Ricci flow on Kähler-Einstein manifolds

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Aug. 26, 2000
Revised on May 20, 2001

Contents

1 Introduction

This is the continuation of our earlier paper [8]. For any Kähler-Einstein surfaces with positive scalar curvature, if the initial metric has positive bisectional curvature, then we proved [8] that the Kähler-Ricci flow converges exponentially to a unique Kähler-Einstein metric in the end. This answers partially to a long standing problem in Ricci flow: on a compact Kähler-Einstein manifold, does the Kähler Ricci flow converge to a Kähler-Einstein metric if the initial metric has positive bisectional curvature? In this paper, we will give a complete affirmative answer to this problem.

Theorem 1.1. *Let M be a Kähler-Einstein manifold with positive scalar curvature. If the initial metric has nonnegative bisectional curvature and positive at least at one point, then Kähler Ricci flow will converge exponentially fast to a Kähler-Einstein metric with constant bisectional curvature.*

Remark 1.2. *This problem was completely solved by R. Hamilton in the case of Riemann surfaces (cf. [15]). We also refer the reader to B. Chow's paper [9] for more developments on this problem.*

As a direct consequence, we have the following:

Corollary 1.3. *The space of Kähler metrics with non-negative bisectional curvature (and positive at least at one point) is path-connected. The space of metrics with non-negative curvature operator (and positive at least at one point) is also path-connected.*

Theorem 1.4. *Let M be any Kähler-Einstein orbifold (cf. Definition 9.2) with positive scalar curvature. If the initial metric has non-negative bisectional curvature and positive at least at one point, then the Kähler Ricci flow converges exponentially to a Kähler-Einstein metric with constant bisectional curvature. Moreover, M is a global quotient of $\mathbb{C}P^n$.*

Clearly, Corollary 1.3 holds in the case of Kähler orbifolds.

Remark 1.5. *What we really need is that the Ricci curvature is positive along the Kähler-Ricci flow. Since the positivity on Ricci curvature may not be preserved under the Ricci flow, we will use the fact that the positivity of the bisectional curvature is preserved.*

Remark 1.6. *In view of the solution of the Frankel conjecture solved by S. Mori [21] and Siu-Yau [24], it suffices to study this problem on a Kähler manifold which is biholomorphic to $\mathbb{C}P^n$. However, we don't need to use the result of Frankel conjecture. Moreover, we do not use explicitly the knowledge of the positive bisectional curvature. We use this condition only when we quote a result of Mok and Bando (cf. [8]), and a classification theorem by M. Berger.*

Remark 1.7. *We need the assumption on the existence of Kähler-Einstein metric because we will use a nonlinear inequality from [27]. Such an inequality is just the Moser-Trudinger-Onofri inequality if the underlying manifold is the Riemann sphere.*

Remark 1.8. *If we assume the existence of a lower bound for the functional $E_1 - E_0$[1], then we shall be able to derive a convergence result similarly. Therefore, it is interesting to study the lower bound of $E_1 - E_0$ among metrics whose bisectional curvature is positive.*

[1]cf. Section 2.3 for definition of E_0, E_1.

Remark 1.9. *We learn from H. D. Cao [5] that the holomorphic orthogonal bisectional curvature [2] is preserved under the Kähler Ricci flow (this will follow from Mok's proof by a simple modification). It is easy to see that positive Ricci curvature is preserved under the flow. Then our proof will extend to this case. Note that the bisectional curvature is not necessary positive during the flow.*

Now let us review briefly the history of Ricci flow. The Ricci flow was first introduced by R. Hamilton in [13], and it has been a subject of intense study ever since. The Ricci flow provides an indispensable tool of deforming Riemannian metrics towards to canonical metrics, such as Einstein ones. It is hoped that by deforming a metric to a canonical metric, one can understand the geometric and topological structures of underlying manifolds. For instance, it was proved [13] that any closed 3-manifold of positive Ricci curvature is diffeomorphic to a spherical space form. We refer the reader to [16] for more information.

If the underlying manifold is a Kähler manifold, the Ricci flow preserves the Kähler class. Following a similar idea of Yau [28], Cao [4] proved that the solution converges to a Kähler-Einstein metric if the first Chern class of the underlying Kähler manifold is zero or negative. Consequently, he re-proved the famous Calabi-Yau theorem[28]. On the other hand, if the first Chern class of the underlying Kähler manifold is positive, the solution of the Kähler Ricci flow may not converge to any Kähler-Einstein metric. This is because there are compact Kähler manifolds with positive first Chern class which do not admit any Kähler-Einstein metrics (cf. [12], [26]). A natural and challenging problem is whether or not the Kähler Ricci flow on a compact Kähler-Einstein manifold converges to a Kähler-Einstein metric. Our theorem settles this problem in the case of Kähler metrics of positive bisectional curvature or positive curvature operator. It was proved by S. Bando [1] for 3-dimensional Kähler manifolds and by N. Mok [20] for higher dimensional Kähler manifolds that the positivity of bisectional curvature is preserved under the Kähler Ricci flow.

The typical method in studying the Ricci flow depends on pointwise bounds of the curvature tensor by using its evolution equation as well as the blow-up analysis. In order to prevent formation of singularities, one blows up the solution of the Ricci flow to obtain profiles of singular solutions. Those profiles involve Ricci solitons and possibly more complicated singular models. Then one tries to exclude formation of singularities by checking that these solitons or models do not exist under appropriate global geometric conditions. It is a common sense that it is very difficult to detect how the global geometry effects those singular models even for a very simple manifold like $\mathbb{C}P^2$. The first step is to classify those singular models and hope to find their geometric information. Of course, it is already a very big task. There have been many exciting works on these (cf. [16]).

Our new contribution is to find a set of new functionals which are the Lagrangians of certain new curvature equations involving various symmetric functions of the Ricci curvature. We show that these functionals decrease essentially along the Kähler Ricci flow and have uniform lower bound. By computing their

[2] It is the bisectional curvature between two any two orthogonal complex plan.

derivatives, we can obtain certain integral bounds on curvature of metrics along the flow.

For the reader's convenience, we will recall what we study in [8] regarding these new functionals. In [8], we proved that the derivative of each E_k along an orbit of automorphisms gives rise to a holomorphic invariant $\Im_k$, including the well-known Futaki invariant as a special one. When M admits a Kähler-Einstein metric, all these invariants $\Im_k$ vanish, and the functionals E_k are invariant under the action of automorphisms.

Next we proved in [8] that these E_k are bounded from below. We then computed the derivatives of E_k along the Kähler-Ricci flow. Recall that the Kähler Ricci flow is given by

$$\frac{\partial \varphi}{\partial t} = \log \frac{(\omega + \partial\overline{\partial}\varphi)^n}{\omega^n} + \varphi - h_\omega, \tag{1.1}$$

where h_ω depends only on ω. The derivatives of these functionals are all bounded uniformly from above along the Kähler Ricci flow. Furthermore, we found that E_0 and E_1 decrease along the Kähler Ricci flow. These play a very important role in this and the preceding paper. We can derive from these properties of E_k integral bounds on curvature, e.g. for almost all Kähler metrics $\omega_{\varphi(t)}$ along the flow, we have

$$\int_M (R(\omega_{\varphi(t)}) - r)^2 \, {\omega_{\varphi(t)}}^n \to 0, \tag{1.2}$$

where $R(\omega_{\varphi(t)})$ denotes the scalar curvature and r is the average scalar curvature.

In complex dimension 2, using the above integral bounds on the curvature with Cao's Harnack inequality and the generalization of Klingenberg's estimate, we can bound the curvature uniformly along the Kähler Ricci flow in the case of Kähler-Einstein surfaces. However, it is not enough in high dimension, since the formula (1.2) is not scaling invariant. We must find a new way of utilizing this inequality in higher dimensional manifolds. Following the work of C. Sprouse [25], J. Cheeger and T. Colding [6] of deriving a uniform upper-bound on the diameter, we then use a result of Li-Yau [18] and a theorem of C. Croke [10] to derive a uniform upper bound on both the Sobolev constant and the Poincare constant on the evolved Kähler metric. Once these two important constants are bounded uniformly, we can use the Moser iteration to obtain C^0 estimate along the modified Kähler Ricci flow. A priori, this curve of evolved Kähler-Einstein metrics is not even differentiable on the level of potentials in terms of time parameter. This gives us a lot of troubles in deriving the desired C^0 estimates. What we need is to re-adjust this curve of automorphisms so that it is at least C^1 uniform on the level of Kähler potentials. Once C^0 estimate is established, it is then possible to obtain the C^2 estimate (following a similar calculation of Yau [28]) and Calabi's C^3 estimates. Eventually, we can prove that the modified Kähler Ricci flow converges exponentially to the unique Kähler-Einstein metric.

Unlike [8], we don't use any pointwise estimate on curvature; in particular, we don't need to use the Harnack inequality. It appears to us that the fact that the set of functionals we found being essentially decreasing along the Kähler Ricci flow and having a uniform lower bound at the same time, has already exclude the possibilities of formation of singularities. In higher dimensional manifolds, this idea of having integral estimates on curvature terms, may prove to be an effective and attractive alternative (vs. the usual pointwise estimates).

In this paper, we also extend our results to Kähler-Einstein orbifolds with positive bisectional curvature. Note that the limit metric of the Kähler-Ricci flow on orbifolds must be Einstein metric with positive bisectional curvature. M. Berger's theorem [3] then implies that it must be of constant bisectional curvature. We then use the exponential map to explicitly prove that such an orbifold must be a global quotient of $\mathbb{C}P^n$.

The organization of our paper is roughly as follows: In Section 2, we review briefly some basics in Kähler geometry and some results we obtained in [8]. In Section 3, we prove that for any Kähler metric in the canonical class with non-negative Ricci curvature, if the scalar curvature is sufficiently closed to the average in the L^2 sense, then it has uniform diameter bound. Next using the old results of Li-Yau and the result of C. Croke, we bound both the Sobolev constant and the Poincaré constant. In Section 4, we prove C^0 estimates for all time over the modified Kähler Ricci flow. In Section 5, we prove that we can choose a uniform gauge. In Section 6, we obtain both C^2 and C^3 estimates. In section 7, we prove the exponential convergence to the unique Kähler-Einstein metric with constant bisectional curvature. In Section 8, we prove that any orbifold supports a Kähler metric with positive constant bisectional curvature is globally a quotient of $\mathbb{C}P^n$. In Section 9, we prove Theorem 1.5 and make some concluding remarks and propose some open questions.

2 Setup and known results

2.1 Setup of notations

Let M be an n-dimensional compact Kähler manifold. A Kähler metric can be given by its Kähler form ω on M. In local coordinates $z_1, \cdots, z_n$, this ω is of the form

$$\omega = \sqrt{-1} \sum_{i,j=1}^{n} g_{i\overline{j}} d\, z^i \wedge d\, z^{\overline{j}} > 0,$$

where $\{g_{i\overline{j}}\}$ is a positive definite Hermitian matrix function. The Kähler condition requires that ω is a closed positive (1,1)-form. In other words, the following holds

$$\frac{\partial g_{i\overline{k}}}{\partial z^j} = \frac{\partial g_{j\overline{k}}}{\partial z^i} \qquad \text{and} \qquad \frac{\partial g_{k\overline{i}}}{\partial z^{\overline{j}}} = \frac{\partial g_{k\overline{j}}}{\partial z^{\overline{i}}} \qquad \forall\, i, j, k = 1, 2, \cdots, n.$$

The Kähler metric corresponding to ω is given by

$$\sqrt{-1}\,\sum_1^n\, g_{\alpha\overline{\beta}}\, d\,z^{\alpha}\ \otimes d\,z^{\overline{\beta}}.$$

For simplicity, in the following, we will often denote by ω the corresponding Kähler metric. The Kähler class of ω is its cohomology class $[\omega]$ in $H^2(M,\mathbb{R})$. By the Hodge theorem, any other Kähler metric in the same Kähler class is of the form

$$\omega_{\varphi} = \omega + \sqrt{-1}\sum_{i,j=1}^n\, \frac{\partial^2\varphi}{\partial z^i \partial z^{\overline{j}}} > 0$$

for some real valued function φ on M. The functional space in which we are interested (often referred as the space of Kähler potentials) is

$$\mathcal{P}(M,\omega) = \{\varphi \ \mid\ \omega_{\varphi} = \omega + \sqrt{-1}\partial\overline{\partial}\varphi > 0\ \text{ on } M\}.$$

Given a Kähler metric ω, its volume form is

$$\omega^n = \frac{1}{n!}\ \left(\sqrt{-1}\right)^n \det\left(g_{i\overline{j}}\right) d\,z^1\wedge d\,z^{\overline{1}}\wedge\cdots\wedge d\,z^n\wedge d\,z^{\overline{n}}.$$

Its Christoffel symbols are given by

$$\Gamma^k_{i\,j} = \sum_{l=1}^n\, g^{k\overline{l}}\frac{\partial g_{i\overline{l}}}{\partial z^j} \quad \text{and} \quad \Gamma^{\overline{k}}_{\overline{i}\,\overline{j}} = \sum_{l=1}^n\, g^{\overline{k}l}\frac{\partial g_{l\overline{i}}}{\partial z^{\overline{j}}}, \qquad \forall\, i,j,k = 1,2,\cdots n.$$

The curvature tensor is

$$R_{i\overline{j}k\overline{l}} = -\frac{\partial^2 g_{i\overline{j}}}{\partial z^k \partial z^{\overline{l}}} + \sum_{p,q=1}^n g^{p\overline{q}}\frac{\partial g_{i\overline{q}}}{\partial z^k}\frac{\partial g_{p\overline{j}}}{\partial z^{\overline{l}}}, \qquad \forall\, i,j,k,l = 1,2,\cdots n.$$

We say that ω is of nonnegative bisectional curvature if

$$R_{i\overline{j}k\overline{l}}v^i v^{\overline{j}} w^k w^{\overline{l}} \geq 0$$

for all non-zero vectors v and w in the holomorphic tangent bundle of M. The bisectional curvature and the curvature tensor can be mutually determined. The Ricci curvature of ω is locally given by

$$R_{i\overline{j}} = -\frac{\partial^2 \log\det(g_{k\overline{l}})}{\partial z_i \partial \overline{z}_j}.$$

So its Ricci curvature form is

$$\mathrm{Ric}(\omega) = \sqrt{-1}\sum_{i,j=1}^n\ R_{i\overline{j}}(\omega) d\,z^i\wedge d\,z^{\overline{j}} = -\sqrt{-1}\partial\overline{\partial}\log\ \det(g_{k\overline{l}}).$$

It is a real, closed (1,1)-form. Recall that $[\omega]$ is called a canonical Kähler class if this Ricci form is cohomologous to $\lambda\,\omega$, for some constant λ.

