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RICCI FLOW AND CURVATURE ON THE VARIETY OF FLAGS ON THE TWO DIMENSIONAL PROJECTIVE SPACE OVER THE COMPLEXES, QUATERNIONS AND OCTONIONS

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ABSTRACT. For homogeneous metrics on the spaces over the complexes, quaternions and octonions, it is shown that the Ricci flow can move a metric of strictly positive sectional curvature to one with some negative sectional curvature and one of positive definite Ricci tensor to one with indefinite signature. A variant of the method of Böhm and Wilking is given, proving that one can flow a metric of positive sectional curvature to one with Ricci curvature of indefinite signature in the quaternionic and octonian cases. A proof is given that this cannot occur in the complex case.

1. INTRODUCTION

Ricci flow has been an important tool in the study of the geometry and topology of a space. Hamilton, in a groundbreaking paper [6], proved that a compact 3-manifold with positive Ricci curvature is deformed to a space of constant positive sectional curvature. This implies that if a simply connected compact 3-manifold has a metric with positive Ricci curvature, it is diffeomorphic to the sphere $\mathbb{S}^3$. Furthermore, the nonnegativity of the curvature operator is preserved under Ricci flow in all dimensions (see [5]). Hamilton conjectured that compact manifolds in all dimensions with positive curvature forms must be diffeomorphic to space forms. This is confirmed by Böhm and Wilking in [3].

However, it is not true in general that Ricci flow preserves positivity. Ni [8] has shown that for some noncompact Riemannian manifolds with bounded nonnegative sectional curvature, Ricci flow does not preserve the nonnegativity. Two years later, Knopf [7] also showed that nonnegativity of Ricci curvature is not preserved for certain complex surfaces under the Kähler-Ricci flow. Later, Böhm and Wilking [2] showed that on $Sp(3)/Sp(1)\times Sp(1)\times Sp(1)$ (which will be called the 12 dimensional example in this paper), Ricci flow deforms a metric of positive sectional curvature to metrics with mixed Ricci curvature. In their paper, there is a remark asserting that their method fails for $SU(3)/T^2$ (the 6 dimensional case). This is the beginning of our investigation.

In this note, we will show that a metric of the homogeneous Riemannian manifold $SU(3)/T^2$ with strictly positive sectional curvature is deformed to a metric with some negative sectional curvature by the Ricci flow. We also develop a variant of

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the method of [2], giving a proof of their result and showing that the same is true for $F_4/Spin(8)$ (the 24 dimensional example). Our technique allows us to show that for the 6 dimensional example a homogeneous metric of positive sectional curvature never leaves the metrics of positive Ricci curvature under the flow. We show that if we initiate the Ricci flow at a metric on the boundary of the metrics with positive sectional, then the derivative of the flow of sectional curvature at any plane of zero curvature is negative for the all of the examples in [10]. We also show that for all the examples in [10] (dimensions 6, 12, 24) the Ricci flow can cause the Ricci tensor to go from positive definite to signature $(d, 2d)$ ($d = 2, 4, 8$). In the second-to-last section we give a simple variant of Valiev's necessary and sufficient condition for a homogeneous metric on one of these spaces to have strictly positive sectional curvature. In the last section we give our proof of the result in [2] and the assertion that no such example exists for the 6 dimensional case.

2. Setup

In this section, we will set up the notation for the main calculations and establish the Ricci flow equations in terms of the metric parameters. Set $G = SU(3), Sp(3)$ or compact F_4 and let K be respectively a maximal torus, T^2, of $SU(3)$, $Sp(1) \times Sp(1) \times Sp(1)$ in $Sp(3)$ or $Spin(8)$ in compact F_4. Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{k}$ be the Lie algebra of a K. Let $\mathfrak{p}$ be the $Ad(K)$–invariant complement to $\mathfrak{k}$ in $\mathfrak{g}$. Then $\mathfrak{p}$ can be decomposed into a direct sum of three irreducible inequivalent K-invariant subspaces $\mathfrak{p} = V_1 \oplus V_2 \oplus V_3$. Consider the $Ad(G)$-invariant inner product $\langle X, Y \rangle_0 = -1/2 \text{ Re tr}(X, Y)$ on $\mathfrak{g}$ for the first two examples and in the case of $G = F_4$ the unique $Ad(G)$-invariant inner product that agrees with our choice for the imbedded $Sp(3)$ that is compatible with the decompositions. The dimensions of the real vector spaces V_i are the same in each case and are respectively $d = 2, 4$, or 8. In each case we may identify the spaces V_i with the fields over $\mathbb{R}$: $\mathbb{C}$, $\mathbb{H}$ (the quaternions), $\mathbb{O}$ (the octonions) such that the inner product is $\text{Re}(z\overline{w})$. If $z \in V_1, w \in V_2$, then $[z, w] \in V_3$, and under our identification it corresponds to $\overline{z}w$ in V_3. Similarly with sign changes as in the cross-product $[V_i, V_j] \subseteq V_k$ if i, j, k are distinct. Schur's Lemma implies that any K-invariant inner product on $\mathfrak{p}$ is given by

