\mathrm{Rm} = \{R_{ijkl}\}$ of a Riemannian manifold can be decomposed into three orthogonal components which have the same symmetries as Rm:

$$\mathrm{Rm} = W + V + U. \tag{1.1}$$

Here $W = \{W_{ijkl}\}$ is the Weyl conformal curvature tensor, whereas $V = \{V_{ijkl}\}$ and $U = \{U_{ijkl}\}$ denote the traceless Ricci part and the scalar curvature part respectively. Let $g = \{g_{ij}\}$ be the metric on M and denote by $\mathrm{Rc} = \{R_{ij}\}$ and R the Ricci tensor and the scalar curvature. Furthermore we write r for the average of the scalar curvature, $r = \int_M R\,d\mu / \int_M d\mu$, and define the norm of a tensor:

$$|T|^2 = |T_{ijkl}|^2 = g^{im}g^{jn}g^{kp}g^{lq}T_{ijkl}T_{mnpq}.$$

Then we have

1.1. Theorem. *Let $n \geqslant 4$. If the curvature tensor of a smooth compact n-dimensional Riemannian manifold of positive scalar curvature satisfies*

$$|W|^2 + |V|^2 < \delta_n \cdot |U|^2 \tag{1.2}$$

with

$$\delta_4 = \frac{1}{5}, \qquad \delta_5 = \frac{1}{10}, \qquad \delta_n = \frac{2}{(n-2)(n+1)}, \quad n \geqslant 6, \tag{1.3}$$

then the evolution equation

$$\frac{\partial}{\partial t} g_{ij}(t) = \frac{2}{n} r(t) \cdot g_{ij}(t) - 2R_{ij}(t), \qquad g_{ij}(0) = g_{ij}, \tag{1.4}$$

has a solution for all times $0 \leqslant t < \infty$ and $g_{ij}(t)$ converges to a smooth metric of constant positive curvature in the C^∞-topology as $t \to \infty$.

The constant δ_n, $n \geqslant 6$, is optimal in a certain sense: Consider the Riemann curvature tensor as a symmetric transformation $\mathrm{Rm}\colon \Lambda^2 \to \Lambda^2$ of two-forms with eigenvalues $\rho_1, \cdots, \rho_N$, $N = n(n-1)/2$. Then (1.2) describes a certain cone around the diagonal $\{x_1 = x_2 = \cdots = x_N\}$ in $\mathbf{R}^N$ in which the vector $(\rho_1, \cdots, \rho_N)$ has to lie. We will see in §2 that $2/(n-2)(n+1)$ corresponds to the cone which touches the faces $\{x_i = 0\}$. In particular it will follow that this constant in (1.2) still forces all eigenvalues of the operator $(U + W)\colon \Lambda^2 \to \Lambda^2$ to be positive, which is absolutely necessary for our proof in dimension $n \geqslant 4$. The constants δ_4 and δ_5 can be improved slightly but for the sake of simplicity no attempt has been made to do so. Furthermore it will become clear from the estimates in §3 that the constant in front of $|V|^2$ in (1.2) can be considerably smaller than 1, in fact it has only to be of order n^{-1}.

2. Evolution equations and preliminary results

From now on we will deal with the unnormalized equation

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} \tag{2.1}$$

since a solution of (2.1) defers from a solution of (1.4) only by a change of scale [4]. Hamilton already showed for all n that a solution of (2.1) exists on a finite time interval $[0, T)$ and that the quantity $\sup_{M_t}|\mathrm{Rm}|$ becomes unbounded as $t \to T$ (see also [2] for the local existence result). Throughout the paper we adopt the notation of [4], in particular

$$S = |\mathrm{Rc}|^2, \qquad B_{ijkl} = g^{pr}g^{qs}R_{piqj}R_{rksl} = R^{r\ s}_{\ i\ j}R_{rksl}.$$

Then we have the following evolution equations which were calculated in [4] for all n:

2.1. Lemma. *If the metric $g_{ij}(t)$ satisfies* (2.1), *then we have*

$$\begin{aligned}\frac{\partial}{\partial t}R_{ijkl} &= \Delta R_{ijkl} + 2\left(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}\right)\\ &\quad -\left(R_{pjkl}R^p{}_i + R_{ipkl}R^p{}_j + R_{ijpl}R^p{}_k + R_{ijkp}R^p{}_l\right),\\ \frac{\partial}{\partial t}R_{ik} &= \Delta R_{ik} + 2R_{piqk}R^{pq} - 2R_{pi}R^p{}_k, \qquad \frac{\partial}{\partial t}R = \Delta R + 2S.\end{aligned}$$

From this we deduce

2.2. Lemma. *The full norm of the Riemann curvature tensor satisfies the evolution equation*

$$\frac{\partial}{\partial t}|\mathrm{Rm}|^2 = \Delta|\mathrm{Rm}|^2 - 2|\partial\mathrm{Rm}|^2 + 8R_{lmqp}R^{ljqh}R^{mp}{}_{jh} + 2R_{lqmp}R^{lqih}R^{mp}{}_{ih}.$$

Proof. We obtain from Lemma 2.1 and (2.1)

$$\begin{aligned}\frac{\partial}{\partial t}|\mathrm{Rm}|^2 &= 4\left(\frac{\partial}{\partial t}g^{ir}\right)g^{js}g^{kt}g^{lp}R_{ijkl}R_{rstp} + 2g^{ir}g^{js}g^{kt}g^{lp}\left(\frac{\partial}{\partial t}R_{ijkl}\right)R_{rstp}\\ &= \Delta|\mathrm{Rm}|^2 - 2|\partial\mathrm{Rm}|^2 + 4R^{ijkl}\left(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}\right).\end{aligned}$$

The Bianchi identity yields

$$2\left(B_{ijkl} - B_{ijlk}\right) = R_{ijmn}R^{mn}{}_{kl},$$

$$R^{ijkl}\left(B_{ikjl} - B_{iljk}\right) = \tfrac{1}{2}R_{mink}R^{m\ n}_{\ j\ l}R^{ijkl}$$

and the conclusion follows.