2.2 The Kähler Ricci flow

Now we assume that the first Chern class $c_1(M)$ is positive. The normalized Ricci flow (c.f. [13] and [14]) on a Kähler manifold M is of the form

$$\frac{\partial g_{i\bar{j}}}{\partial t} = g_{i\bar{j}} - R_{i\bar{j}}, \qquad \forall\, i,\ j = 1, 2, \cdots, n, \tag{2.1}$$

if we choose the initial Kähler metric ω with $c_1(M)$ as its Kähler class. The flow (2.1) preserves the Kähler class $[\omega]$. It follows that on the level of Kähler potentials, the Ricci flow becomes

$$\frac{\partial \varphi}{\partial t} = \log \frac{{\omega_\varphi}^n}{\omega^n} + \varphi - h_\omega, \tag{2.2}$$

where h_ω is defined by

$$\mathrm{Ric}(\omega) - \omega = \sqrt{-1}\partial\overline{\partial} h_\omega, \text{ and } \int_M (e^{h_\omega} - 1)\omega^n = 0.$$

As usual, the flow (2.2) is referred as the Kähler Ricci flow on M.

The following theorem was proved by S. Bando for 3-dimensional compact Kähler manifolds. This was later proved by N. Mok in [20] for all dimenisonal Kähler manifolds. Their proofs used Hamilton's maximum principle for tensors. The proof for higher dimensions is quite intrigue.

Theorem 2.1. *[1] [20] Under the Kähler Ricci flow, if the initial metric has nonnegative bisectional curvature, then the evolved metrics also have non-negative bisectional curvature. Furthermore, if the bisectional curvature of the initial metric is positive at least at one point, then the evolved metric has positive bisectional curvature at all points.*

Before Bando and Mok, R. Hamilton proved (by using his maximum principle for tensors)

Theorem 2.2. *Under the Ricci flow, if the initial metric has nonnegative curvature operator, then the evolved metrics also has non-negative curvature operator. Furthermore, if the curvature operator of the initial metric is positive at least at one point, then the evolved metric has positive curvature operator at all points.*

2.3 Results from the previous paper [8]

In this subsection, we collect a few results in our earlier paper [8]. First, we introduce the new functionals $E_k = E_k^0 - J_k (k = 0, 1, 2 \cdots, n)$ where E_k^0 and J_k are defined below.

Definition 2.3. *For any $k = 0, 1, \cdots, n$, we define a functional E_k^0 on $\mathcal{P}(M, \omega)$ by*

$$E_{k,\omega}^0(\varphi) = \frac{1}{V} \int_M \left(\log \frac{{\omega_\varphi}^n}{\omega^n} - h_\omega\right) \left(\sum_{i=0}^{k} \mathrm{Ric}(\omega_\varphi)^i \wedge \omega^{k-i}\right) \wedge {\omega_\varphi}^{n-k} + c_k,$$

where

$$c_k = \frac{1}{V}\int_M h_\omega \left(\sum_{i=0}^{k} \mathrm{Ric}(\omega)^i \wedge \omega^{k-i}\right) \wedge \omega^{n-k}.$$

Definition 2.4. *For each $k = 0, 1, 2, \cdots, n-1$, we will define $J_{k,\omega}$ as follows: Let $\varphi(t)$ $(t \in [0,1])$ be a path from 0 to φ in $\mathcal{P}(M,\omega)$, we define*

$$J_{k,\o}(\varphi) = -\frac{n-k}{V}\int_0^1 \int_M \frac{\partial\varphi}{\partial t}\left({\omega_\varphi}^{k+1} - \omega^{k+1}\right) \wedge {\omega_\varphi}^{n-k-1} \wedge dt.$$

Put $J_n = 0$ for convenience in notations.

Remark 2.5. *In a non canonical Kähler class, we need to modify the definition slightly since h_ω is not defined. For any $k = 0, 1, \cdots, n$, we define*

$$\begin{aligned} E_{k,\omega}(\varphi) &= \tfrac{1}{V}\int_M \log\frac{{\omega_\varphi}^n}{\omega^n}\left(\sum_{i=0}^{k} \mathrm{Ric}(\omega_\varphi)^i \wedge \mathrm{Ric}(\omega)^{k-i}\right) \wedge {\omega_\varphi}^{n-k} \\ &\quad - \tfrac{n-k}{V}\int_M \varphi\left(\mathrm{Ric}(\omega)^{k+1} - \omega^{k+1}\right) \wedge \omega^{n-k-1} - J_{k,\omega}(\varphi). \end{aligned}$$

The second integral on the right hand side is to offset the change from ω to $Ric(\omega)$ in the first term. The derivative of this functional is exactly the same as in the canonical Kähler class. In other words, the Euler-Lagrange equation is not changed.

If $\omega \in c_1(M)$, then we assume $E_k = E_{k,\omega}$. Direct computations lead to

Theorem 2.6. *For any $k = 0, 1, \cdots, n$, we have*

$$\begin{aligned} \frac{dE_k}{dt} &= \frac{k+1}{V}\int_M \Delta_\varphi\left(\frac{\partial\varphi}{\partial t}\right) \mathrm{Ric}(\omega_\varphi)^k \wedge {\omega_\varphi}^{n-k} \\ &\quad - \frac{n-k}{V}\int_M \frac{\partial\varphi}{\partial t}\left(\mathrm{Ric}(\omega_\varphi)^{k+1} - {\omega_\varphi}^{k+1}\right) \wedge {\omega_\varphi}^{n-k-1}. \end{aligned} \tag{2.3}$$

Here $\{\varphi(t)\}$ is any path in $\mathcal{P}(M,\omega)$.

Proposition 2.7. *Along the Kähler Ricci flow, we have*

$$\frac{dE_k}{dt} \le -\frac{k+1}{V}\int_M (R(\omega_\varphi) - r)\mathrm{Ric}(\omega_\varphi)^k \wedge {\omega_\varphi}^{n-k}. \tag{2.4}$$

When $k = 0, 1$, we have

$$\begin{aligned} \frac{dE_0}{d\,t} &= -\frac{n\sqrt{-1}}{V}\int_M \partial\frac{\partial\varphi}{\partial t} \wedge \overline{\partial}\frac{\partial\varphi}{\partial t}{\omega_\varphi}^{n-1} \le 0, \qquad (2.5) \\ \frac{dE_1}{dt} &\le -\frac{2}{V}\int_M (R(\omega_\varphi) - r)^2 {\omega_\varphi}^n \le 0. \end{aligned}$$

In particular, both E_0 and E_1 are decreasing along the Kähler Ricci flow.

We then prove that the derivatives of these functionals along a holomorphic automorphisms give rise to holomorphic invariants. For any holomorphic vector field X, and for any Kähler metric ω, there exists a potential function θ_X such that

$$L_X \omega = \sqrt{-1}\partial\bar{\partial}\theta_X.$$

Here L_X denotes the Lie derivative along a vector field X and θ_X is defined up to the addition of any constant. Now we define $\Im_k(X,\omega)$ for each $k = 0, 1, \cdots, n$ by

$$\begin{aligned}\Im_k(X,\omega) &= (n-k)\int_M \theta_X\ \omega^n \\ &+ \int_M \Big((k+1)\Delta\theta_X\ \mathrm{Ric}(\omega)^k \wedge \omega^{n-k} - (n-k)\ \theta_X\ \mathrm{Ric}(\omega)^{k+1} \wedge \omega^{n-k-1}\Big).\end{aligned}$$

Here and in the following, Δ denotes the Laplacian of ω. Clearly, the integral is unchanged if we replace θ_X by $\theta_X + c$ for any constant c.

The next theorem assures that the above integral gives rise to a holomorphic invariant.

Theorem 2.8. *The integral $\Im_k(X,\omega)$ is independent of choices of Kähler metrics in the Kähler class $[\omega]$. That is, $\Im_k(X,\omega) = \Im_k(X,\omega')$ so long as the Kähler forms ω and ω' represent the same Kähler class. Hence, the integral $\Im_k(X,\omega)$ is a holomorphic invariant, which will be denoted by $\Im_k(X,[\omega])$.*

Corollary 2.9. *The above invariants $\Im_k(X, c_1(M))$ all vanish for any holomorphic vector fields X on a compact Kähler-Einstein manifold. In particular, these invariants all vanish on $\mathbb{C}P^n$.*

Corollary 2.10. *For any Kähler Einstein manifold, $E_k (k = 0, 1, \cdots, n)$ is invariant under actions of holomorphic automorphisms.*

One crucial step in [8] is to modify the Kähler-Einstein metric so that the evolved Kähler form is centrally positioned with respect to this new Kähler Einstein metric. For the convenience of a reader, we include the definition of "centrally positioned" here.

Definition 2.11. *Any Kähler form ω_φ is called centrally positioned with respect to some Kähler-Einstein metric $\omega_\rho = \omega + \sqrt{-1}\partial\overline{\partial}\rho$ if it satisfies the following:*

$$\int_M (\varphi - \rho)\ \theta\ {\omega_\rho}^n = 0, \qquad \forall\ \theta \in \Lambda_1(\omega_\rho). \tag{2.6}$$

Proposition 2.12. *Let $\varphi(t)$ be the evolved Kähler potentials. For any $t > 0$, there always exists an automorphism $\sigma(t) \in \mathrm{Aut}(M)$ such that $\omega_{\varphi(t)}$ is centrally positioned with respect to $\omega_{\rho(t)}$. Here*

$$\sigma(t)^*\omega_1 = \omega_{\rho(t)} = \omega + \sqrt{-1}\partial\bar{\partial}\rho(t),$$

where ω_1 is an Kähler-Einstein metric.

Remark 2.13. *In [8], we proved that the existence of at least one Kähler-Einstein metric $\omega_{\rho(t)}$ such that $\omega_{\varphi(t)}$ is centrally positioned with respect to $\omega_{\rho(t)}$. As a matter of fact, such a Kähler-Einstein metric is unique. However, a priori we don't know if the curve $\rho(t)$ is differentiable or not.*

Proposition 2.14. *On a Kähler-Einstein manifold, the K-energy ν_ω is uniformly bounded from above and below along the Kähler Ricci flow. Moreover, there exists a uniform constant C such that*

$$\begin{array}{rcl} |J_{k,\omega_{\rho(t)}}(\varphi(t)-\rho(t))| & \leq & \{\nu_\omega(\varphi(t))+C\}^{\frac{1}{\delta}}, \\ \log \frac{{\omega_\varphi}^n}{{\omega_{\rho(t)}}^n} & \geq & -4C''\, e^{2(\nu_\omega(\varphi(t))+C)^{\frac{1}{\delta}}+C')}, \\ E_k(\varphi(t)) & \geq & -e^{c\left(1+\max\{0,\nu_\omega(\varphi(t))\}+(\nu_\omega(\varphi(t))+C)^{\frac{1}{\delta}}\right)}, \end{array}$$

where c, C, C' and C'' are some uniform constants. And $\rho(t)$ is defined in the preceding proposition.