$$x_1 \langle \ldots, \ldots \rangle_0 |_{V_1} + x_2 \langle \ldots, \ldots \rangle_0 |_{V_2} + x_3 \langle \ldots, \ldots \rangle_0 |_{V_3}, \tag{1}$$

where x_1, x_2, x_3 are positive constants. Let g be the Riemannian structure on M corresponding to (x_1, x_2, x_3). We will write $g \longleftrightarrow (x_1, x_2, x_3)$. In [1] it was proved that if $x_1 = x_2 = 1$, then for all examples above the sectional curvature is strictly positive if $0 < x_3 < 1$ or $1 < x_3 < \frac{4}{3}$. We note

Lemma 1. *If $x_1 = x_2$, then the sectional curvature is strictly positive if $0 < \frac{x_3}{x_1} < 1$ or $1 < \frac{x_3}{x_1} < \frac{4}{3}$ and there is some strictly negative curvature if $\frac{x_3}{x_1} > \frac{4}{3}$.*

Proof. We need only prove the assertion about negative curvature. We may assume that $x_1 = x_2 = 1$. We consider the embedding of $SU(3)$ into G so that T^2 imbeds in K and the imbedding of the complement to $Lie(T^2)$ in $Lie(SU(3)$, $\mathfrak{q}$, imbeds in $\mathfrak{p}$ as $\mathbb{C}$ imbeds in $\mathbb{H}$ or $\mathbb{O}$. We note that if $u, v \in \mathfrak{q}$, then the formula in Lemma 7.3 of

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[10] reduces the calculation to the case of $SU(3)$. We compute a specific curvature

$$u = \begin{bmatrix} 0 & u_1 & u_2 \\ -u_1 & 0 & u_3 \\ -u_2 & -u_3 & 0 \end{bmatrix}, \quad v = \begin{bmatrix} 0 & v_1 & v_2 \\ -v_1 & 0 & v_3 \\ -v_2 & -v_3 & 0 \end{bmatrix}$$

with $u_1 = 1, v_1 = -1, u_2 = v_2 = 1/\sqrt{1+x^2}, u_3 = v_3 = x/\sqrt{1+x^2}$ with $x \in \mathbb{R}$. Then with $x_1 = x_2 = 1, x_3 = 1+t$,

$$g(R(u,v)v,u) = \frac{2}{1+x^2}(1 - 3t + (1+t)^2x^2).$$

So if $t = \frac{1}{3} + s$ with $s > 0$ and

$$0 < x < \sqrt{\frac{3s}{(1 + (\frac{4}{3}+s)^2)}},$$

then the curvature corresponding to the two plane $span_{\mathbb{R}}(u,v)$ is negative. This shows that there is negative Gaussian curvature for any $t > \frac{1}{3}$, so our condition is necessary and sufficient. □

We also note that Schur's lemma implies that the Ricci curvature of g, denoted $\mathrm{Ric}(g)$, is given by

$$\text{(2)} \qquad \mathrm{Ric}(g) = x_1 r_1 \langle \ldots, \ldots \rangle_0 |_{V_1} + x_2 r_2 \langle \ldots, \ldots \rangle_0 |_{V_2} + x_3 r_3 \langle \ldots, \ldots \rangle_0 |_{V_3}.$$

Using the (first) Lemma 7.1 in [10] it is easily seen that r_i is given by

$$\text{(3)} \qquad r_i = \frac{x_i^2 d - x_j^2 d - x_k^2 d + (10d-8)x_j x_k}{2x_1 x_2 x_3},$$

where $\{i,j,k\} = \{1,2,3\}$.