The proof of Theorem 1.1 depends on a careful examination of the absolute terms in the evolution equation above. To this end we decompose the Riemann curvature tensor as in (1.1). Let us also agree to denote the traceless Ricci tensor by $\mathring{\mathrm{R}}\mathrm{c}$ and the scalar curvature free curvature tensor by $\mathring{\mathrm{R}}\mathrm{m}$:

$$\mathring{\mathrm{R}}\mathrm{c} = \{\mathring{R}_{ij}\} = \{R_{ij} - \tfrac{1}{n}Rg_{ij}\},$$

$$\mathring{\mathrm{R}}\mathrm{m} = \{\mathring{R}_{ijkl}\} = \{R_{ijkl} - U_{ijkl}\} = \{V_{ijkl} + W_{ijkl}\}.$$

Then we have the formulas

$$U_{ijkl} = \frac{1}{n(n-1)}R\{g_{ik}g_{jl} - g_{il}g_{jk}\}, \tag{2.2}$$

$$V_{ijkl} = \frac{1}{n-2}\{\mathring{R}_{ik}g_{jl} - \mathring{R}_{il}g_{jk} - \mathring{R}_{jk}g_{il} + \mathring{R}_{jl}g_{ik}\},$$

and an easy calculation shows

2.3. Lemma. *We have the identities*

$$|U|^2 = \frac{2}{n(n-1)}R^2, \qquad |V|^2 = \frac{4}{n-2}\left(S - \frac{1}{n}R^2\right),$$

$$|\mathrm{Rm}|^2 = |W|^2 + \frac{4}{n-2}S - \frac{2}{(n-1)(n-2)}R^2, \tag{2.3}$$

$$|\mathring{\mathrm{R}}\mathrm{m}|^2 = |W|^2 + |V|^2 = |W|^2 + \frac{4}{n-2}\left(S - \frac{1}{n}R^2\right).$$

We will also need the following estimates for symmetric 2-tensors:

2.4. Lemma. *Let $T = \{T_{ij}\}_{1 \leqslant i,j \leqslant m}$ be a symmetric tracefree operator with eigenvalues $\lambda_1, \cdots, \lambda_m$:*

$$\sum_{i=1}^{m} \lambda_i = 0, \qquad |T|^2 = \sum_{i=1}^{m} \lambda_i^2.$$

Then we have

(i) $$\lambda_i^2 \leqslant \frac{m-1}{m}|T|^2, \qquad 1 \leqslant i \leqslant m,$$

(ii) $$|\mathrm{tr}(T^3)| = \left|\sum_{i=1}^{m} \lambda_i^3\right| \leqslant \frac{(m-2)}{\sqrt{m(m-1)}}|T|^3.$$

Proof. The first estimate follows from

$$(m-1)|T|^2 - m\lambda_i^2 = (m-1)\sum_{\substack{j=1\\ j\neq i}}^{m} \lambda_j^2 - \lambda_i^2$$

$$= (m-1)\sum_{\substack{j=1\\ j\neq i}}^{m}\left(\lambda_j + \frac{\lambda_i}{m-1}\right)^2 \geqslant 0.$$

Since the second inequality is homogeneous we can confine ourselves to the case $|T|^2 = 1$ and calculate the extrema of the function $f(\lambda_1,\cdots,\lambda_m) = \sum_{i=1}^m \lambda_i^3$ on the codimension-2 submanifold of $\mathbf{R}^m$ given by $\sum_{i=1}^m \lambda_i^2 = 1$, $\sum_{i=1}^m \lambda_i = 0$. An easy computation shows that f is extremal if, for example,

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{m-1} = \pm\frac{1}{\sqrt{m(m-1)}}, \qquad \lambda_m = \mp\sqrt{\frac{m-1}{m}}.$$

Then f equals $\mp(m-2)/\sqrt{m(m-1)}$ which proves the lemma.

We can now show that condition (1.2) in Theorem 1.1 forces the eigenvalues of the curvature operator to be positive. A tensor $T = \{T_{ijkl}\}$ having the same symmetries as the curvature tensor defines a symmetric operator $T\colon \Lambda^2 \to \Lambda^2$ on the space of two-forms by

$$(Tu)_{kl} = \tfrac{1}{2}T^{ij}{}_{kl}u_{ij}, \qquad u = \{u_{ij}\} \in \Lambda^2.$$

With this definition k is an eigenvalue of T if $T^{ij}{}_{kl}u_{ij} = 2ku_{kl}$ for some $0 \neq u \in \Lambda^2$ and we have $\|T\|^2_{\Lambda_2} = \frac{1}{4}|T|^2$.