Corollary 2.15. *The energy functional $E_k(k=0,1,\cdots,n)$ has a uniform lower bound from below along the Kähler Ricci flow.*

Corollary 2.16. *For each $k=0,1,\cdots,n$, there exists a uniform constant C such that the following holds (for any $T\leq\infty$) along the Kähler Ricci flow:*

$$\int_0^T \frac{k+1}{V}\int_M \left(R(\omega_{\varphi(t)})-r\right)\ \mathrm{Ric}(\omega_{\varphi(t)})^k\wedge{\omega_{\varphi(t)}}^{n-k}\,d\,t\leq C.$$

When $k=1$, we have

$$\int_0^\infty \frac{1}{V}\int_M (R(\omega_{\varphi(t)})-r)^2\ {\omega_{\varphi(t)}}^n\,d\,t\leq C<\infty.$$

3 Estimates of Sobolev and Poincare constants

In this section, we will prove that for any Kähler metric in the canonical Kähler class, if the scalar curvature is close enough to a constant in L^2 sense and if the Ricci curvature is non-negative, then there exists a uniform upper bound for both the Poincaré constant and the Sobolev constant. We first follow an approach taken by C. Sprouse [25] to obtain a uniform upper bound on the diameter.

In [6], J. Cheeger and T. Colding proved an interesting and useful inequality which converts integral estimates along geodesic to integral estimates on the whole manifold. In this section, we assume $m=\dim(M)$.

Lemma 3.1. *[6] Let $A_1,\ A_2$ and W be open subsets of M such that $A_1,A_2\subset W$, and all minimal geodesics $r_{x,y}$ from $x\in A_1$ to $y\in A_2$ lie in W. Let f be any non-negative function. Then*

$$\begin{aligned}&\int_{A_1\times A_2}\int_{r_{x,y}} f(r(s))\,d\,s\,d\,vol_{A_1\times A_2}\\ &\quad\leq C(m,k,\Re)(\mathrm{diam}(A_2)vol(A_1)+\mathrm{diam}(A_1)vol(A_2))\int_W f\,d\,vol,\end{aligned}$$

where for $k \leq 0$,

$$C(m,k,\Re) = \frac{\text{area}(\partial B_k(x,\Re))}{\text{area}(\partial B_k(x,\frac{\Re}{2}))}, \tag{3.1}$$

$$\Re \geq \sup\{d(x,y) \mid (x,y) \in (A_1 \times A_2)\}, \tag{3.2}$$

and $B_k(x,r)$ denotes the ball of radius r in the simply connected space of constant sectional curvature k.

In this paper, we always assume $\text{Ric} \geq 0$ on M, and thus $C(n,k,\Re) = C(n)$. Using this theorem of Cheeger and Colding, C. Sprouse [25] proved an interesting lemma:

Lemma 3.2. *[25] Let (M,g) be a compact Riemannian manifold with* $\text{Ric} \geq 0$. *Then for any $\delta > 0$ there exists $\epsilon = \epsilon(n,\delta)$ such that if*

$$\frac{1}{V}\int_M ((m-1) - Ric_-)_+ < \epsilon(m,\delta), \tag{3.3}$$

then the $diam(M) < \pi + \delta$. Here Ric_- *denotes the lowest eigenvalue of the Ricci tensor; For any function f on M, $f_+(x) = \max\{f(x), 0\}$.*

Remark 3.3. *Note that the right hand side of equation (3.3) is not scaling correct. A scaling correct version of this lemma should be: For any positive integer $a > 0$, if*

$$\frac{1}{V}\int_M |Ric - a| \, d\,vol < \epsilon(m,\delta)\cdot a,$$

then the diameter has a uniform upper bound.

Remark 3.4. *It is interesting to see what the optimal constant $\epsilon(m,\delta)$ is. Following this idea, the best constant should be*

$$\epsilon(m,\delta) = \sup_{N>2} \frac{N-2}{8C(m)N^m}.$$

However, it will be interesting to figure out the best constant here.

Adopting his arguments, we will prove the similar lemma,

Lemma 3.5. *Let (M,ω) be a polarized Kähler manifold and $[\omega]$ is the canonical Kähler class. Then there exists a positive constant ϵ_0 which only depends on the dimension, such that if the Ricci curvature of ω is non-negative and if*

$$\frac{1}{V}\int_M (R-n)^2\omega^n \leq \epsilon_0^2,$$

then there exists a uniform upper bound on diameter of the Kähler metric ω. Here r is the average of the scalar curvature.

Proof. We first prove that the Ricci form is close to its Kähler form in the L^1 sense (after proper rescaling). Note that

$$\mathrm{Ric}(\omega) - \omega = \sqrt{-1}\partial\bar{\partial} f$$

for some real valued function f. Thus

$$\int_M (\mathrm{Ric}(\omega) - \omega)^2 \wedge \omega^{n-2} = \int_M \left(\sqrt{-1}\partial\bar{\partial} f\right)^2 \wedge \omega^{n-2} = 0.$$

On the other hand, we have

$$\int_M (\mathrm{Ric}(\omega) - \omega)^2 \wedge \omega^{n-2} = \frac{1}{n(n-1)} \int_M \left((R-n)^2 - |\mathrm{Ric}(\omega) - \omega|^2\right) \omega^n.$$

Here we already use the identity $tr_\omega\,(\mathrm{Ric}(\omega) - \omega) = R - n$. Thus

$$\int_M |\mathrm{Ric}(\omega) - \omega|^2 \,\omega^n = \int_M (R-n)^2 \,\omega^n.$$

This implies that

$$\begin{aligned}\left(\int_M |\mathrm{Ric} - 1|\, \omega^n\right)^2 &\le \int_M |\mathrm{Ric}(\omega) - \omega|^2 \,\omega^n \cdot \int_M \omega^n \\ &= \int_M (R-n)^2 \,\omega^n \cdot V \\ &\le \epsilon_0^2 \cdot V \cdot V = \epsilon_0^2 \cdot V^2,\end{aligned}$$

which gives

$$\frac{1}{V}\int_M |\mathrm{Ric} - 1|\, \omega^n \le \epsilon_0. \tag{3.4}$$

The value of ϵ_0 will be determined later.

Using this inequality (3.4), we want to show that the diameter must be bounded from above. Note that in our setting, $m = \dim(M) = 2n$. Unlike in [25], we are not interested in obtaining a sharp upper bound on the diameter.

Let A_1 and A_2 be two balla of small radius and $W = M$. Let $f = |\mathrm{Ric} - 1| = \sum_{i=1}^{m} |\lambda_i - 1|$, where λ_i is the eigenvalue of the Ricci tensor. We assume also that all geodesics are parameterized by arc length. By possibly removing a set of measure 0 in $A_1 \times A_2$, there is a unique minimal geodesic from x to y for all $(x, y) \in A_1 \times A_2$. Let p, q be two points on M such that

$$d(p, q) = \mathrm{diam}(M) = D.$$

We also used $d\,vol$ to denote the volume element in the Riemannian manifold M and V denote the total volume of M. For $r > 0$, put $A_1 = B(p, r)$ and

$A_2 = B(q,r)$. Then Lemma 3.1 implies that

$$\int_{A_1\times A_2}\int_{r_{x,y}} |\mathrm{Ric}-1|\,d\,s\,d\,vol_{A_1\times A_2}$$
$$\le C(n,k,R)(diam(A_2)vol(A_1)+diam(A_1)vol(A_2))\int_W |\mathrm{Ric}-1|\,d\,vol.$$

Taking infimum over both sides, we obtain

$$\begin{aligned} &\inf_{(x,y)\in A_1\times A_2}\int_{r_{x,y}} |\mathrm{Ric}-1|\,d\,s \\ \le\ & 2\,r\,C(n)(\frac{1}{vol(A_1)}+\frac{1}{vol(A_2)})\int_W |\mathrm{Ric}-1|\,d\,vol \\ \le\ & 4rC(n)\frac{D^n}{r^n}\frac{1}{V}\int_M |\mathrm{Ric}-1|\,d\,vol, \end{aligned} \tag{3.5}$$

where the last inequality follows from the relative volume comparison. We can then find a minimizing unit-speed geodesic γ from $x\in\overline{A_1}$ and $y\in\overline{A_2}$ which realizes the infimum, and will show that for $L=d(x,y)$ much larger than π, γ can not be minimizing if the right hand side of (3.5) is small enough.

Let $E_1(t), E_2(t), \cdots E_m(t)$ be a parallel orthonormal basis along the geodesic γ such that $E_1(t)=\gamma'(t)$. Set now $Y_i(t)=\sin\left(\frac{\pi t}{L}\right)E_i(t), i=2,3,\cdots m$. Denote by $L_i(s)$ the length functional of a fixed endpoint variation of curves through γ with variational direction Y_i, we have the 2nd variation formula

$$\begin{aligned} & \sum_{i=2}^m \frac{d^2L_i(s)}{d\,s^2}\,|_{s=0} \\ =\ & \sum_{i=2}^m\int_0^L (g(\nabla_{\gamma'}Y_i,\nabla_{\gamma'}Y_i)-R(\gamma',Y_i,\gamma',Y_i))\,d\,s \\ =\ & \int_0^L (m-1)\left(\frac{\pi^2}{L^2}\cos^2\left(\frac{\pi t}{L}\right)\right)-\sin^2\left(\frac{\pi t}{L}\right)\mathrm{Ric}(\gamma',\gamma')\,d\,s \\ =\ & \int_0^L\left((m-1)\frac{\pi^2}{L^2}\cos^2\left(\frac{\pi t}{L}\right)-\sin^2\left(\frac{\pi t}{L}\right)\right)\,d\,s \\ & +\int_0^L \sin^2\left(\frac{\pi t}{L}\right)(1-\mathrm{Ric}(\gamma',\gamma'))\,\,d\,s \\ =\ & -\tfrac{L}{2}\left(1-(m-1)\tfrac{\pi^2}{L^2}\right) \\ & +\int_0^L \sin^2\left(\frac{\pi t}{L}\right)(1-\mathrm{Ric}(\gamma',\gamma'))\,\,d\,s. \end{aligned}$$

Note that

$$1-\mathrm{Ric}(\gamma',\gamma')\le|\mathrm{Ric}-1|.$$

Combining the above calculation and the inequality (3.5), we obtain

$$\begin{aligned}
&\sum_{i=2}^{n} \frac{d^2 L_i(s)}{d\,s^2}\mid_{s=0} \\
\le\ & -\frac{L}{2}\left(1-(m-1)\frac{\pi^2}{L^2}\right)+\int_0^L \sin^2\left(\frac{\pi t}{L}\right)|\mathrm{Ric}-1|\,d\,s \\
\le\ & -\frac{L}{2}\left(1-(m-1)\frac{\pi^2}{L^2}\right)+4rC(n)\frac{D^n}{r^n}\frac{1}{V}\int_M |\mathrm{Ric}-1|\,d\,vol. \qquad (3.6)
\end{aligned}$$

Here in the last inequality, we have already used the fact that γ is a geodesic which realizes the infimum of the left side of inequality (3.5). For any fixed positive larger number $N > 4$, let $D = N \cdot r$. Set $c = \frac{1}{V}\int_M |\mathrm{Ric}-1|\ d\,vol.$ Note that

$$L = d(x,y) \ge d(p,q) - 2r = D(1-\frac{2}{N}) \ge \frac{D}{2}.$$

Then the above inequality (3.6) leads to

$$\begin{aligned}
\frac{1}{D}\sum_{i=2}^{n} \frac{d^2 L_i(s)}{d\,s^2}\mid_{s=0} \ &\le\ -\frac{1-\frac{2}{N}}{2}\left(1-(m-1)\frac{\pi^2}{L^2}\right)+4C(n)\ \frac{N^{m-1}}{V}\cdot c\cdot V \\
&=\ 4C(n)\ N^{m-1}\left(c-\frac{(N-2)}{2N}\frac{1}{4C(n)N^{m-1}}\right)+\frac{1-\frac{2}{N}}{2}(m-1)\frac{\pi^2}{L^2}.
\end{aligned}$$

Note that the second term in the right hand side can be ignored if $L \ge \frac{D}{2}$ is large enough. Set

$$\epsilon_0 = \frac{(N-2)}{2N}\cdot\frac{1}{4C(n)N^{m-1}} = \frac{N-2}{8C(n)N^m}.$$

Then if

$$\frac{1}{V}\int_M (R-n)^2\omega^n \le \epsilon_0^2,$$

by the argument at the beginning of this proof, we have the inequality (3.4):

$$\frac{1}{V}\int_M (R-n)^2\omega^n \le \epsilon_0^2,$$

$$\frac{1}{V}\int_M |\mathrm{Ric}-1|\,d\,vol < \epsilon_0,$$

which in turns imply

$$\frac{1}{D}\sum_{i=2}^{n} \frac{d^2 L_i(s)}{d\,s^2}\mid_{s=0} < 0,$$

for D large enough. Thus, if the diameter is too large, γ cannot be a length minimizing geodesic. This contradicts our earlier assumption that γ is a minimizing geodesic. Therefore, the diameter must have a uniform upper bound. □

According to the work of. C. Croke [10], Li-Yau [18] and Li [17]), we state the following lemma on the upper bound of the Sobolev constant and Poincare constant:

Lemma 3.6. *Let (M,ω) be any compact polarised Kähler manifold where $[\omega]$ is the canonical class. If $Ric(\omega) \geq 0,\ V = \int_M \omega^n \geq \nu > 0$ and the diameter has a uniform upper bound, then there exists a constant $\sigma = \sigma(\epsilon_0, \nu)$ such that for all function $f \in C^\infty(M)$, we have*

$$\left(\int_M \mid f \mid^{\frac{2n}{n-1}} \ \omega^n\right)^{\frac{n-1}{n}} \leq \sigma \left(\int_M \mid \nabla f \mid^2 \ \omega^n + \int_M f^2 \, \omega^n\right).$$

Furthermore, there exists a uniform Poincaré constant $c(\epsilon_0)$ such that the Poincaré inequality holds

$$\int_M \left(f - \frac{1}{V}\int_M f \,\omega^n\right)^2 \omega^n \leq c(\epsilon_0) \int_M \mid \nabla f \mid^2 \ \omega^n.$$

Here ϵ_0 is the constant appeared in Lemma 3.5.