We note that the Ricci flow preserves left invariant metrics on the spaces G/K and hence can be considered to be the ordinary differential equation

$$\text{(4)} \qquad \frac{dx_i}{dt} = -2r_i x_i.$$

In particular, we see that the set of metrics with $x_i = x_j$ for some i, j is preserved by the Ricci flow. Also, permutation of the indices of the x_i preserves the solutions.

3. The sectional curvature

In this section we will prove that the Ricci flow deforms some metric g with strictly positive curvature into a metric with some negative sectional curvature. To start with, we investigate the metric $g_0 \longleftrightarrow (1,1,\frac{4}{3})$ which, in light of Lemma 1, is of nonnegative sectional curvature and $g \longleftrightarrow (1,1,u)$, $u > \frac{4}{3}$, has some strictly negative curvature. Using the symmetric invariance of the system (2.4) we note that if we start with $g_0 \longleftrightarrow (1,1,\frac{4}{3})$ under the Ricci flow the metric $g_t \longleftrightarrow (x_1(t), x_2(t), x_3(t))$ satisfies $x_1(t) = x_2(t)$. Our strategy to prove that some curvature turns negative is to show that

$$\text{(5)} \qquad \frac{d}{dt}_{t=0} \frac{x_3(t)}{x_1(t)} > 0.$$

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This will show that there exists $\varepsilon > 0$ such that $\frac{1}{x_3(-\varepsilon)} g_{-\varepsilon} \longleftrightarrow (1,1,u)$ with $1 < u < \frac{4}{3}$ and $\frac{1}{x_3(\varepsilon)} g_\varepsilon \longleftrightarrow (1,1,v)$ with $v > \frac{4}{3}$. So Lemma 1 implies our assertion. We now carry out the calculation

$$\frac{d}{dt}\frac{x_3(t)}{x_1(t)} = \frac{x_3'(t)x_1(t) - x_3(t)x_1'(t)}{x_1(t)^2},$$

so (2.4) implies that

$$\frac{d}{dt}\frac{x_3(t)}{x_1(t)} = -2\frac{x_3(t)}{x_1(t)}\left(r_3 - r_1\right). \tag{6}$$

In the three cases ($d = 2, 4, 8$) we have $-2(r_3 - r_1)_{|t=0} = -2 + \frac{4d}{3} > 0$.

We have proved

Theorem 2. *On the three examples of* [10] *the Ricci flow deforms certain positively curved metrics into metrics with mixed sectional curvatures.*

We note that this result for the 12 dimensional example follows from [2]. Böhm has pointed out (in the 6 dimensional case) that if we continue the Ricci flow, then entropy considerations would imply that the metric would approach the Kähler metric in the set of homogeneous ones with $x_1 = x_2$. This metric doesn't have nonnegative sectional curvature.

4. Change in Ricci curvature

We first indicate why the method of the last section doesn't work for Ricci curvature. We consider the case when $x_1 = x_2$ and calculate

$$2(r_1 - r_3) = \frac{-2(1 - \frac{x_3}{x_1})((4d-4) - d\frac{x_3}{x_1})}{x_1 x_3}. \tag{7}$$

We therefore see (in light of (6)) that if $0 < \frac{x_3(t)}{x_1(t)} < 1$, then $\frac{d}{dt}\frac{x_3(t)}{x_1(t)} < 0$. So if we started the Ricci flow with a (positive curvature) initial condition $x_1 = x_2, \frac{x_3}{x_1} < 1$, then $\frac{x_3}{x_1}$ is decreasing. If initially $1 < \frac{x_3}{x_1} < \frac{4(d-1)}{d}$, then under the flow we would have $\frac{d}{dt}\frac{x_3(t)}{x_1(t)} > 0$. If $\frac{4(d-1)}{d} < \frac{x_3}{x_1}$, then $\frac{d}{dt}\frac{x_3(t)}{x_1(t)} < 0$. Thus $\frac{x_3}{x_1} = 1$ is a repelling (i.e. unstable) fixed point and $\frac{x_3}{x_1} = \frac{4(d-1)}{d}$ is an attractor. The upshot is that if the initial condition is $x_1 = x_2$ and the sectional curvature is positive, then $\frac{x_3}{x_1} < \frac{4(d-1)}{d}$ for the entire Ricci flow. On the other hand, the Ricci tensor for $x_1 = x_2$ is given by