Since the manifold M is compact, the strict inequality (1.2) and (2.3) imply that there is some $\varepsilon > 0$ such that

$$|W|^2 + |V|^2 \leqslant \delta_n\cdot(1-\varepsilon)^2\cdot\frac{2}{n(n-1)}R^2 \tag{2.4}$$

holds everywhere on M. Thus we have the following consequence of Lemma 2.4(i):

2.5. Corollary. *Let k be an eigenvalue of the symmetric operator $T\colon \Lambda^2 \to \Lambda^2$,* trace$(T) = 0$. *Then*

$$k^2 \leqslant \frac{N-1}{N}\|T\|^2_{\Lambda^2} = \frac{(n-2)(n+1)}{4n(n-1)}|T|^2. \tag{2.5}$$

In particular, if k is an eigenvalue of $\mathring{\mathrm{R}}\mathrm{m}$ *and* (2.4) *is satisfied with δ_n as in* (1.3), *then we have $|k| \leqslant (1-\varepsilon)R/n(n-1)$ and since all eigenvalues of U are equal to $R/n(n-1)$ we have $R^{ijkl}u_{ij}u_{kl} \geqslant 2\varepsilon|u|^2$ for all $u \in \Lambda^2$.*

3. The eigenvalues of the curvature operator

In this section we will show that the inequality (2.4) remains valid as long as the solution of (2.1) exists. Furthermore we show that the eigenvalues of the curvature operator approach each other at least at points where the scalar curvature becomes large. In the case $n = 3$ Hamilton [4] used the quantity $S - \frac{1}{3}R^2 = |\mathring{\mathrm{R}}\mathrm{c}|^2$ which measures how far the eigenvalues of the Ricci tensor diverge from each other. In the higher dimensional case the analogous quantity is

$$|\mathring{\mathrm{R}}\mathrm{m}|^2 = |\mathrm{Rm}| - \frac{2}{n(n-1)}R^2.$$

3.1. Theorem. *If inequality* (2.4) *that is*

$$|\mathring{\mathrm{R}}\mathrm{m}|^2 \leqslant \delta_n(1-\varepsilon)^2\frac{2}{n(n-1)}R^2, \qquad \varepsilon > 0,$$

holds at time $t = 0$, *then it remains so on* $0 \leqslant t < T$. *Moreover, there are constants* $C < \infty$ *and* $\delta > 0$ *depending on* n *and the initial metric such that* $|\mathring{\mathrm{R}}\mathrm{m}|^2 \leqslant C \cdot R^{2-\delta}$ *holds on* $0 \leqslant t < T$.

To prove the theorem, we consider the functions $f_\sigma = |\mathring{\mathrm{R}}\mathrm{m}|^2/R^{2-\sigma}$ for small $\sigma \geqslant 0$. The evolution equations for $|\mathrm{Rm}|^2$ and for R in Lemmas 2.1 and 2.2 imply an evolution equation for f_σ. The calculations are analogous to those in [4, Lemma 10.3] and we derive

3.2. Lemma. *For any* $0 \leqslant \sigma < \frac{1}{2}$ *we have on* $0 \leqslant t < T$ *the identity*

$$\frac{\partial}{\partial t}f_\sigma = \Delta f_\sigma + \frac{2(1-\sigma)}{R}\langle \partial_i f_\sigma, \partial_i R\rangle - \frac{2}{R^{4-\sigma}}|\partial_i R_{jklm}R - \partial_i R R_{jklm}|^2$$
$$- \frac{\sigma(1-\sigma)}{R^{4-\sigma}}|\mathring{\mathrm{R}}\mathrm{m}|^2|\partial R|^2 + \frac{4}{R^{3-\sigma}}\left\{P + \frac{1}{2}\sigma|\mathring{\mathrm{R}}\mathrm{m}|^2 S\right\},$$

where

$$P = 2RR_{ijkl}R^{imkn}R_{m\ n}^{\ j\ l} + \tfrac{1}{2}RR_{ijkl}R^{klmn}R_{mn}^{\ \ \ ij} - |\mathring{\mathrm{R}}\mathrm{m}|^2 S. \tag{3.1}$$

In order to apply the maximum principle, we have to estimate the absolute term in this equation, i.e., the polynomial P.

3.3. Theorem. *If inequality* (2.4) *holds for some small* $\varepsilon > 0$ *then we have the estimate* $P \leqslant -\varepsilon/n \cdot R^2 \cdot |\mathring{\mathrm{R}}\mathrm{m}|^2$.

Before proving the estimate, let us see how it implies Theorem 3.1:

At time $t = 0$ we have the strict inequalities

$$f_0 < \delta_n(1-\varepsilon+\eta)^2 \cdot \frac{2}{n(n-1)}$$

for all $0 < \eta \leqslant \varepsilon/2$ and the first part of Theorem 3.1 is proved if this remains true for all time $0 \leqslant t < T$. Suppose it does not for some η. Then there is a first time $0 < t_0 < T$ and some $x_0 \in M$ such that

$$f_0(x_0, t_0) = \delta_n(1-\varepsilon+\eta)^2\frac{2}{n(n-1)}.$$

At the point (x_0, t_0) we have $\partial f_0/\partial t \geqslant 0$, $\Delta f_0 \leqslant 0$ and $\partial_i f_0 = 0$. But by Theorem 3.3 we have at (x_0, t_0)

$$P \leqslant -\frac{(\varepsilon-\eta)}{n} R^2 |\mathring{\mathrm{R}}\mathrm{m}|^2$$
$$= -\frac{(\varepsilon-\eta)}{n} \cdot \delta_n (1-\varepsilon+\eta)^2 \frac{2}{n(n-1)} R^4 < 0,$$

a contradiction to the evolution equation of f_0. The second part of Theorem 3.1 now follows immediately from the first part, Theorem 3.3 and the evolution equation of f_σ if we choose $\delta = \sigma_0 < \varepsilon/n$ and $C = \sup_M f_{\sigma_0}|_{t=0}$.

Note that by Corollary 2.5 the curvature operator and therefore the Ricci curvature is positive, in particular we have $S \leqslant R^2$.