Proof. Note that (M,ω) has a uniform upper bound on the diameter. Moreover, it has a lower volume bound and it has non-negative Ricci curvature. Following a proof in [17] which is based on a result of C. Croke [10], we obtain a uniform upper bound on the Sobolev constant (independent of metric!).

Recall a theorem of Li-Yau [18] which gives a positive lower bound of the first eigenvalue in terms of the diameter when Ricci curvature is nonnegative:

$$\lambda_1(\omega) \geq \frac{\pi^2}{4D^2},$$

here $\lambda_1,\ D$ denote the first eigenvalue and the diameter of the Kähler metric ω. Now D has a uniform upper bound according to Lemma 3.5. Thus the first eigenvalue of ω has a uniform positive lower bound; which, in turn, implies that there exists a uniform Poincaré constant. □

4 C^0 estimates

Let us first prove a general lemma on C^0 estimate:

Lemma 4.1. *Let ω_ψ be a Kähler metric such that*

$$\sup_M \psi \leq C_1,$$

and

$$\int_M (-\psi)\omega_\psi^n \leq C_2.$$

If the Sobolev constant and the Poincaré constant of ω_ψ are bounded from above by C_3, then there exists a uniform constant C_4 which depends only on the dimension and the constants C_1, C_2 and C_3 such that

$$|\psi| \leq C_4.$$

We will use this lemma several times, so we include a proof here for the convenience of the reader.

Proof. Denote by Δ_ψ the Laplacian of $\o_\psi$. Then, because $\o + \partial\bar{\partial}\psi > 0$, we see that $\o = \o_\psi - \partial\bar{\partial}\psi > 0$. Taking the trace of this latter expression with respect to $\o_\psi$, we get

$$n - \Delta_\psi \psi = \mathrm{tr}_{\o_\psi} \o > 0.$$

Define now $\psi_-(x) = \max\{-\psi(x), 1\} \geq 1$. It is clear that

$$\psi_-{}^p (n - \Delta_\psi \psi) \geq 0.$$

Integrating this inequality, we get

$$\begin{aligned} 0 &\leq \frac{1}{V} \int_M \psi_-{}^p (n - \Delta_\psi \psi) \o_\psi^n \\ &= \frac{n}{V} \int_M \psi_-{}^p \o_\psi^n + \frac{1}{V} \int_M \nabla_\psi \psi_-{}^p \nabla_\psi \psi \o_\psi^n \\ &= \frac{n}{V} \int_M \psi_-{}^p \o_\psi^n + \frac{1}{V} \int_{\{\psi \leq -1\}} \nabla_\psi \psi_-{}^p \nabla_\psi \psi \o_\psi^n \\ &= \frac{n}{V} \int_M \psi_-{}^p \o_\psi^n + \frac{1}{V} \int_M \nabla_\psi \psi_-{}^p \nabla_\psi (-\psi_-) \o_\psi^n \\ &= \frac{n}{V} \int_M \psi_-{}^p \o_\psi^n - \frac{1}{V} \frac{4p}{(p+1)^2} \int_M |\nabla_\psi \psi_-{}^{\frac{p+1}{2}}|^2 \o_\psi^n, \end{aligned}$$

which yields, using the fact that $\psi_- \geq 1$ and hence $\psi_-{}^p \leq \psi_-{}^{p+1}$,

$$\frac{1}{V} \int_M \left|\nabla_\psi \psi_-{}^{\frac{p+1}{2}}\right|^2 \o_\psi^n \leq \frac{n(p+1)^2}{4pV} \int_M \psi_-{}^{p+1} \o_\psi^n.$$

Since the Sobolev constant of ω_ψ is bounded from above, we can use the Sobolev inequality,

$$\frac{1}{V} \left(\int_M |\psi_-|^{\frac{(p+1)n}{n-1}} \o_\psi^n \right)^{\frac{n-1}{n}} \leq \frac{c(p+1)}{V} \int_M \psi_-{}^{p+1} \o_\psi^n.$$

Moser's iteration will show us that

$$\sup_M \psi_- = \lim_{p \to \infty} \|\psi_-\|_{L^{p+1}(M, \o_\psi)} \leq C \|\psi_-\|_{L^2(M, \o_\psi)}.$$

Since the Poincare constant is uniformly bounded from above, we can use the Poincaré inequality

$$\frac{1}{V}\int_M\left(\psi_- - \frac{C}{V}\int_M \psi_- ø_\psi^n\right)^2 ø_\psi^n \leq \frac{1}{V}\int_M |\nabla\psi_-|^2 ø_\psi^n$$
$$\leq \frac{C'}{V}\int_M \psi_- ø_\psi^n,$$

where we have set $p = 1$ and used the same reasoning as before. This then implies that

$$\max\{-\inf_M \psi, 1\} = \sup_M \psi_- \leq \frac{C}{V}\int_M \psi_- ø_\psi^n.$$

Since $\int_M e^{-h_\varphi+\psi} ø_\psi^n = V$, we can easily deduce $\displaystyle\int_{\psi>0} \psi ø_\psi^n \leq C$. Combining this together with the above, we get

$$-\inf_M \psi \leq \frac{C}{V}\int_M (-\psi) ø_\psi^n + C,$$

which proves the lemma. □

Lemma 4.2. *Along the Kähler Ricci flow, the diameter of the evolving metric is uniformly bounded.*

Proof. In our first work [8], we proved that

$$\int_0^\infty d\,t \int_M (R-r)^2\,\omega_\varphi^n \leq C.$$

Thefore, for any sequence $s_i \to \infty$, and for any fixed time period T, there exists $t_i \to \infty$ and $0 < s_i - t_i < T$ such that

$$\lim_{t_i\to\infty} \frac{1}{V}\int_M (R-r)^2\,\omega_\varphi^n = 0. \tag{4.1}$$

Now for this sequence of t_i, applying lemma 3.5, we show there exists a uniform constant D such that the diameters of $\omega_{\varphi(t_i)}$ are uniformly bounded by $\frac{D}{2}$. Recalled that the Ricci curvature is uniformly positive along the flow so that diameter of evolving metric increased at most exponentially since

$$\frac{\partial}{\partial t} g_{i\bar{j}} = g_{i\bar{j}} - R_{i\bar{j}} \leq g_{i\bar{j}}.$$

Now $t_{i+1} - t_i < 2T$ for all $i > 0$, this implies that the Diameters of the evolving metric along the entire flow is controlled by $e^{2T}\frac{D}{2} \leq D$ (choose T small enough in the first place. □

Combining this with Lemma 3.6, we obtain

Theorem 4.3. *Along the Kähler Ricci flow, the evolving Kähler metric $\omega_{\varphi(t)}$ has a uniform upper bound on the Sobolev constant and Poincaré constant.*

Before we go on any further, we want to review some results we obtained in previous paper [8].

Let $\varphi(t)$ be the global solution of the Kähler Ricci flow. In the level of Kähler potentials, the evolution equation is:

$$\frac{\partial \varphi(t)}{\partial t} = \log\left(\frac{\omega^n_{\varphi(t)}}{\omega^n}\right) + \varphi(t) - h_\omega.$$

According to Lemma 6.5 of [8], there exists a one parameter family of Kähler-Einstein metrics $\omega_{\rho(t)} = \omega + \sqrt{-1}\partial\bar{\partial}\rho(t)$ such that $\omega_{\varphi(t)}$ is centrally positioned with respect to $\omega_{\rho(t)}$ for any $t \geq 0$. Suppose that $\omega_{\varphi(0)}$ is already centrally positioned with the Kähler-Einstein metric $\omega_1 = \omega + \sqrt{-1}\partial\bar{\partial}\rho(0)$. Normalize the value of $\rho(t)$ such that

$$\omega^n_{\rho(t)} = e^{-\rho(t)+h_\omega}\ \omega^n,$$

or equivalently

$$\ln\left(\frac{\omega^n_{\varphi_{\rho(t)}}}{\omega^n}\right) = -\rho(t) + h_\omega. \tag{4.2}$$

Then the Kähler Ricci flow equation can be re-written as ,

$$\frac{\partial \varphi(t)}{\partial t} = \log\left(\frac{\omega^n_{\varphi(t)}}{\omega^n_{\rho(t)}}\right) + \varphi(t) - \rho(t). \tag{4.3}$$

Sometimes we may refer this equation as the modified Kähler Ricci flow. Next we are ready to prove the C^0 estimates for both the Kähler potentials and the volume form when $t = t_i$.

Theorem 4.4. *There exists a uniform constant C such that*

$$\mid \varphi(t) - \rho(t) \mid < C, \qquad \text{and } \mid\frac{\partial\varphi}{\partial t}\mid \leq C.$$

In particular, we have

$$\mid \ln\det\left(\frac{\omega^n_{\varphi(t)}}{\omega^n_{\rho(t)}}\right)\mid < C.$$

We need a lemma on L^1 integral of the Kähler potentials.

Lemma 4.5. *Along the Kähler Ricci flow on a Kähler Einstein manifold, there exists a uniform bound C such that*

$$-C \leq \int_M (\varphi(t) - \rho(t))\,\omega^n_{\varphi(t)} \leq C.$$

Proof. As in section 11 of [8], we define

$$c(t) = \int_M \frac{\partial \varphi(t)}{\partial t}\, \omega^n_{\varphi(t)}.$$

In a Kähler Einstein manifold, the K-energy has a uniform lower bound along the Kähler Ricci flow. Thus

$$\int_0^\infty \int_M \mid \nabla \frac{\partial \varphi(t)}{\partial t} \mid^2_{\varphi(t)}\ \omega^n_{\varphi(t)}\, d\,t \leq C.$$

Therefore, we can normalize the initial value of Kähler potential so that

$$c(0) = \int_0^\infty e^{-t} \int_M \mid \nabla \frac{\partial \varphi(t)}{\partial t} \mid^2_{\varphi(t)}\ \omega^n_{\varphi(t)}\, d\,t \leq C.$$

According to Lemma 11.1 of [8], we have $c(t) > 0$ and

$$\lim_{t\to\infty} c(t) = 0.$$

In particular, this implies that there exists a constant C such that

$$\begin{aligned} C &\geq c(t) = \int_M \frac{\partial \varphi(t)}{\partial t}\, \omega^n_{\varphi(t)} \\ &= \int_M \left(\log \left(\frac{\omega^n_{\varphi(t)}}{\omega^n_{\rho(t)}} \right) + \varphi(t) - \rho(t) \right) \omega^n_{\varphi(t)} > 0. \end{aligned}$$

In the last inequality we have used the fact that $c(t) > 0$. According to Theorem 2.14, we have

$$-C \leq \int_M \log \left(\frac{{\omega_\varphi}^n}{{\omega_{\rho(t)}}^n} \right) \omega^n_{\varphi(t)} \leq C.$$

Combining this with the previous inequality, we arrive at

$$-C \leq \int_M (\varphi(t) - \rho(t))\ \omega^n_{\varphi(t)} < C.$$

Here C is a constant which may be different from line to line. □

Next we return to the proof of Theorem 4.4.