$$\frac{(10d - 8 - d\frac{x_3}{x_1})}{2}(\langle \ldots, \ldots \rangle_0 |_{V_1} + \langle \ldots, \ldots \rangle_0 |_{V_2}) + \frac{(8d-8) - d\left(\frac{x_3}{x_1}\right)^2}{2}\langle \ldots, \ldots \rangle_0 |_{V_3}.$$

Thus if we begin the Ricci flow with a metric of positive curvature and $x_1 = x_2$, then $\frac{x_3}{x_1} < \frac{4(d-1)}{d}$, which implies that

$$\frac{(10d - 8 - d\frac{x_3}{x_1})}{2} > \frac{3d-2}{d} > 0$$

and

$$\frac{(8d-8) - d\left(\frac{x_3}{x_1}\right)^2}{2} > \frac{4(d-1)(3d-2)}{d} > 0. \tag{8}$$

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This indicates how delicate the methods of [2] must be.

We observe that if the initial condition satisfies $x_2 > x_1 > x_3$, then the flow will stay among the homogeneous metrics satisfying this condition. Since Ricci curvature is invariant under constant scalar multiples of the metric, we may assume that our initial metric corresponds to $x_1 = 1, x_2 = 1 + r, x_3 = s$ and $s < 1$ (notice that Lemma 1 implies that if $s < 1$ is fixed and r is sufficiently small, then the metric has positive sectional curvature). We also note that (3) implies that if $x_1 = 1, x_2 = 1 + r, x_3 = s$, then the coefficients of the Ricci curvature are given by

$$r_1 x_1 = \frac{-2rd - dr^2 + (10d-8)s + (10d-8)rs - ds^2}{2(1+r)s},$$

$$r_2 x_2 = \frac{dr + dr^2 + (10d-8)s - ds^2}{2s},$$

and

$$r_3 x_3 = \frac{(8d-8) + (8d-8)r - dr^2 + ds^2}{2(1+r)}.$$

Thus if $s < 1$ and $0 < r < 1$, then $r_2 x_2$ and $r_3 x_3$ are strictly positive. If we solve the quadratic equation for $r_1 x_1 = 0$, then we have for the cases $d = 2, 4, 8$, respectively,

$$r = \sqrt{1 + 8s^2} - (1 - 3s),$$

$$r = \sqrt{1 + 15s^2} - (1 - 4s)$$

and

$$r = \sqrt{1 + \frac{77}{4}s^2} - (1 - \frac{9}{2}s).$$

We note that if we substitute these values of r into the above coefficients of the Ricci tensor, then we find that if $s < 1$, then $r_2 x_2 > 0$ and $r_3 x_3 > 0$. So if we show that we can take our initial condition such that $\frac{dr_1}{dt} < 0$, then we will have shown that the Ricci flow transitions from positive definite to signature $(d, 2d)$ (d negatives). We therefore study

$$-2 \sum r_i x_i \frac{\partial r_1}{\partial x_i}$$

at these values. We find that if $d = 2$, then this expression is negative for $0 < s < 1 - \sqrt{\frac{5}{8}}$ ($\approx 0.20943058\dots$); for $d = 4$ the expression is negative for $0 < s < \frac{30+5\sqrt{21}-3\sqrt{5(21+4\sqrt{21})}}{30}$ ($\approx 0.361437\dots$); and for $d = 8$ the expression is negative for

$$0 < s < \frac{693 + 11\sqrt{2737} - 7\sqrt{22(511 + 9\sqrt{2737})}}{616}$$

($\approx 0.389089\dots$). This proves

Theorem 3. *For all the examples in* [10] *(i.e. the manifold of flags in the two dimensional projective space over* $\mathbb{C}$, $\mathbb{H}$ *or* $\mathbb{O}$*) the Ricci flow of a metric with positive definite Ricci tensor can flow to one with signature* $(d, 2d)$.