Proof of Theorem 3.3. The decomposition (1.1) yields

$$2R_{ijkl}R^{imkn}R_{m\ n}^{\ j\ l} + \tfrac{1}{2}R_{ijkl}R^{klmn}R_{mn}^{\ \ \ ij}$$
$$= \left(U_{ijkl} + V_{ijkl} + W_{ijkl}\right) \cdot \left(2R^{imkn}R_{m\ n}^{\ j\ l} + \tfrac{1}{2}R^{klmn}R_{mn}^{\ \ \ ij}\right).$$

The Bianchi identity $R_{ijkl} + R_{iklj} + R_{iljk} = 0$ is valid for all components of the curvature tensor and using the formulas (2.2) we derive after long but simple calculations

$$U_{ijkl}\left(2R^{imkn}R_{m\ n}^{\ j\ l} + \tfrac{1}{2}R^{klmn}R_{mn}^{\ \ \ ij}\right) = \frac{2}{n(n-1)} R \cdot S,$$

$$V_{ijkl}\left(2R^{imkn}R_{m\ n}^{\ j\ l} + \tfrac{1}{2}R^{klmn}R_{mn}^{\ \ \ ij}\right)$$
$$= \frac{4}{n-2} R_{ik}\mathring{R}_{jk}R^{ijkl}$$
$$= \frac{4}{n(n-1)} R\left(S - \frac{1}{n}R^2\right) - \frac{8}{(n-2)^2}\mathring{R}_{ij}\mathring{R}^j{}_k\mathring{R}^{ik} + \frac{4}{n-2}\mathring{R}_{ik}\mathring{R}_{jl}W^{ijkl},$$

$$W_{ijkl}\left(2R^{imkn}R_{m\ n}^{\ j\ l} + \tfrac{1}{2}R^{klmn}R_{mn}^{\ \ \ ij}\right)$$
$$= 2W_{ijkl}W^{imkn}W_{m\ n}^{\ j\ l} + \tfrac{1}{2}W_{ijkl}W^{klmn}W_{mn}^{\ \ \ ij} + \frac{2}{n-2}\mathring{R}_{ik}\mathring{R}_{jl}W^{ijkl}.$$

An expression for $|\mathrm{Rm}|^2$ is given in Lemma 2.3 and we obtain from (3.1)

$$P = -\tfrac{1}{n}R^2|W|^2 + 2RW_{ijkl}W^{imkn}W_{m\ n}^{\ j\ l} + \tfrac{1}{2}RW_{ijkl}W^{klmn}W_{mn}^{\ \ \ ij}$$
$$(3.2) \qquad - \frac{4}{n(n-1)(n-2)}R^2\left(S - \frac{1}{n}R^2\right) - \frac{4}{n-2}\left(S - \frac{1}{n}R^2\right)^2$$
$$- \frac{8}{(n-2)^2}R\mathring{R}_{ij}\mathring{R}^j{}_k\mathring{R}^{ik} - |W|^2\left(S - \frac{1}{n}R^2\right) + \frac{6}{n-2}R\mathring{R}_{ik}\mathring{R}_{jl}W^{ijkl}.$$

3.4. Lemma. *We have the estimate*

$$\left|\frac{6}{n-2}\mathring{R}_{ik}\mathring{R}_{jl}W^{ijkl}\right| \leqslant \frac{3\sqrt{2}}{\sqrt{(n-1)(n-2)}}|W|\left(S-\frac{1}{n}R^2\right).$$

Proof. From formula (2.2) we calculate

$$\begin{aligned}\frac{6}{n-2}\mathring{R}_{ik}\mathring{R}_{jl}W^{ijkl} &= \frac{3}{4}(n-2)W^{ijkl}V_{klmn}V^{mn}{}_{ij}\\ &= \frac{3}{4}(n-2)\left\langle W_{ijkl}, (V\circ V)_{ijkl}\right\rangle.\end{aligned}$$

The tensor $(V\circ V)_{ijkl} = V_{ijmn}V^{mn}{}_{kl}$ has the same symmetries as the Riemann curvature tensor and therefore also decomposes into three orthogonal parts $V\circ V = T_1 + T_2 + T_3$. Here T_1 denotes the 'scalar' part, T_3 the 'Weyl' part and T_2 the 'traceless Ricci' part. The conclusion of the lemma follows if we show that

$$|T_3|^2 \leqslant \frac{32}{(n-1)(n-2)^3}\left(S-\frac{1}{n}R^2\right)^2$$

since T_1 and T_2 are orthogonal to W. From (2.2) we derive

$$\begin{aligned}(V\circ V)_{ijkl} = \frac{2}{(n-2)^2}\{&\mathring{R}_{im}\mathring{R}^m{}_k g_{jl} - \mathring{R}_{im}\mathring{R}^m{}_l g_{jk}\\ &+\mathring{R}_{jm}\mathring{R}^m{}_l g_{ik} - \mathring{R}_{jm}\mathring{R}^m{}_k g_{il} + 2(\mathring{R}_{ik}\mathring{R}_{jl} - \mathring{R}_{il}\mathring{R}_{jk})\},\end{aligned}$$

$$g^{ik}(V\circ V)_{ijkl} = \frac{2}{(n-2)^2}\left\{(n-4)\mathring{R}_{jk}\mathring{R}^k{}_l + \left(S-\frac{1}{n}R^2\right)g_{jl}\right\},$$

$$g^{jl}g^{ik}(V\circ V)_{ijkl} = \frac{4}{n-2}\left(S-\frac{1}{n}R^2\right).$$

Now we introduce the notation $Z = \mathring{R}_{ij}\mathring{R}^j{}_k\mathring{R}^k{}_l\mathring{R}^{il}$ and obtain analogous to the formulas in Lemma 2.3