Proof. According to Proposition 2.14, we have

$$J_{\omega_{\rho(t)}}(\omega_{\varphi(t)}) < C.$$

Then

$$0 \leq (I - J)(\omega_{\varphi(t)}, \omega_{\rho(t)}) \leq (n+1) \cdot J_{\omega_{\rho(t)}}(\omega_{\varphi(t)}) < (n+1)C.$$

By definition, this implies that

$$0 \leq \int_M (\varphi(t) - \rho(t))(\omega^n_{\rho(t)} - \omega^n_{\varphi(t)}) \leq C.$$

Combining this with Lemma 4.6 we obtain

$$-C \le \int_M (\varphi(t) - \rho(t))\omega_{\rho(t)}^n \le C.$$

Since $\triangle_{\rho(t)}(\varphi(t) - \rho(t)) \ge -n,$ by the Green formula, we have

$$\begin{aligned} & \sup_M(\varphi(t) - \rho(t)) \\ \le & \tfrac{1}{V}\int_M (\varphi(t) - \rho(t))\omega_{\rho(t)}^n - \max_{x\in M}\left(\frac{1}{V}\int_M (G(x,\cdot) + C_4)\triangle_{\rho(t)}(\varphi(t) - \rho(t))\omega_{\rho(t)}^n(y)\right) \\ \le & \tfrac{1}{V}\int_M (\varphi(t) - \rho(t))\omega_{\rho(t)}^n + nC_4, \end{aligned}$$

where $G(x,y)$ is the Green function associated to ω_ρ satisfying $G(x,\cdot) \ge 0.$ Therefore, there exists a uniform constant C such that

$$\sup_M\, (\varphi(t) - \rho(t)) \le C.$$

By Lemma 4.5, we have

$$-C \le \int_M (\varphi(t) - \rho(t))\, \omega_{\varphi(t)}^n \le C.$$

Furthermore, according to Theorem 4.3, the Kähler metrics $\omega_{\varphi(t)}$ have a uniform upper bound on both the Sobolev constant and the Poincaré constant. Now using Lemma 4.1, we conclude that there exists a uniform constant C such that

$$-C \le (\varphi(t) - \rho(t)) \le C.$$

Next we consider the following

$$\begin{aligned} \tfrac{\partial\varphi(t)}{\partial t} &= \log\left(\tfrac{\omega_{\varphi(t)}^n}{\omega_{\rho(t)}^n}\right) + (\varphi(t) - \rho(t)) \\ &\ge -C, \end{aligned}$$

for some uniform constant C. Recall that $|c(t)| = |\int_M \frac{\partial\varphi(t)}{\partial t}\, \omega_\varphi^n|$ is uniformly bounded. Therefore, there is some uniform constant C such that

$$\int_M |\frac{\partial\varphi}{\partial t}|\omega_\varphi^n \le C.$$

In view of the fact the K energy is uniformly bounded below, we arrive at

$$\int_0^\infty d\,t\int_M |\nabla\frac{\partial\varphi}{\partial t}|^2\, \omega_\varphi^n < \infty.$$

Since the Poincare constant of the evolving Kähler metric is bounded, we have

$$\int_a^{a+1} dt\int_M \left(\frac{\partial\varphi}{\partial t}\right)^2\, \omega_\varphi^n \le C,$$

where $c > 0$ is a constant independent of $a > 0$.

Note that $\frac{\partial \varphi(t)}{\partial t}$ satisfies the following evolution equation

$$\frac{\partial}{\partial t}\frac{\partial \varphi(t)}{\partial t} = \triangle_{\varphi} \frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial t}$$

and the fact that both the Sobolev and the Poincare constants of the evolving metrics are uniformly bounded. Applying Lemma 4.7, a parabolic version of Lemma 4.1 below, we prove that there exists a uniform constant C such that

$$-C \leq \frac{\partial \varphi(t)}{\partial t} = \log\left(\frac{\omega^n_{\varphi(t)}}{\omega^n_{\rho(t)}}\right) + (\varphi(t) + \rho(t)) < C.$$

It follows that

$$-C \leq \log\left(\frac{\omega^n_{\varphi(t)}}{\omega^n_{\rho(t)}}\right) < C.$$

□

By Proposition 2.14, there exists a one parameter family of $\sigma(t) \in \mathrm{Aut}(M)$ such that $\omega_{\varphi(t)}$ is centrally positioned with respect to the Kähler-Einstein metric $\omega_{\rho(t)}$. Here

$$\sigma(t)^*\omega_1 = \omega_{\rho(t)} = \omega + \sqrt{-1}\partial\overline{\partial}\rho(t). \tag{4.4}$$

This condition "centrally positioned" plays an important role in deriving Proposition 2.14 there. However, it is no longer needed once we have Proposition 2.14.

Lemma 4.6. *There exists a uniform constant C such that for all integers $i = 1, 2, \cdots, \infty$, we have*

$$|\rho(i) - \rho(i+1)| < C.$$

Moreover,

$$|\ \sigma(i+1)\sigma(i)^{-1}\ |_{\hbar} < C.$$

Here $\hbar$ is the left invariant metric in $\mathrm{Aut}(M)$.

Proof. The modified Kähler Ricci flow is

$$\frac{\partial}{\partial t}(\varphi - \rho) = \varphi - \rho + \log\frac{{\omega_{\varphi}}^n}{{\omega_{\rho(t)}}^n} - \frac{\partial \rho}{\partial t}.$$

Since $\frac{\partial \varphi}{\partial t}$ is uniformly bounded, we arrive at

$$\begin{aligned} & |\rho(i) - \rho(i+1)| \\ \leq\ & |\rho(i) - \varphi(i)| + |\rho(i+1) - \varphi(i+1)| + |\varphi(i) - \varphi(i+1)| \\ \leq\ & C. \end{aligned}$$

Since $\omega_{\rho(t)}$ is a Kähler-Einstein metric for any time t, we have (cf. equation (4.2))

$$|\log \frac{{\omega_{\rho(i+1)}}^n}{{\omega_{\rho(i)}}^n}| = |\rho(i) - \rho(i+1)| < C$$

and

$$|\ \sigma(i+1)\sigma(i)^{-1}\ |_\hbar < C.$$

□

This lemma allows us to do the following modification on the curve $\sigma(t) \in \mathrm{Aut}(M)$ (we also modify the curve $\rho(t)$ by the equation (4.4)): Fix all of the integer points $(\sigma(i), i = 1, 2, \cdots)$of the curve $\sigma(t)$ first. At each unit interval, replace the original curve in $\mathrm{Aut}(M)$ by a straight line which connects the two end points in $\mathrm{Aut}(M)$. Such a new curve in $\mathrm{Aut}(M)$ will satisfy all the estimates listed below (for convenience, we still denote it as $\sigma(t), \rho(t)$ respectively):

1. Theorem 4.4 still holds for this new curve $\rho(t)$ since we only change $\rho(t)$ by a uniformly controlled amount (fix at each integer points, and adapt linear intepolation between them).

2. The new curves $\sigma(t)$ and $\rho(t)$ are Lipschitz with a uniform Lipschitz constant for all the time $t \in [0, \infty)$. In fact, $\sigma(t)$ is a infinite long piecewise linear in $\mathrm{Aut}_r(M)$.

3. There exists a uniform constant C such that

$$|\left(\frac{d}{dt}\sigma(t)\right) \cdot \sigma(t)^{-1}| < C, \qquad \text{for any} \qquad t \neq \text{ integer}.$$

In the remaining of this section, we want to give a technical lemma required by the proof of Theorem 4.4.

Lemma 4.7. [3] *If the Poincare constant and the Sobolev constant of the evolving Kähler metrics are both uniformly bounded along the Kähler Ricci flow, and if* $\frac{\partial \varphi}{\partial t}$ *is bounded from below and if* $\int_a^{a+1} dt \int_M \left(\frac{\partial \varphi}{\partial t}\right)^2 \omega^n_{\varphi(t)}$ *is uniformly bounded from above for any* $a \geq 0$, *then* $\frac{\partial \varphi}{\partial t}$ *is uniformly bounded from above and below.*

Proof. Since $\frac{\partial \varphi}{\partial t}$ has a uniform lower bound, there is a constant c such that $u = \frac{\partial \varphi}{\partial t} + c > 1$ holds all the time. Now u satisfies the equation:

$$\frac{\partial}{\partial t}\ u = \triangle_\varphi\ u + u - c.$$

Set $d\,\mu(t) = \omega^n_{\varphi(t)}$ as the evolving volume element. Then

$$\frac{\partial}{\partial t}\, d\,\mu(t) = \triangle_\varphi u\ d\,\mu(t).$$

[3]This is a parabolic verison of Moser iteration arguments. We give a detailed proof here for the convenience of the readers.

For any $a < b < \infty$, define η to be any positive increasing function which vanishes at a. Set

$$\psi(t,x) = \eta^2 u^{\beta-1}$$

for any $\beta > 2$. Then (Here $\partial_t d\mu(t) = \triangle_\varphi u d\,\mu(t)$)

$$\begin{array}{rl} & \int_a^b \,d\,t \int_M \,(\partial_t u)\psi d\,\mu(t) \\ = & \int_a^b \,d\,t \left(\partial_t \int_M \,u\psi d\,\mu(t) - \int_M \,u\,\frac{\partial\psi}{\partial t} - \int_M \,\psi\,u\,\triangle_\varphi\,u\right) \\ = & \eta(b)^2 \int_M \,u^\beta - \int_a^b \,d\,t \{\int_M \left(u 2\eta\eta' u^{\beta-1} + u\eta^2(\beta-1)u^{\beta-2}\partial_t u\right) + \int_M \,\psi\,u\,\triangle_\varphi\,u\}. \end{array}$$

Thus,

$$\begin{array}{rl} & \int_a^b \,d\,t \left(\int_M \,\beta(\partial_t u)\psi d\,\mu(t) + \int_M \,\psi\,u\,\triangle_\varphi\,u\right) \\ = & \eta(b)^2 \int_M \,u^\beta - \int_a^b \,d\,t \int_M 2\eta\eta' u^\beta \\ = & \int_a^b \,d\,t \left(\int_M \,\beta(\triangle_\varphi u + u - c)\eta^2 u^{\beta-1} + \int_M \,\eta^2 u^{\beta-1}\,u\,\triangle_\varphi\,u\right) \\ \leq & -\int_a^b \,d\,t \left(\int_M \,\beta(\beta-1)u^{\beta-2}|\nabla u|^2\eta^2 - \int_M \,\beta\eta^2 u^\beta\right). \end{array}$$

Therefore, we have

$$\eta(b)^2 \int_M \,u^\beta + \int_a^b \,d\,t \int_M \,\beta(\beta-1)u^{\beta-2}|\nabla u|^2\eta^2 \leq \int_a^b \,d\,t \int_M \,\beta\eta^2 u^\beta + \int_a^b \,d\,t \int_M 2\eta\eta' u^\beta.$$

In other words,

$$\begin{array}{rl} & \eta(b)^2 \int_M \,u^\beta + \int_a^b \,d\,t \int_M \,4(1-\frac{1}{\beta})|\nabla u^{\frac{\beta}{2}}|^2\eta^2 \\ \leq & \int_a^b \,d\,t \int_M \,\beta(\eta^2 + 2\eta\eta')u^\beta \end{array}$$

or

$$\begin{array}{rl} & \eta(b)^2 \int_M \,u^\beta + \int_a^b \,d\,t \int_M \,4(1-\frac{1}{\beta})\left(|\nabla u^{\frac{\beta}{2}}|^2\eta^2 + u^\beta\eta^2\right) \\ \leq & \int_a^b \,d\,t \int_M \,\beta(2\eta^2 + 2\eta\eta')u^\beta. \end{array}$$

In particular, this implies that

$$\max_{a\leq t\leq b} \int_M \,\eta(t)^2 u^\beta \leq \int_a^b \,d\,t \int_M \,\beta(2\eta^2 + 2\eta\eta')u^\beta.$$

Let us first state a lemma.

Lemma 4.8. *(Sobolev inequality) Assume* $0 \leq a < b$ *and* $v : M \times [a,b] \to \mathbf{R}$ *is a measurable function such that*

$$\sup_{a\leq t\leq b} \,|v(\cdot,t)|_{L^2(M,d\mu(t))} < \infty$$

and

$$\int_a^b \int_M \,|\nabla v|^2 \,d\,\mu d\,t < \infty,$$

then we have ($m = 2n = \dim(M)$)

$$\int_a^b \,d\,t \int_M |v|^{\frac{2(m+2)}{m}} \,d\,\mu(t) \leq \sigma \sup_{a\leq t\leq b} \,|v(\cdot,t)|^{\frac{4}{m}}_{L^2(M,d\mu(t))} \int_a^b \,d\,t \int_M \,\left(|\nabla v|^2 + v^2\right) \,d\,\mu(t).$$

Here σ *is the Sobolev constant.*

Proof. For any $a \le t \le b$, we have

$$\begin{aligned}
|v(\cdot,t)|_{L^{\frac{2(m+2)}{m}}(M,d\mu(t))} &\le |v(\cdot,t)|^{\frac{2}{m+2}}_{L^2(M,d\mu(t))}\, |v(\cdot,t)|^{\frac{m}{m+2}}_{L^{\frac{2m}{m-2}}(M,d\mu(t))} \\
&\le |v(\cdot,t)^{\frac{2}{m+2}}_{L^2(M,d\mu(t))} \left(\sigma \int_M \left(|\nabla v|^2 + v^2\right) d\mu(t)\right)^{\frac{m}{2(m+2)}}.
\end{aligned}$$

The lemma follows by taking $\frac{2(m+2)}{m}$ power on both sides and integrating over $[a,b]$. ☐

Now we return to the proof of main theorem. Let $v = \eta u^{\frac{\beta}{2}}$, we have

$$\begin{aligned}
&\left(\int_a^b dt \int_M |\eta^2 u^\beta|^{\frac{m+2}{m}}\right)^{\frac{m}{m+2}} \\
\le\ & \sigma^{\frac{m}{m+2}} \sup_{a\le t\le b} |v(\cdot,t)|^{\frac{4}{m+2}}_{L^2(M,d\mu(t))} \left(\int_a^b dt \int_M \left(|\nabla v|^2 + v^2\right) d\mu(t)\right)^{\frac{m}{m+2}} \\
\le\ & C(m) \left(\int_a^b dt \int_M \beta(2\eta^2 + 2\eta\eta')u^\beta\right)^{\frac{2}{m+2}} \left(\int_a^b dt \int_M \beta(2\eta^2 + 2\eta\eta')u^\beta\right)^{\frac{m}{m+2}} \\
\le\ & C(m)\beta \int_a^b dt \int_M (\eta^2 + \eta\eta')u^\beta.
\end{aligned}$$

Here $C(m)$ is a constant depending only on the Sobolev constant of $(M, g(t))$ and dimension of manifold.