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5. Sectional curvature and Ricci curvature, I

In this section we will give a necessary and sufficient condition for the metric corresponding to (x_1, x_2, x_3) to have positive curvature for the three types of examples that we have been studying. We compare this condition to what is necessary for positive Ricci curvature, and one, therefore, gets a better understanding of the result in [2].

We first observe that the permutation action of the symmetric group permutes the (x_1, x_2, x_3) that correspond to strictly positive curvature among themselves. We have also completely described the (x_1, x_2, x_3) with some pair $x_i = x_j$ with $i \neq j$. Thus we are left with the cases where

$$\prod_{i<j}(x_i - x_j) \neq 0.$$

Using the action of the symmetric group just described we may assume that $x_2 > x_1 > x_3 > 0$ (we choose this order to be consistent with the results of [2]). Since multiplication by a positive scalar doesn't change the sign of curvature, we may assume that $x_1 = 1, x_2 = 1 + r$ and $x_3 = s$ with $r > 0$ and $s < 1$. The following result follows directly from Theorem 3 a) in [9].

Proposition 4. *With the notation above a necessary and sufficient condition that the sectional curvature be positive is* $r < \frac{s-2+2\sqrt{1-s+s^2}}{3}$.

Remark. We note that if $0 < s < 1$, then

$$\frac{s^2}{4} < \frac{s-2+2\sqrt{1-s+s^2}}{3} < \frac{s^2}{3},$$

and the expression estimated is monotone increasing. This can be seen in Figure 1.

The axes are horizontal, s, and vertical, r; the set of points under each curve represent the r values for each s value such that $(1, 1+r, s)$ with $r > 0$ and

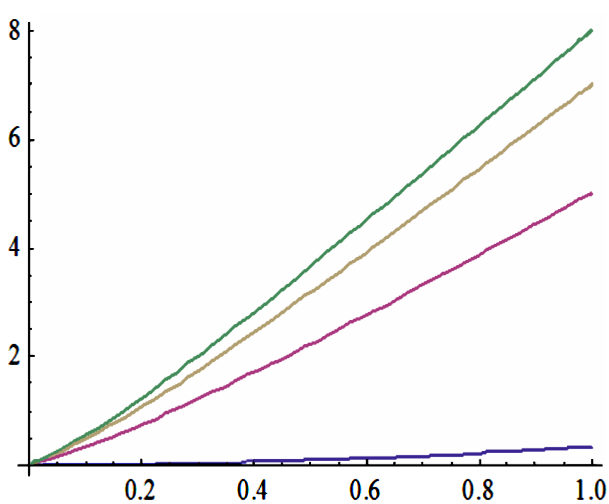

Figure 1. Comparison between the condition for positive sectional sectional curvature and the condition for positive Ricci curvature for the dim $= 6, 12, 24$ case.

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$0 < s < 1$, respectively, satisfy the necessary condition above (lowest curve), and the necessary and sufficient condition for positive Ricci curvature for the 6 (second curve), 12 (third curve) and 24 (top curve) dimensional examples. We note that to get the full set of metrics with strictly positive curvature satisfying the inequalities $x_3 \leq x_1 \leq x_2$, one must allow the points $(s, 0)$, $0 < s < 1$, and $(1, r)$ with $0 < r < \frac{1}{3}$ (that is, added to the original set given in [1]).

In the argument in [2] Böhm and Wilking start their Ricci flow at a metric corresponding to (x_1, x_2, x_3) such that $(x_1, x_2, x_3)/x_1 = (1, 1+r, s)$ (in our notation) and $r > 0$, $0 < s < 1$ (the reason for our strange condition). In light of the above, (r, s) must be below the bottom curve in Figure 1. In the Ricci flow (normalized or not) the set $x_2 > x_1 > x_3$ is preserved. If $(x_1(t), x_2(t), x_3(t))$ is a point in the flow and $(x_1(t), x_2(t), x_3(t))/x_1(t) = (1, 1+r(t), s(t))$, then the curve $(s(t), r(t))$ starts at $t = 0$ under the lowest curve (so as to have positive curvature) and must eventually cross the second highest curve in order to have some negative Ricci curvature.