$$|V\circ V|^2 = \frac{16}{(n-2)^4}\left\{(n-8)Z + 3\left(S-\frac{1}{n}R^2\right)^2\right\},$$

$$|T_1|^2 = \frac{32}{n(n-1)(n-2)^2}\left(S-\frac{1}{n}R^2\right)^2,$$

$$\begin{aligned}|T_2|^2 &= \frac{4}{n-2}\cdot\frac{4}{(n-2)^4}\left\{(n-4)^2 Z + (3n-8)\left(S-\frac{1}{n}R^2\right)^2\right\}\\ &\quad - \frac{4}{n(n-2)}\cdot\frac{16}{(n-2)^2}\left(S-\frac{1}{n}R^2\right)^2\\ &= \frac{16(n-4)}{(n-2)^5}\left\{Z - \frac{1}{n}\left(S-\frac{1}{n}R^2\right)^2\right\}.\end{aligned}$$

Thus we conclude

$$\begin{aligned} |T_3|^2 &= |V \circ V|^2 - |T_1|^2 - |T_2|^2 \\ &= \frac{32}{(n-2)^5}\left\{-nZ + \frac{(n^2-3n+3)}{(n-1)}\left(S - \frac{1}{n}R^2\right)^2\right\} \\ &\leqslant \frac{32}{(n-1)(n-2)^3}\left(S - \frac{1}{n}R^2\right)^2, \end{aligned}$$

which proves the lemma.

From Lemma 2.4(ii) we obtain

$$\left|\frac{8}{(n-2)^2}\mathring{R}_{ij}\mathring{R}^j{}_k\mathring{R}^{ik}\right| \leqslant \frac{8}{n-2}\frac{1}{\sqrt{n(n-1)}}\left(S - \frac{1}{n}R^2\right)^{3/2},$$

such that

$$\begin{aligned} P \leqslant & -\frac{1}{n}R^2|W|^2 + 2RW_{ijkl}W^{imkn}W_{m\,n}^{\,j\,l} + \frac{1}{2}RW_{ijkl}W^{klmn}W_{mn}{}^{ij} \\ & - \frac{4}{n(n-1)(n-2)}R^2\left(S - \frac{1}{n}R^2\right) - \frac{4}{n-2}\left(S - \frac{1}{n}R^2\right)^2 \\ & - |W|^2\left(S - \frac{1}{n}R^2\right) + \frac{8}{n-2}\frac{1}{\sqrt{n(n-1)}}R\left(S - \frac{1}{n}R^2\right)^{3/2} \qquad (3.3) \\ & + \frac{3\sqrt{2}}{\sqrt{(n-1)(n-2)}}R|W|\left(S - \frac{1}{n}R^2\right). \end{aligned}$$

We have now to distinguish the cases $n \geqslant 6$, $n = 5$ and $n = 4$.

(i) $n \geqslant 6$: We use an idea of Tachibana in [10] and define for fixed m, n, p, q a local skew symmetric tensor field $u_{ij}^{(mnpq)}$ by

$$\begin{aligned} u_{ij}^{(mnpq)} = & W_{inpq}g_{jm} + W_{mipq}g_{jn} + W_{mniq}g_{jp} + W_{mnpi}g_{jq} \\ & - W_{jnpq}g_{im} - W_{mjpq}g_{in} - W_{mnjq}g_{ip} - W_{mnpj}g_{iq}. \end{aligned}$$

Then it is not hard to see that

$$|u|^2 = \left\langle u_{ij}^{(mnpq)}, u_{ij}^{(mnpq)}\right\rangle = 8(n-1)|W|^2,$$

$$2W_{ijkl}W^{imkn}W_{m\,n}^{\,j\,l} + \tfrac{1}{2}W_{ijkl}W^{klmn}W_{mn}{}^{ij} = \frac{1}{16}\left\langle W^{ij}{}_{kl}u_{ij}^{(mnpq)}, u_{kl}^{(mnpq)}\right\rangle.$$

In view of Corollary 2.5 this can now be estimated by

$$\frac{1}{16}\sqrt{\frac{(n-2)(n+1)}{n(n-1)}}\,|W|\cdot|u|^2 = \frac{n-1}{2}\sqrt{\frac{(n-2)(n+1)}{n(n-1)}}\,|W|^3.$$

Using then the inequality

$$xy \leqslant \tfrac{1}{4}\eta^{-1}x^2 + \eta y^2, \qquad \eta > 0, \tag{3.4}$$

with $\eta = (1-\varepsilon)\cdot\{n(n-1)(n-2)(n+1)\}^{-1/2}$ we obtain from assumption (2.4) that

$$\begin{aligned} &2RW_{ijkl}W^{imkn}W_{m\,n}^{\,j\,l} + \frac{1}{2}RW_{ijkl}W^{klmn}W_{mn}^{\;\;ij} \\ &\leqslant \frac{1}{n}(1-\varepsilon)R^2|W|^2 - \frac{(n-1)(n+1)}{2}|W|^2\Big(S - \frac{1}{n}R^2\Big). \end{aligned} \tag{3.5}$$