Now for any $a \le b_0 < b \le a+1$, define

$$b_k = b - \frac{b - b_0}{2^k}$$

for any $k \in \mathbf{Z}_+$. Fix a function $\eta_0 \in C^\infty(\mathbf{R}, \mathbf{R})$ such that $0 \le \eta_0 \le 1, \eta_0' \ge 0, \eta_0(t) = 0$ for $t \le 0$ and $\eta_0(t) = 1$ for $t \ge 1$. For each integer $k > 0$, we let $\eta(t) = \eta_0(\frac{t-b_k}{b_{k+1}-b_k})$ and $\beta = 2\left(\frac{m+2}{m}\right)^k$. Then $(b_{k+1} - b_k = \frac{b-b_0}{2^{k+1}})$

$$\begin{aligned}
&\left(\int_{b_{k+1}}^{a+1} dt \int_M u^{2(\frac{m+2}{m})^{k+1}} d\mu\right)^{\frac{1}{2}(\frac{m}{m+2})^{k+1}} \\
\le\ & C(m)^{\frac{1}{2}(\frac{m}{m+2})^k} \left(\frac{m+2}{m}\right)^{k(\frac{m}{m+2})^k} \left(\frac{2^{k+1}}{b-b_0}\right)^{\frac{1}{2}(\frac{m}{m+2})^k} \left(\int_{b_k}^{a+1} dt \int_M u^{2\left(\frac{m+2}{m}\right)^k} d\mu\right)^{\frac{1}{2}\left(\frac{m}{m+2}\right)^k}.
\end{aligned}$$

The iteration shows that for any integer $k > 0$, we have

$$\begin{aligned}
&\left(\int_b^{a+1} dt \int_M u^{2(\frac{m+2}{m})^{k+1}} d\mu\right)^{\frac{1}{2}(\frac{m}{m+2})^{k+1}} \\
\le\ & \frac{C(m)}{(b-b_0)^{\frac{m+2}{4}}} \left(\int_a^b dt \int_M u^2 d\mu\right)^{\frac{1}{2}}.
\end{aligned}$$

Here again $C(m)$ is a uniform constant which depends only on the sobleve constant of the evolving metrics and the dimension m. Since the last term is uniformly bounded, this implies that as $k \to \infty$, we have

$$\sup_{b\le t\le a+1} u \le \frac{C(m)}{(b-b_0)^{\frac{m+2}{4}}} \left(\int_a^b dt \int_M u^2 d\mu\right)^{\frac{1}{2}}.$$

☐

5 Uniform bounds on gauge

In order to use this uniform C^0 estimate and the flow equation (5.3) to derive the desired C^2 estimate, we still need to control the size of $\frac{\partial \rho}{\partial t}$. However, from our earlier modification above, we can not determine $\frac{\partial \rho}{\partial t}$ at any integer point. For any non-integer points, we have a uniform bound C such that

$$\mid \frac{\partial \rho}{\partial t} \mid_t < C, \qquad \forall\, t \neq \text{integer}.$$

Note that $\sigma(t)$ is an infinite long broken line in $\mathrm{Aut}_r(M)$. Next we want to further modify the curve $\sigma(t)$ by smoothing the corner at the integer points. Let us first set up some notations. Let $\jmath$ be the Lie algebra of $\mathrm{Aut}(M)$. As before, suppose $\hbar$ is the left invariant metric on $\mathrm{Aut}(M)$. Denote id the identity element in $\mathrm{Aut}(M)$ and exp is the exponential map at the identity. Use B_r to denote the ball centered at the identity element with radius r.

After the modification of last section, $\sigma(t)$ is an infinite long broken line in $\mathrm{Aut}(M)$. We can write down this curve explicitly: For any integer $i = 0, 1, 2, \cdots, \infty$, we have

$$\sigma(t) = \sigma(i) \cdot \exp((t-i)X_i), \qquad \forall\, t \in [i, i+1]. \tag{5.1}$$

Here $\{X_i\}$ is a sequence of vector fields in $\jmath$ with a uniform upper bound C on their lengths:

$$\|X_i\|_\hbar \leq C, \qquad \forall\, i = 0, 1, 2, \cdots, \infty. \tag{5.2}$$

Then there exists a uniform positive number $\frac{1}{4} > \delta > 0$ such that for any integer $i > 0$, we have

$$\sigma(t) \in \sigma(i) \cdot B_{\frac{1}{2}}, \qquad \forall\, t \in (i-\delta, i+\delta).$$

Note that δ depends on $\|X_i\|_\hbar$. Since the latter has a uniform upper bound, δ must have a uniform lower bound. We then can choose one $\delta > 0$ for all i.

At each ball $\sigma_i \cdot B_1$, we want to replace the curve segment $\sigma(t)$ ($t \in [i-\delta, i+\delta]$) by a new smooth curve $\tilde{\sigma}(t)$ such that:

1. The two end points and their derivatives are not changed[4]:

$$\tilde{\sigma}(i \pm \delta) = \sigma(i \pm \delta)$$

and

$$\left(\left(\frac{d}{dt}\tilde{\sigma}(t)\right)\tilde{\sigma}(t)^{-1}\right)_{t=i\pm\delta} = \left(\left(\frac{d}{dt}\sigma(t)\right)\sigma(t)^{-1}\right)_{t=i\pm\delta}.$$

[4]In a Euclidean ball, we can use the 4th order polynomial to achieve this. In any unit ball of any finite dimensional Riemannian manifold, we can always do this uniformly, as long as the metric and other data involved are uniformly bounded.

2. There exists a uniform bound C' which depends only on the upper bound of $\|X_i\|_\hbar$ and δ such that

$$\| \left(\frac{d}{d\,t}\tilde{\sigma}(t)\right) \tilde{\sigma}(t)^{-1}\|_\hbar \leq C', \qquad \forall\, t \in [i-\delta, i+\delta].$$

3. For any $t \in [i-\delta, i+\delta]$, we have $\tilde{\sigma}(t) \in \sigma(i)B_1$. In other words, there exists a uniform constant C such that:

$$|\tilde{\sigma}(t)\sigma(t)^{-1}|_\hbar < C.$$

The last step is to set $\tilde{\sigma}(t) = \sigma(t)$ for all other time. Then the new curve $\tilde{\sigma}(t)$ has all the properties we want:

1. There exists a uniform constant C such that $|\tilde{\sigma}(t)\sigma(t)^{-1}| < C$ for all $t \in [0, \infty)$.

2. There exists a uniform constant C such that

$$| \left(\frac{d}{d\,t}\tilde{\sigma}(t)\right) \cdot \tilde{\sigma}(t)^{-1}| < C, \qquad \text{for any} \qquad t \geq 0.$$

Denote by $\tilde{\sigma}(t)^*\omega_1 = \omega_{\tilde{\rho}(t)} = \omega + \sqrt{-1}\partial\bar{\partial}\tilde{\rho}(t)$. Then, $\omega_{\tilde{\rho}(t)}$ is a Kähler-Einstein metric

$$\omega^n_{\tilde{\rho}(t)} = e^{-\tilde{\rho}(t)+h_\omega}\ \omega^n.$$

There exists a uniform constant C such that

$$|\rho(t) - \tilde{\rho}(t)| < C \tag{5.3}$$

and

$$|\frac{\partial\tilde{\rho}(t)}{\partial t}| < C$$

hold for all t.

Now the inequality (5.3) implies that

$$|\log\left(\det\frac{\omega^n_{\rho(t)}}{\omega^n_{\tilde{\rho}(t)}}\right)| \leq C.$$

Combining this with Theorem 4.4, we arrive at

Theorem 5.1. *There exists a one parameter family of Kähler Einstein metrics* $\omega_{\tilde{\rho}(t)} = \omega + \sqrt{-1}\partial\bar{\partial}\tilde{\rho}(t)$, *which is essentially parallel to the initial family of Kähler Einstein metrics, and a uniform constant* C *such that the following holds*

$$|\varphi(t) - \tilde{\rho}(t)| \leq C,$$

$$-C < \log\frac{{\omega_{\varphi(t)}}^n}{{\omega_{\tilde{\rho}(t)}}^n} < C$$

and

$$|\frac{\partial\tilde{\rho}(t)}{\partial t}| < C$$

over the entire modified Kähler Ricci flow.

6 C^2 and higher order derivative estimates

Consider the modified Kähler Ricci flow

$$\frac{\partial}{\partial t}(\varphi - \tilde{\rho}) = \varphi - \tilde{\rho} + \log \frac{{\omega_{\varphi}}^n}{{\omega_{\tilde{\rho}(t)}}^n} - \frac{\partial \tilde{\rho}}{\partial t}. \tag{6.1}$$

By Theorem 5.1, we have a uniform bound on both $\mid (\varphi - \tilde{\rho}) \mid$ and $\mid \frac{\partial}{\partial t}(\varphi - \tilde{\rho}) \mid$. This fact will play an important role in deriving C^2 estimate on the evolved relative Kähler potential $(\varphi - \tilde{\rho})$ in this section:

Theorem 6.1. *If the C^0 norms of $\mid (\varphi - \tilde{\rho}) \mid$ and $\mid \frac{\partial}{\partial t}(\varphi - \tilde{\rho}) \mid$ are uniformly bounded (independent of time t), then there exists a uniform constant C such that*

$$0 \leq n + \tilde{\triangle}(\varphi - \tilde{\rho}) < C,$$

where $\tilde{\triangle}$ is the Laplacian operator corresponding to the evolved Kähler-Einstein metrics $\omega_{\rho(t)}$.

We set up some notations first. Let $\triangle'$ be the Laplacian operator corresponding to the evolved Kähler metric $\omega_{\varphi(t)}$ respectively. Let $\Box = \triangle' - \frac{\partial}{\partial t}$. Put $\omega_{\tilde{\rho}(t)} = \sqrt{-1} h_{\alpha\bar{\beta}} d\, z^{\alpha} \otimes z^{\bar{\beta}}$ and $\omega_{\varphi(t)} = \sqrt{-1} g'_{\alpha\bar{\beta}} d\, z^{\alpha} \otimes d\, z^{\bar{\beta}}$ where

$$g'_{\alpha\bar{\beta}} = h_{\alpha\bar{\beta}} + \frac{\partial^2 \left(\varphi(t) - \tilde{\rho}(t)\right)}{\partial z^{\alpha} \partial z^{\bar{\beta}}}.$$

Then

$$\triangle' = \sum_{\alpha,\beta=1}^{n} g'^{\alpha\bar{\beta}} \frac{\partial^2}{\partial z^{\alpha} \partial z^{\bar{\beta}}}, \qquad \tilde{\triangle} = \sum_{\alpha,\beta=1}^{n} h^{\alpha\bar{\beta}} \frac{\partial^2}{\partial z^{\alpha} \partial z^{\bar{\beta}}}.$$

and

$$[\frac{\partial}{\partial t}, \tilde{\triangle}] = -\sum_{a,b,c,d=1}^{n} h^{a\bar{b}} \frac{\partial^2 \frac{\partial \tilde{\rho}}{\partial t}}{\partial z^c \partial z^{\bar{b}}} h^{c\bar{d}} \frac{\partial^2}{\partial z^a \partial z^{\bar{d}}}.$$

Furthermore, we have

$$\tilde{\triangle} \frac{\partial \tilde{\rho}}{\partial t} = -\frac{\partial \tilde{\rho}}{\partial t}.$$

Thus the Hessian of $\frac{\partial \tilde{\rho}}{\partial t}$ with respect to the evolved Kähler Einstein metric $\omega_{\tilde{\rho}(t)}$ is uniformly bounded from above since $|\frac{\partial \tilde{\rho}}{\partial t}|$ is uniformly bounded from above.

Proof. of Theorem 6.1: We want to use the maximum principle in this proof. Let us first calculate $\Box \left(n + \tilde{\triangle}(\varphi - \tilde{\rho}) \right)$.