6. Sectional curvature and Ricci curvature, II

We have seen that if $x_1 = 1, x_2 = 1 + r, x_3 = s$ with $0 < s < 1$ and $r > 0$, then the sectional curvature is strictly positive if and only if

$$r < \frac{s - 2 + 2\sqrt{1 - s + s^2}}{3}.$$

We note that if $0 < s < 1$, then

$$\frac{s^2}{4} < \frac{s - 2 + 2\sqrt{1 - s + s^2}}{3} < \frac{s^2}{3}.$$

We now consider what happens when we start with a metric in the indicated range and apply the Ricci flow. Note that we have $1 + r = \frac{x_2}{x_1}$ and $s = \frac{x_3}{x_1}$. This implies that under the Ricci flow we have

$$r' = \frac{-2(x_2' x_1 - x_1' x_2)}{x_1^2} = 2(1 + r)(r_1 - r_2) = g(d, r, s)$$

and

$$s' = \frac{-2(x_3' x_1 - x_1' x_3)}{x_1^2} = h(d, r, s),$$

with

$$g(d, r, s) = \begin{cases} -4\frac{r}{s}(2 + r - 3s), d = 2, \\ -8\frac{r}{s}(2 + r - 4s), d = 4, \\ -8\frac{r}{s}(4 + 2r - 9s), d = 8 \end{cases}$$

and

$$h(d, r, s) = \begin{cases} 4\frac{1-s}{1+r}(-2 - 3r + s), d = 2, \\ 8\frac{1-s}{1+r}(-3 - 4r + s), d = 4, \\ 8\frac{1-s}{1+r}(-7 - 9r + 2s), d = 8. \end{cases}$$

We note that this implies for $d = 2, 4, 8$ that if (say) $0 < s < \frac{1}{2}$, then $g(d, r, s) < 0$ for all $r > 0$, and that if $0 < s < 1$ and $r > 0$, then $h(d, r, s) < 0$. We can thus think of r as a function of s in this range and have

$$r'(s) = \frac{r'(t)}{s'(t)} = \frac{r}{s} f(d, r, s)$$

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with

$$f(d,r,s)=\frac{g(d,r,s)}{h(d,r,s)}=\frac{1+r}{1-s}\begin{cases}\frac{2+r-3s}{2+3r-s}, d=2,\\ \frac{2+r-4s}{3+4r-s}, d=4,\\ \frac{4+2r-9s}{7+9r-2s}, d=8.\end{cases}$$

We note that if (say) $0<s<\frac{1}{2}$, then $f(d,r,s)>0$ for all $r>0$.

We will use the following simple calculus result.

Lemma 5. *Suppose that we have a solution to the Ricci flow with initial condition* $s_o>0, r(s_o)>0$ *and* $r(s)$ *is defined for* $0<s_1\leq s\leq s_o$.

1. If $f(d,r(s),s)\geq C>0$ *in this range, then we have*

$$r(s)\leq s^C\frac{r(s_o)}{s_o^C}, s_1\leq s\leq s_o.$$

2. If $0<f(d,r(s),s)\leq C$ *in this range, we have*

$$r(s)\geq s^C\frac{r(s_o)}{s_o^C}, s_1\leq s\leq s_o.$$

Proof. We note that

$$\frac{r'(s)}{r(s)}=\frac{1}{s}f(d,r(s),s).$$

So in case 1 we have $s_1<s$,

$$\frac{r'(s)}{r(s)}\geq\frac{C}{s};$$

thus if $s_1\leq s\leq s_o$, then

$$\int_s^{s_o}\frac{r'(u)}{r(u)}du\geq C\int_s^{s_o}\frac{du}{u}.$$

Thus $\log r(s_o)-\log r(s)\geq C(\log(s_o)-\log(s))$. Hence

$$\log r(s)\leq C\log s-C\log(s_o)+\log r(s_o).$$

Exponentiating both sides of the equation yields the result. Case 2 is proved in the same way. □

We also observe

Lemma 6. *If* $d=2$, *then* $r_2,r_3>0$ *if* $0<s<1$ *and* $0<r<2(1+\sqrt{2})$.