Moreover, choosing $\eta = \{2\sqrt{n+1}\}^{-1}$ we get

$$\begin{aligned} &\frac{8}{n-2}\,\frac{1}{\sqrt{n(n-1)}}R\Big(S - \frac{1}{n}R^2\Big)^{3/2} - \frac{4}{n-2}\Big(S - \frac{1}{n}R^2\Big)^2 \\ &\leqslant \frac{4}{n(n-1)(n-2)}\Big(\frac{2}{\sqrt{n+1}} - \frac{1}{n+1}\Big)R^2\Big(S - \frac{1}{n}R^2\Big) \\ &\quad -(\sqrt{n+1}-1)\Big(S - \frac{1}{n}R^2\Big)|W|^2, \end{aligned} \tag{3.6}$$

where we used (2.4) again. Thus we derive

$$\begin{aligned} P \leqslant &-\frac{\varepsilon}{n}R^2|W|^2 - \frac{4}{n(n-1)(n-2)}\Big\{1 + \frac{1}{n+1} - \frac{2}{\sqrt{n+1}}\Big\}R^2\Big(S - \frac{1}{n}R^2\Big) \\ &-\Big(\sqrt{n+1} + \frac{(n-1)(n+1)}{2}\Big)|W|^2\Big(S - \frac{1}{n}R^2\Big) \\ &+\frac{3\sqrt{2}}{\sqrt{(n-1)(n-2)}}R|W|\Big(S - \frac{1}{n}R^2\Big). \end{aligned}$$

Once again we use (3.4) to estimate the last term in this expression and we have only to check that

$$\frac{9n}{8}\Big\{1 + \frac{1}{n+1} - \frac{2}{\sqrt{n+1}}\Big\}^{-1} < \Big\{\sqrt{n+1} + \frac{(n-1)(n+1)}{2}\Big\}. \tag{3.7}$$

This strict inequality holds for $n \geqslant 6$, which proves Theorem 3.3 in that case since ε is small. Moreover it becomes clear from (3.5), (3.6) and (3.7) that instead of 1 we could have some constant of order n^{-1} in front of the term $|V|^2$ in assumption (2.4).

(ii) $n = 5$: We proceed as in (i) but now the stronger assumption

$$|W|^2 + |V|^2 \leqslant (1-\varepsilon)^2\cdot\frac{1}{10}|U|^2 = (1-\varepsilon)^2\cdot\frac{1}{100}R^2$$

leads to

$$2RW_{ijkl}W^{imkn}W_{m\,n}^{\;j\;l} + \tfrac{1}{2}RW_{ijkl}W^{klmn}W_{mn}^{\;\;\;ij}$$
$$\leqslant (1-\varepsilon)\tfrac{1}{5}R^2|W|^2 - 16{,}5|W|^2\left(S - \tfrac{1}{5}R^2\right).$$

Furthermore we get

$$\frac{4}{3\sqrt{5}}R\left(S - \tfrac{1}{5}R^2\right)^{3/2} \leqslant \frac{1}{25}R^2\left(S - \tfrac{1}{5}R^2\right) - 2{,}5|W|^2\left(S - \tfrac{1}{5}R^2\right),$$

such that

$$P \leqslant -\tfrac{\varepsilon}{5}R^2|W|^2 - \tfrac{1}{40}R^2\left(S - \tfrac{1}{5}R^2\right) - 19|W|^2\left(S - \tfrac{1}{5}R^2\right)$$
$$+\sqrt{\tfrac{3}{2}}\,R|W|\left(S - \tfrac{1}{5}R^2\right),$$

and the conclusion is easily checked.

(iii) $n = 4$: In the four-dimensional case we have the following improvement:

3.5. Lemma. *On a four-dimensional manifold we have the estimate*

$$\left|2W_{ijkl}W^{imkn}W_{m\,n}^{\;j\;l} + \tfrac{1}{2}W_{ijkl}W^{klmn}W_{mn}^{\;\;\;ij}\right| \leqslant \tfrac{1}{4}\sqrt{6}\,|W|^3. \tag{3.8}$$

Proof. As in §2 we regard the Weyl conformal curvature tensor as a linear symmetric transformation on the bundle of two-forms Λ_2 defined by

$$W(e_i \wedge e_j) = \tfrac{1}{2}W^{ijkl}e_k \wedge e_l.$$

The theorem on the normal form of the Weyl tensor on a four-dimensional Riemannian manifold (see for example [5]) then states that there exists a local orthonormal basis $e_1,\cdots,e_4$ such that relative to the corresponding basis $\{e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_3 \wedge e_4, e_4 \wedge e_2, e_2 \wedge e_3\}$ of Λ_2, W takes the form $\left[\begin{smallmatrix} A & B \\ B & A \end{smallmatrix}\right]$, where

$$A = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{bmatrix}, \qquad B = \begin{bmatrix} \mu_1 & & 0 \\ & \mu_2 & \\ 0 & & \mu_3 \end{bmatrix}.$$

We have $\sum \mu_i = 0$ in view of the Bianchi identity and $\sum \lambda_i = 0$ since the trace of W vanishes. In this setting the expression in (3.8) is a cubic polynomial in μ_i, λ_i and we obtain

$$2W_{ijkl}W^{imkn}W_{m\,n}^{\;j\;l} + \tfrac{1}{2}W_{ijkl}W^{klmn}W_{mn}^{\;\;\;ij}$$
$$= 48\{\lambda_1\lambda_2\lambda_3 + \lambda_1\mu_2\mu_3 + \lambda_2\mu_1\mu_3 + \lambda_3\mu_1\mu_2\}$$
$$+8\{\lambda_1^3 + \lambda_2^3 + \lambda_3^3\} + 24\{\lambda_1\mu_1^2 + \lambda_2\mu_2^2 + \lambda_3\mu_3^2\}$$
$$=: 8P(\lambda_i, \mu_i).$$

Inequality (3.8) is invariant under scaling and so we may confine ourselves to the case where $1 = |W|^2 = 8\sum_{i=1}^{3}(\lambda_i^2 + \mu_i^2)$.