Let us choose a coordinate so that at a fixed point both $\omega_{\tilde{\rho}(t)} = \sqrt{-1} h_{\alpha\bar{\beta}} d\, z^{\alpha} \otimes d\, z^{\bar{\beta}}$ and the complex Hessian of $\varphi(t) - \tilde{\rho}(t)$ are in diagonal forms. In particular, we assume that $h_{i\bar{j}} = \delta_{i\bar{j}}$ and $\left(\varphi(t) - \tilde{\rho}(t)\right)_{i\bar{j}} = \delta_{i\bar{j}} \left(\varphi(t) - \tilde{\rho}(t)\right)_{i\bar{i}}$. Thus

$$g'^{i\bar{s}} = \frac{\delta_{i\bar{s}}}{1 + \left(\varphi(t) - \tilde{\rho}(t)\right)_{i\bar{i}}}.$$

For convenience, put

$$F = \frac{\partial}{\partial t}(\varphi - \tilde{\rho}) - (\varphi - \tilde{\rho}) + \frac{\partial \tilde{\rho}}{\partial t}.$$

Note that F has a uniform bound. The modified Kähler Ricci flow (6.1) can be reduced to

$$\log \frac{{\omega_{\varphi}}^n}{{\omega_{\tilde{\rho}(t)}}^n} = F,$$

or, equivalently

$$\left(\omega_{\tilde{\rho}(t)} + \partial\bar{\partial}(\varphi - \tilde{\rho})\right)^n = e^F \ \o^n,$$

i.e.,

$$\log\det\left(h_{i\bar{j}} + \frac{\partial^2(\varphi - \tilde{\rho})}{\partial z_i \partial z_{\bar{j}}}\right) = F + \log\det(h_{i\bar{j}}).$$

For convenience, set

$$\psi(t) = \varphi(t) - \tilde{\rho}(t)$$

in this proof. Note that both $|\ \psi(t)\ |$ and $|\ \frac{\partial \psi(t)}{\partial t}\ |$ are uniformly bounded (cf. Theorem 5.1). We first follow the standard calculation of C^2 estimates in [28]. Differentiate both sides with respect to $\frac{\partial}{\partial z_k}$

$$(g')^{i\bar{j}}\left(\frac{\partial h_{i\bar{j}}}{\partial z_k} + \frac{\partial^3 \psi(t)}{\partial z_i \partial \bar{z}_j \partial z_k}\right) - h^{i\bar{j}}\frac{\partial h_{i\bar{j}}}{\partial z_k} = \frac{\partial F}{\partial z_k},$$

and differentiating again with respect to $\frac{\partial}{\partial \bar{z}_l}$ yields

$$\begin{aligned}&(g')^{i\bar{j}}\left(\frac{\partial^2 h_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} + \frac{\partial^4 \psi(t)}{\partial z_i \partial \bar{z}_j \partial z_k \partial \bar{z}_l}\right) + h^{t\bar{j}}h^{i\bar{s}}\frac{\partial h_{t\bar{s}}}{\partial \bar{z}_l}\frac{\partial h_{i\bar{j}}}{\partial z_k} - h^{i\bar{j}}\frac{\partial^2 h_{i\bar{j}}}{\partial z_k \partial \bar{z}_l}\\ &-(g')^{t\bar{j}}(g')^{i\bar{s}}\left(\frac{\partial h_{t\bar{s}}}{\partial \bar{z}_l} + \frac{\partial^3 \psi(t)}{\partial z_t \partial \bar{z}_s \partial \bar{z}_l}\right)\left(\frac{\partial h_{i\bar{j}}}{\partial z_k} + \frac{\partial^3 \psi(t)}{\partial z_i \partial \bar{z}_j \partial z_k}\right) = \frac{\partial^2 F}{\partial z_k \partial \bar{z}_l}.\end{aligned}$$

Assume that we have normal coordinates at the given point, i.e., $h_{i\bar{j}} = \delta_{ij}$ and the first order derivatives of g vanish. Now taking the trace of both sides results in

$$\begin{aligned}\widetilde{\Delta} F = {}& h^{k\bar{l}}(g')^{i\bar{j}}\left(\frac{\partial^2 h_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} + \frac{\partial^4 \psi(t)}{\partial z_i \partial \bar{z}_j \partial z_k \partial \bar{z}_l}\right)\\ &- h^{k\bar{l}}(g')^{t\bar{j}}(g')^{i\bar{s}}\frac{\partial^3 \psi(t)}{\partial z_t \partial \bar{z}_s \partial \bar{z}_l}\frac{\partial^3 \psi(t)}{\partial z_i \partial \bar{z}_j \partial z_k} - h^{k\bar{l}}h^{i\bar{j}}\frac{\partial^2 h_{i\bar{j}}}{\partial z_k \partial \bar{z}_l}.\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}\Delta'(\widetilde{\Delta}\psi(t)) &= (g')^{k\bar{l}}\frac{\partial^2}{\partial z_k \partial \bar{z}_l}\left(h^{i\bar{j}}\frac{\partial^2 \psi(t)}{\partial z_i \partial \bar{z}_j}\right)\\ &= (g')^{k\bar{l}}h^{i\bar{j}}\frac{\partial^4 \psi(t)}{\partial z_i \partial \bar{z}_j \partial z_k \partial \bar{z}_l} + (g')^{k\bar{l}}\frac{\partial^2 h^{i\bar{j}}}{\partial z_k \partial \bar{z}_l}\frac{\partial^2 \psi(t)}{\partial z_i \partial \bar{z}_j},\end{aligned}$$

and we will substitute $\frac{\partial^4\psi(t)}{\partial z_i\partial\bar z_j\partial z_k\partial\bar z_l}$ in $\Delta'(\widetilde\Delta\psi(t))$ so that the above reads

$$\Delta'(\widetilde\Delta\psi(t)) = -h^{k\bar l}(g')^{i\bar j}\frac{\partial^2 h_{i\bar j}}{\partial z_k\partial\bar z_l} + h^{k\bar l}(g')^{t\bar j}(g')^{i\bar s}\frac{\partial^3\psi(t)}{\partial z_t\partial\bar z_s\partial\bar z_l}\frac{\partial^3\psi(t)}{\partial z_i\partial\bar z_j\partial z_k}$$
$$+ h^{k\bar l}h^{i\bar j}\frac{\partial^2 h_{i\bar j}}{\partial z_k\partial\bar z_l} + \widetilde\Delta F + (g')^{k\bar l}\frac{\partial^2 h^{i\bar j}}{\partial z_k\partial\bar z_l}\frac{\partial^2\psi(t)}{\partial z_i\partial\bar z_j},$$

which we can rewrite after substituting $\frac{\partial^2 h_{i\bar j}}{\partial z_k\partial\bar z_l} = -R_{i\bar j k\bar l}$ and $\frac{\partial^2 h^{i\bar j}}{\partial z_k\partial\bar z_l} = R_{j\bar i k\bar l}$ as

$$\begin{aligned}\Delta'(\widetilde\Delta\psi(t)) \quad &= \quad \widetilde\Delta F + h^{k\bar l}(g')^{t\bar j}(g')^{i\bar s}\psi(t)_{t\bar s l}\psi(t)_{i\bar j k}\\ &\quad +(g')^{i\bar j}h^{k\bar l}R_{i\bar j k\bar l} - h^{i\bar j}h^{k\bar l}R_{i\bar j k\bar l} + (g')^{k\bar l}R_{j\bar i k\bar l}\psi(t)_{i\bar j}.\end{aligned}$$

Restrict to the coordinates we chose in the beginning so that both g and $\psi(t)$ are in diagonal form. The above transforms to

$$\begin{aligned}\Delta'(\widetilde\Delta\psi(t)) \quad &= \quad \tfrac{1}{1+\psi(t)_{i\bar i}}\tfrac{1}{1+\psi(t)_{j\bar j}}\psi(t)_{i\bar j k}\psi(t)_{\bar i j\bar k} + \widetilde\Delta F\\ &\quad +R_{i\bar i k\bar k}(-1 + \tfrac{1}{1+\psi(t)_{i\bar i}} + \tfrac{\psi(t)_{i\bar i}}{1+\psi(t)_{k\bar k}}).\end{aligned}$$

Set now $C = \inf_{i\neq k} R_{i\bar i k\bar k}$ and observe that

$$\begin{aligned}R_{i\bar i k\bar k}(-1 + \frac{1}{1+\psi(t)_{i\bar i}} + \frac{\psi(t)_{i\bar i}}{1+\psi(t)_{k\bar k}}) &= \frac12 R_{i\bar i k\bar k}\frac{(\psi(t)_{k\bar k} - \psi(t)_{i\bar i})^2}{(1+\psi(t)_{i\bar i})(1+\psi(t)_{k\bar k})}\\ &\geq \frac{C}{2}\frac{(1+\psi(t)_{k\bar k} - 1 - \psi(t)_{i\bar i})^2}{(1+\psi(t)_{i\bar i})(1+\psi(t)_{k\bar k})}\\ &= C\left(\frac{1+\psi(t)_{i\bar i}}{1+\psi(t)_{k\bar k}} - 1\right),\end{aligned}$$

which yields

$$\begin{aligned}\Delta'(\widetilde\Delta\psi(t)) \quad &\geq \quad \tfrac{1}{(1+\psi(t)_{i\bar i})(1+\psi(t)_{j\bar j})}\psi(t)_{i\bar j k}\psi(t)_{\bar i j\bar k} + \widetilde\Delta F\\ &\quad +C\left((n+\widetilde\Delta\psi(t))\textstyle\sum_i \tfrac{1}{1+\psi(t)_{i\bar i}} - 1\right).\end{aligned}$$

We need to apply one more trick to obtain the requested estimates. Namely,

$$\begin{aligned}\Delta'(e^{-l\psi(t)}(n+\widetilde\Delta\psi(t))) \quad &= \quad e^{-l\psi(t)}\Delta'(\widetilde\Delta\psi(t)) + 2\nabla' e^{-l\psi(t)}\nabla'(n+\widetilde\Delta\psi(t))\\ &\quad +\Delta'(e^{-l\psi(t)})(n+\widetilde\Delta\psi(t))\\ &= \quad e^{-l\psi(t)}\Delta'(\widetilde\Delta\psi(t)) - le^{-l\psi(t)}(g')^{i\bar i}\psi(t)_i(\widetilde\Delta\psi(t))_{\bar i}\\ &\quad -le^{-l\psi(t)}(g')^{i\bar i}\psi(t)_{\bar i}(\widetilde\Delta\psi(t))_i\\ &\quad -le^{-l\psi(t)}\Delta'\psi(t)(n+\widetilde\Delta\psi(t))\\ &\quad +l^2e^{-l\psi(t)}(g')^{i\bar i}\psi(t)_i\psi(t)_{\bar i}(n+\widetilde\Delta\psi(t))\\ &\geq \quad e^{-l\psi(t)}\Delta'(\widetilde\Delta\psi(t))\\ &\quad -e^{-l\psi(t)}(g')^{i\bar i}(n+\widetilde\Delta\psi(t))^{-1}(\widetilde\Delta\psi(t))_i(\widetilde\Delta\psi(t))_{\bar i}\\ &\quad -le^{-l\psi(t)}\Delta'\psi(t)(n+\widetilde\Delta\psi(t)),\end{aligned}$$

which follows from the Schwarz Lemma applied to the middle two terms. We will write out one term here, the other goes in an analogous way

$$
\begin{aligned}
&(le^{-\frac{l}{2}\psi(t)}\psi(t)_i(n+\widetilde{\Delta}\psi(t))^{\frac{1}{2}})(e^{-\frac{l}{2}\psi(t)}(\widetilde{\Delta}\psi(t))_{\bar{i}}(n+\widetilde{\Delta}\psi(t))^{-\frac{1}{2}})\\
\leq&\frac{1}{2}(l^2e^{-l\psi(t)}\psi(t)_i\psi(t)_{\bar{i}}(n+\widetilde{\Delta}\psi(t))\\
&+e^{-l\psi(t)}(\widetilde{\Delta}\psi(t))_{\bar{i}}(\widetilde{\Delta}\psi(t))_i(n+\widetilde{\Delta}\psi(t))^{-1}).
\end{aligned}
$$

Consider now the following

$$
\begin{aligned}
&-(n+\widetilde{\Delta}\psi(t))^{-1}\frac{1}{1+\psi(t)_{i\bar{i}}}(\widetilde{\Delta}\psi(t))_i(\widetilde{\Delta}\psi(t))_{\bar{i}}+\Delta'\widetilde{\Delta}\psi(t)\geq\\
&-(n+\widetilde{\Delta}\psi(t))^{-1}\frac{1}{1+\psi(t)_{i\bar{i}}}|\psi(t)_{k\bar{k}i}|^2+\widetilde{\Delta}F\\
&+\frac{1}{1+\psi(t)_{i\bar{i}}}\frac{1}{1+\psi(t)_{k\bar{k}}}\psi(t)_{k\bar{i}\bar{j}}\psi(t)_{i\bar{k}j}+C(n+\widetilde{\Delta}\psi(t))\frac{1}{1+\psi(t)_{i\bar{i}}}.
\end{aligned}
$$