Proof. $r_2=\frac{2r+r^2+6s-s^2}{(1+r)s}>0$ if $r>0$ and $0<s<6$, and $r_3=\frac{4+4r-r^2+s^2}{(1+r)s}>0$ if $0<s<1$ and $0<r<2(1+\sqrt{2})$. □

Lemma 7. *If* $d=2, 0<s<1$ *and* $r(s)>s$, *then* $r'(s)>0$. *Suppose that* $0<s_o<1$, $s_o<r(s_o)\leq 2s_o$ *and* $0<s_1<s_o$ *is such that* $r(s)$ *is defined and* $r(s)>s$ *for* $s_1\leq s\leq s_o$. *Then* $r(s)<2s$.

Proof. We note that if $0<s<1$ and $r>s$, then $f(2,r,s)\geq 1$. Thus Lemma 5 with $C=1$ implies that in the indicated range

$$r(s)\leq s\frac{r(s_o)}{s_o}\leq 2s.$$ □

This implies

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Theorem 8. *If g_o is a homogeneous Riemannian structure on the 6 dimensional example with strictly positive sectional curvature, then under the Ricci flow it retains strictly positive Ricci curvature.*

Proof. We note that the discussion in the previous section implies that if there were such a g_o, then it would, up to permutation of indices and normalization, correspond to $1, 1+r, s$ with $r > 0$ (constrained as above) and $0 < s < 1$. The condition for some negative Ricci curvature for metrics in this range is

$$r > \sqrt{1+8s^2} - (1-3s) > 3s.$$

An initial condition with strictly positive sectional curvature must satisfy

$$r < \frac{s-2+2\sqrt{1-s+s^2}}{3} < \frac{s^2}{3}.$$

The previous lemma implies that $r(s)$ can never pass $2s$. □

We will now study the 12 and 24 dimensional examples using the above parametrization of solutions to the Ricci flow. We first note that the result of Sesum (cf. [4], Theorem 6.4) implies that if $0 < s_o < 1$ and $0 < r_o < 1$, then the solution to the Ricci flow equation with initial condition $r(s_o) = r_o$ has the property that it is defined for all $0 < s \le s_o$. The proof of the next result was inspired by the argument of [2] for the 12 dimensional case.

Theorem 9. *There exist homogeneous metrics of strictly positive sectional curvature on the 12 and 24 dimensional examples that deform under the Ricci flow to metrics with some negative Ricci curvature.*

Proof. If $0 < s < \frac{1}{2}$, then $f(d,r,s) > 0$ for $r > 0$. This implies that we can choose an initial condition with $0 < r(s_o) < \frac{s_o-2+2\sqrt{1-s_o+s_o^2}}{3}$ and s_o so small that if $\varepsilon > 0$, then

$$f(d,r,s) < \begin{cases} \frac{2}{3}+\varepsilon, d=4, \\ \frac{4}{7}+\varepsilon, d=8 \end{cases}$$

for all $0 < s < s_o$ and all $0 < r < r(s_o)$. We also note that if (say) $0 < s < 1/2$, then $h(d,r,s) < -8$ if $d = 4$ or 8. This implies that

$$s'(t) < -8s(t).$$

Hence

$$\lim_{t\to+\infty} s(t) = 0.$$

Fix an initial condition such that ε can be chosen to be (say) $\frac{1}{6}$. Then in both cases we have $f(d,r,s) < \frac{5}{6}$. We can apply Lemma 5 and find that

$$r(s) \ge s^{\frac{5}{6}} \frac{r_o}{s_o^{\frac{5}{6}}}$$

for $0 < s < s_o$. As in the proof of Theorem 8 the condition for some negative Ricci curvature is

$$r > \sqrt{1+15s^2} - (1-4s), d=4$$

and

$$r > \sqrt{1+\frac{77}{4}s^2} - (1-\frac{9}{2}s), d=8,$$

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and we note that if $0 < s < 1$, then

$$0 < \sqrt{1 + 15s^2} - (1 - 4s) < 7s$$

and

$$0 < \sqrt{1 + \frac{77}{4}s^2} - (1 - \frac{9}{2}s) < 8s.$$

Thus if s is sufficiently small, the metric corresponding to $1, 1 + r(s), s$ has Ricci curvature of signature $(d, 2d)$ (here negatives come first) since $r(s)$ can be made to be larger than any fixed multiple of s for sufficiently small values of s. □

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