Obviously it is then enough to show inequality (3.8) for all stationary points (λ_i, μ_i) of P which lie in the set given by

$$\sum \lambda_i = 0, \qquad \sum \mu_i = 0, \qquad \sum (\lambda_i^2 + \mu_i^2) = \frac{1}{8}. \tag{3.9}$$

For such a stationary point (λ_i, μ_i) of P we must have

$$\begin{aligned} 2\lambda_2\lambda_3 + 2\mu_2\mu_3 + \lambda_1^2 + \mu_1^2 &= \alpha_1\lambda_1 + \alpha_2, \\ 2\lambda_1\lambda_3 + 2\mu_1\mu_3 + \lambda_2^2 + \mu_2^2 &= \alpha_1\lambda_2 + \alpha_2, \\ 2\lambda_1\lambda_2 + 2\mu_1\mu_2 + \lambda_3^2 + \mu_3^2 &= \alpha_1\lambda_3 + \alpha_2, \\ 2\lambda_2\mu_3 + 2\lambda_3\mu_2 + 2\lambda_1\mu_1 &= \alpha_1\mu_1 + \alpha_3, \\ 2\lambda_1\mu_3 + 2\lambda_3\mu_1 + 2\lambda_2\mu_2 &= \alpha_1\mu_2 + \alpha_3, \\ 2\lambda_1\mu_2 + 2\lambda_2\mu_1 + 2\lambda_3\mu_3 &= \alpha_1\mu_3 + \alpha_3, \end{aligned} \tag{3.10}$$

where α_1, α_2, α_3 are Lagrange multipliers. If we multiply the first equation by λ_1, the second by λ_2 and so on, then we see after summation from (3.9) that $\alpha_1 = 8P$. Furthermore, summing up the first three equations and then the last three equations we conclude from $\Sigma\mu_i = \Sigma\lambda_i = (\Sigma\lambda_i)^2 = 0$ that $\alpha_2 = \alpha_3 = 0$. Now we subtract the first three equations from each other to obtain

$$\alpha_1 = \begin{cases} -3\lambda_3 + 3\mu_3\dfrac{(\mu_2 - \mu_1)}{(\lambda_1 - \lambda_2)} & \text{if } \lambda_1 \neq \lambda_2, \\[2ex] -3\lambda_2 + 3\mu_2\dfrac{(\mu_3 - \mu_1)}{(\lambda_1 - \lambda_3)} & \text{if } \lambda_1 \neq \lambda_3, \\[2ex] -3\lambda_1 + 3\mu_1\dfrac{(\mu_2 - \mu_3)}{(\lambda_3 - \lambda_2)} & \text{if } \lambda_2 \neq \lambda_3 \end{cases} \tag{3.11}$$

and similarly, after subtracting the last three equations from each other, we obtain

$$\alpha_1 = \begin{cases} -3\lambda_3 + 3\mu_3\dfrac{(\lambda_2 - \lambda_1)}{(\mu_1 - \mu_2)} & \text{if } \mu_1 \neq \mu_2, \\[2ex] -3\lambda_2 + 3\mu_2\dfrac{(\lambda_3 - \lambda_1)}{(\mu_1 - \mu_3)} & \text{if } \mu_1 \neq \mu_3, \\[2ex] -3\lambda_1 + 3\mu_1\dfrac{(\lambda_2 - \lambda_3)}{(\mu_3 - \mu_2)} & \text{if } \mu_2 \neq \mu_3. \end{cases} \tag{3.12}$$

It is not hard to see that these six equations can only hold simultaneously if either $(\mu_i - \mu_j) = +(\lambda_i - \lambda_j)$ or $(\mu_i - \mu_j) = -(\lambda_i - \lambda_j)$ for all i, j. But summation then shows that in this case $\alpha_1 = 8P = 0$ and the inequality (3.8) holds trivially. So let us assume that, for example, $\mu_1 = \mu_2$. If $\mu_i \equiv 0$ it follows from (3.11) that two of the λ_i's have to be equal, say $\lambda_1 = \lambda_2$, $\lambda_3 = -2\lambda_1$. Then from (3.9) we get

$$\frac{1}{8} = \sum \lambda_i^2 = 6\lambda_1^2 \Rightarrow \lambda_1 = \pm\frac{1}{4\sqrt{3}}.$$

In this case we have from the second equation in (3.11) that $8P = \alpha_1 = -3\lambda_2 = -3\lambda_1 = \mp \frac{1}{4}\sqrt{3}$ and the inequality (3.8) is satisfied. If $\mu_i \not\equiv 0$, then $0 \neq \mu_3 = -2\mu_1$ and we obtain after subtracting the fifth from the fourth equation in (3.10) that $\lambda_1 = \lambda_2$, $\lambda_3 = -2\lambda_1$. We may assume $\lambda_1 \neq 0$, since otherwise $P = 0$. Subtracting the fourth from the first equation in (3.10) then yields

$$2(\lambda_1 - \mu_1)^2 = 3\lambda_1(\lambda_1 - \mu_1).$$

If $\lambda_1 \neq \mu_1$, then $-2\mu_1 = \mu_3 = \lambda_1$ which leads quickly to a contradiction if we subtract the first from the sixth equation in (3.10). So we have $\lambda_i = \mu_i$, $\lambda_1 = \lambda_2 = -\frac{1}{2}\lambda_3$ and we get from (3.9) that now

$$\frac{1}{8} = 2 \cdot \sum_{i=1}^{3} \lambda_i^2 = 12\lambda_1^2 \Rightarrow \lambda_1 = \pm\frac{1}{4\sqrt{6}}.$$

Then we have $8P = \alpha_1 = -3\lambda_2 - -\mu_2 = -6\lambda_1 = \mp \frac{1}{4}\sqrt{6}$ and inequality (3.8) is satisfied. If we start with $\lambda_1 = \lambda_2$ instead of $\mu_1 = \mu_2$, then we are led to the same cases as considered above which completes the proof of Lemma 3.5.

Now proceeding as in (i) and (ii) we obtain

$$\tfrac{1}{4}\sqrt{6}\,R|W|^3 \leqslant (1-\varepsilon)\tfrac{1}{4}R^2|W|^2 - 14{,}2|W|^2(S - \tfrac{1}{4}R^2).$$

Furthermore, we calculate

$$\frac{2}{\sqrt{3}}R\left(S - \frac{1}{4}R^2\right)^{3/2} - 2\left(S - \frac{1}{4}R^2\right)^2$$
$$\leqslant \frac{1}{30}\{2\sqrt{5} - 1\}R^2\left(S - \frac{1}{4}R^2\right) - (\sqrt{5} - 1)|W|^2\left(S - \frac{1}{4}R^2\right),$$

where we again used assumption (2.4). So we have

$$P \leqslant -\frac{\varepsilon}{4}R^2|W|^2 - \frac{1}{20}R^2\left(S - \frac{1}{4}R^2\right) - 16{,}4|W|^2\left(S - \frac{1}{4}R^2\right)$$
$$+\sqrt{3}\,R|W|\left(S - \frac{1}{4}R^2\right)$$

and the conclusion follows with inequality (3.4).

4. Gradient estimate and conclusion

Once Theorem 3.1 is established, we get the same estimate on the gradient of the scalar curvature as in [4]:

4.1. Theorem. *For every $\eta > 0$ we can find $C(\eta)$ depending only on η, n and the initial value of the metric, such that on $0 \leqslant t < T$ we have $|\partial_i R|^2 \leqslant \eta R^3 + C(\eta)$.*

Proof. The evolution equation for the gradient of the scalar curvature is the same in all dimensions and we take from Lemma 2.1 and [4]:

4.2. Lemma. *We have the equations*

$$\text{(i)} \qquad \frac{\partial}{\partial t}\left(\frac{|\partial_i R|^2}{R}\right) = \Delta\left(\frac{|\partial_i R|^2}{R}\right) - \frac{2}{R^3}\left|R\partial_i\partial_j R - \partial_i R \partial_j R\right|^2 + \frac{4}{R}\langle \partial_i R, \partial_i S\rangle - \frac{2S}{R^2}|\partial_i R|^2,$$

$$\text{(ii)} \qquad \frac{\partial}{\partial t} S = \Delta S - 2|\partial_i R_{jk}|^2 + 4 R_{ij} R_{kl} R^{ikjl}.$$

In particular we have

$$\frac{\partial}{\partial t}\left(S - \frac{1}{n}R^2\right) = \Delta\left(S - \frac{1}{n}R^2\right) - 2\left(|\partial_i R_{jk}|^2 - \frac{1}{n}|\partial_i R|^2\right) + 4 \cdot \mathring{R}_{ij} R_{kl} R^{ikjl},$$

and $\mathring{R}_{ij} R_{kl} R^{ikjl} \leqslant R(S - R^2/n)$ as can be easily seen from assumption (2.4) and decomposition (1.1).

As in [4] we bound the function

$$f = \frac{|\partial_i R|^2}{R} + N \cdot \left(S - \frac{1}{n}R^2\right) - \eta R^2,$$

where N depending only on n is sufficiently large. We need the following higher dimensional analogue of [4, Lemma 11.6].

4.3. Lemma. *We have the inequality*

$$|\partial_i R_{jk}|^2 \geqslant \frac{3n - 2}{2(n-1)(n+2)}|\partial_i R|^2,$$

and therefore

$$|\partial_i R_{jk}|^2 - \frac{1}{n}|\partial_i R|^2 \geqslant \frac{(n-2)^2}{2n(n-1)(n+2)}|\partial_i R|^2.$$

Proof. As in Hamilton's paper we decompose the gradient of the Ricci curvature: $\partial_i R_{jk} = E_{ijk} + F_{ijk}$ with

$$E_{ijk} = \frac{n-2}{2(n-1)(n+2)}\left(g_{ik}\partial_j R + g_{ij}\partial_k R\right) + \frac{n}{(n-1)(n+2)} g_{jk}\partial_i R.$$

Then the contracted second Bianchi identity $g^{ij}\partial_i R_{jk} = \frac{1}{2}\partial_k R$ leads to

$$\langle E_{ijk}, F_{ijk}\rangle = \langle E_{ijk}, \partial_i R_{jk} - E_{ijk}\rangle = 0,$$
$$|E_{ijk}|^2 = \frac{3n-2}{2(n-1)(n+2)}|\partial_i R|^2$$

which proves the lemma.

Using this lemma we derive as in [4] that $\partial f/\partial t \leqslant \Delta f + C(\eta)$. This implies a bound on f since the maximal time T can be estimated as in [4, Lemma 11.11], and therefore proves Theorem 4.1.

Having established the gradient estimate for the scalar curvature, we use Meyer's theorem as in [4] and conclude $R_{\max}/R_{\min} \to 1$ as $t \to T$.

For this step of the proof it is necessary to know that the eigenvalues of the Ricci tensor are bounded from below by εR with some fixed ε independent of time. But we have already seen in Corollary 2.5 that assumption (2.4) implies such a lower bound even for the eigenvalues of the curvature operator, which implies the bound on the Ricci curvature.

All the remaining arguments in [4] carry over almost unchanged to the higher dimensional case.

Added in proof. More recently, the author learned that the higher dimensional Ricci flow has also been studied in [11], [12].

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