On the other hand, using the Schwarz inequality, we have

$$
\begin{aligned}
&(n+\widetilde{\Delta}\psi(t))^{-1}\frac{1}{1+\psi(t)_{i\bar{i}}}|\psi(t)_{k\bar{k}i}|^2\\
&\quad=(n+\widetilde{\Delta}\psi(t))^{-1}\frac{1}{1+\psi(t)_{i\bar{i}}}\left|\frac{\psi(t)_{k\bar{k}i}}{(1+\psi(t)_{k\bar{k}})^{\frac{1}{2}}}(1+\psi(t)_{k\bar{k}})^{\frac{1}{2}}\right|^2\\
&\quad\leq(n+\widetilde{\Delta}\psi(t))^{-1}\left(\frac{1}{1+\psi(t)_{i\bar{i}}}\frac{1}{1+\psi(t)_{k\bar{k}}}\psi(t)_{k\bar{k}i}\psi(t)_{\bar{k}k\bar{i}}\right)\left(1+\psi(t)_{l\bar{l}}\right)\\
&\quad=\frac{1}{1+\psi(t)_{i\bar{i}}}\frac{1}{1+\psi(t)_{k\bar{k}}}\psi(t)_{k\bar{k}i}\psi(t)_{\bar{k}k\bar{i}}\\
&\quad=\frac{1}{1+\psi(t)_{i\bar{i}}}\frac{1}{1+\psi(t)_{k\bar{k}}}\psi(t)_{i\bar{k}k}\psi(t)_{k\bar{i}\bar{k}}\\
&\quad\leq\frac{1}{1+\psi(t)_{i\bar{i}}}\frac{1}{1+\psi(t)_{k\bar{k}}}\psi(t)_{i\bar{k}j}\psi(t)_{k\bar{i}\bar{j}},
\end{aligned}
$$

so that we get

$$
\begin{aligned}
&-(n+\widetilde{\Delta}\psi(t))^{-1}\tfrac{1}{1+\psi(t)_{i\bar{i}}}(\widetilde{\Delta}\psi(t))_i(\widetilde{\Delta}\psi(t))_{\bar{i}}+\Delta'\widetilde{\Delta}\psi(t)\\
&\geq\widetilde{\Delta}F+C(n+\widetilde{\Delta}\psi(t))\tfrac{1}{1+\psi(t)_{i\bar{i}}}.
\end{aligned}
$$

Putting all these together, we obtain

$$
\begin{aligned}
&\triangle'\left(e^{-\lambda\psi(t)}(n+\tilde{\triangle}\psi(t))\right)\\
\geq e^{-\lambda\psi(t)}&\left(\tilde{\triangle}F+C(n+\tilde{\triangle}\psi(t))\sum_{i=1}^{n}\frac{1}{1+\psi(t)_{i\bar{i}}}\right)\\
&-\lambda\,e^{-\lambda\psi(t)}\triangle'\psi(t)\,(n+\tilde{\triangle}\,(\varphi-\tilde{\rho})).
\end{aligned}
\tag{6.2}
$$

Consider

$$
\begin{aligned}
\tilde{\triangle} F &= \tilde{\triangle}\left(\frac{\partial}{\partial t}\psi(t) - \psi(t) + \frac{\partial \tilde{\rho}}{\partial t}\right) \\
&= \tilde{\triangle}\frac{\partial}{\partial t}\psi(t) - (n + \tilde{\triangle}\psi(t)) + n + \tilde{\triangle}\frac{\partial \tilde{\rho}}{\partial t} \\
&\geq \frac{\partial}{\partial t}(n + \tilde{\triangle}\psi(t)) + \sum_{a,b,c,d=1}^{n} h^{a\bar{b}} \frac{\partial^2 \frac{\partial \tilde{\rho}}{\partial t}}{\partial z^c \partial z^{\bar{b}}} h^{c\bar{d}} \frac{\partial^2 \psi(t)}{\partial z^a \partial z^{\bar{d}}} \\
&\qquad - (n + \tilde{\triangle}\psi(t)) + n + \tilde{\triangle}\frac{\partial \tilde{\rho}}{\partial t} \\
&\geq \frac{\partial}{\partial t}(n + \tilde{\triangle}\psi(t)) - c_1(n + \tilde{\triangle}\psi(t)) - c_2
\end{aligned}
$$

for some uniform constants c_1 and c_2. In the last inequality, we have used the fact that $|\frac{\partial \tilde{\rho}}{\partial t}|$ is uniformly bounded and

$$
\left| \frac{\partial^2 \frac{\partial \tilde{\rho}}{\partial t}}{\partial z^c \partial z^{\bar{b}}} \right|_{\tilde{\rho}(t)} \leq c_3 \cdot \left| \frac{\partial \tilde{\rho}}{\partial t} \right|,
$$

and

$$
0 < h_{c\bar{d}} + \frac{\partial^2 \psi(t)}{\partial z^c \partial z^{\bar{d}}} \leq \left(n + \tilde{\triangle}\psi(t)\right) \quad h_{c\bar{d}}
$$

holds as matrix. Here c_3 is some uniform constant.

$$
\begin{aligned}
e^{-\lambda\psi(t)}\tilde{\triangle} F &\geq e^{-\lambda\psi(t)}\frac{\partial}{\partial t}(n + \tilde{\triangle}\psi(t)) \\
&\qquad - c_1 \, e^{-\lambda\psi(t)} \, (n + \tilde{\triangle}\psi(t)) - c_2 \, e^{-\lambda\psi(t)} \\
&\geq \frac{\partial}{\partial t}\left(e^{-\lambda\psi(t)}(n + \tilde{\triangle}\psi(t))\right) + \lambda \frac{\partial}{\partial t}\psi(t) e^{-\lambda\psi(t)}(n + \tilde{\triangle}\psi(t)) \\
&\qquad - c_1 \, e^{-\lambda\psi(t)} \, (n + \tilde{\triangle}\psi(t)) - c_2 \, e^{-\lambda\psi(t)} \\
&\geq \frac{\partial}{\partial t}\left(e^{-\lambda\psi(t)}(n + \tilde{\triangle}\psi(t))\right) - (c_1 + |\lambda| c_4) \, e^{-\lambda\psi(t)} \, (n + \tilde{\triangle}\psi(t)) \\
&\qquad - c_2 \, e^{-\lambda\psi(t)}.
\end{aligned}
$$

Here c_4 is a uniform constant. Plugging this into the inequality (6.2), we obtain

$$
\begin{aligned}
&\Box\left(e^{-\lambda\psi(t)}(n + \tilde{\triangle}\psi(t))\right) \\
&\geq e^{-\lambda\psi(t)}\left(C(n + \tilde{\triangle}\psi(t)) \sum_{i=1}^{n} \frac{1}{1 + \psi(t)_{i\bar{i}}}\right) \\
&\qquad - \lambda \, e^{-\lambda\psi(t)} \triangle' \psi(t) \, (n + \tilde{\triangle}\psi(t)) \\
&\qquad - (c_1 + |\lambda| c_4) \, e^{-\lambda\psi(t)} \, (n + \tilde{\triangle}\psi(t)) - c_2 \, e^{-\lambda\psi(t)}.
\end{aligned}
$$

Now

$$
\triangle' \psi(t) = n - \sum_{i=1}^{n} \frac{1}{1 + (\varphi - \tilde{\rho})_{i\bar{i}}}.
$$

Plugging this into the above inequality, we obtain

$$
\begin{aligned}
&\Box\left(e^{-\lambda\psi(t)}(n + \tilde{\triangle}\psi(t))\right) \\
&\geq e^{-\lambda\psi(t)}\left((C + \lambda)(n + \tilde{\triangle}\psi(t)) \sum_{i=1}^{n} \frac{1}{1 + \psi(t)_{i\bar{i}}}\right) \\
&\qquad - (c_1 + |\lambda| c_4 + n) \, e^{-\lambda\psi(t)} \, (n + \tilde{\triangle}\psi(t)) - c_2 \, e^{-\lambda\psi(t)}.
\end{aligned}
$$

Let $\lambda = -C + 1$, we then have

$$\begin{aligned}
&\Box\left(e^{-\lambda\psi(t)}(n+\tilde{\triangle}\psi(t))\right) \\
&\geq e^{-\lambda\psi(t)}\left((n+\tilde{\triangle}\psi(t))\sum_{i=1}^{n}\frac{1}{1+\psi(t)_{i\bar{i}}}\right) \\
&\qquad -c_5\, e^{-\lambda\psi(t)}\,(n+\tilde{\triangle}\psi(t)) - c_2\, e^{-\lambda(\varphi-\tilde{\rho})}.
\end{aligned}$$

Here c_5 is a uniform constant. Note the following algebraic inequality

$$\begin{aligned}
\textstyle\sum_i \frac{1}{1+\psi(t)_{i\bar{i}}} &\geq \left(\frac{\sum_i(1+\psi(t)_{i\bar{i}})}{\prod_i(1+(\varphi-\tilde{\rho})_{i\bar{i}})}\right)^{\frac{1}{n-1}} \\
&= e^{-\frac{F}{n-1}}(n+\tilde{\Delta}\psi(t))^{\frac{1}{n-1}}.
\end{aligned}$$

This can be verified by taking the $(n-1)$-th power of both sides. So the last term in the above can be estimated by

$$\begin{aligned}
&e^{-l\psi(t)}\textstyle\sum_i \frac{1}{1+\psi(t)_{i\bar{i}}}(n+\Delta\psi(t)) \\
&\qquad \geq e^{-\frac{F}{n-1}}e^{-\frac{l}{n-1}}(e^{-l\psi(t)}(n+\tilde{\Delta}\psi(t)))^{\frac{n}{n-1}}.
\end{aligned}$$

Setting now $u = e^{-l\psi(t)}(n+\tilde{\Delta}\psi(t))$ and recalling that $\psi(t) \leq -1$ and hence $e^{-l\psi(t)} \geq 1$,we finally obtain the following estimate

$$\Box u \geq -c_1 - c_2 u + c_0 u^{\frac{n}{n-1}}.$$

Assume that u achieves its maximum at x_0 and $\frac{\partial u}{\partial t}\mid_{x_0,t}\geq 0$, then at this point, $\Box u = \Delta' u - \frac{\partial u}{\partial t}\mid_{x_0,t}\leq 0$ and therefore the maximum principle gives us an upper bound $u(x_0) \leq C$ which, in turn, gives

$$0 \leq \left(n+\tilde{\Delta}\psi(t)\right)(x) \leq e^{l\psi(t)(x)}u(x_0) \leq C$$

and hence we found a C^2-estimate of $\psi(t)$.

□

Proposition 6.2. *Let $\tilde{\rho}(t)$ be as in Theorem 6.1. Then there exists a uniform constant C such that*

$$\|\varphi(t) - \tilde{\rho}(t)\|_{C^3(\omega_{\tilde{\rho}})} \leq C.$$

Proof. Let

$$g'_{i\bar{j}} = h_{i\bar{j}} + (\varphi-\tilde{\rho})_{i\bar{j}}$$

and

$$S = \sum_{i,j,k,r,s,t=1}^{n} g'^{i\bar{r}}g'^{\bar{j}\ s}g'^{k\bar{t}}\,(\varphi-\tilde{\rho})_{i\bar{j}\ k}\,(\varphi-\tilde{\rho})_{\bar{r}\ s\bar{t}}\,.$$

Using Calabi's computation and Theorem 5.1 as in [28], one can show that $S \leq C$ for some uniform constant C. Consequently, the proposition is proved.

□

7 The proof of main theorems

According to Theorems 5.1, 6.1 and 6.2, we have uniform C^3 estimates on $\varphi(t) - \tilde{\rho}(t)$ along the modified Kähler Ricci flow. It is not difficult to prove the following

Lemma 7.1. *For any integer $l > 0$, there exists a uniform constant C_l such that*

$$\|D^l \ (\varphi(t) - \tilde{\rho}(t))\|_{\omega_{\tilde{\rho}}} \leq C_l,$$

where D^l represents arbitrary $l-th$ derivatives. Consequently, there exists a uniform bound on the sectional curvature and all the derivatives of $\omega_{\varphi(t)}$. The bound may possibly depend on the order of derivatives.

Follow this lemma, we can easily derive that the evolved Kähler metrics $\omega_{\varphi(t)}$ converge to a Kähler metric in the limit (by choosing subsequence). We would like to show that the limit is a Kähler-Einstein metric. Following proposition 2.5 and the fact that E_0 and E_1 have a uniform lower bound, we have

$$\begin{aligned}
\int_0^\infty \frac{n\sqrt{-1}}{V}\int_M \partial\frac{\partial\varphi}{\partial t}\wedge\overline{\partial}\frac{\partial\varphi}{\partial t}{\omega_\varphi}^{n-1}\,dt &= E(0) - E(\infty) < C,\\
\int_0^\infty \frac{2}{V}\int_M (R(\omega_\varphi) - r)^2{\omega_\varphi}^n\,dt &= E_1(0) - E_1(\infty) \leq C.
\end{aligned}$$

Combining this with Lemma 7.1, we prove that for almost all convergence subsequence of the evolved Kähler metrics $\omega_{\varphi(t)}$, the limit metric is of constant scalar curvature metric. From here, it is not difficult to show that any sequence of the evolved Kähler metrics will have a subsequence which converges to a metric of constant scalar curvature. In the canonical class, any metric of constant scalar curvature is a Kähler Einstein metric. We then prove the following

Theorem 7.2. *The modified Kähler Ricci flow converges to some Kähler Einstein metric by taking sub-sequences.*

To prove uniqueness of the limit by sequence, we can follow [8] to first prove the exponential decay of

$$\int_M \left(\frac{\partial\varphi}{\partial t}\right)^2 \ \omega_\varphi^n.$$

In other words, there exists a positive constant α and a uniform constant C such that

$$\int_M \left(\frac{\partial\varphi}{\partial t}\right)^2 \ \omega_\varphi^n < Ce^{-\alpha t}$$

for all evolved metrics over the Kähler Ricci flow. Eventually, we prove the following main proposition (like in [8]):

Proposition 7.3. *For any integer $l > 0$, $\frac{\partial\varphi}{\partial t}$ converges exponentially fast to 0 in any C^l norm. Furthermore, the Kähler Ricci flow converges exponentially fast to a unique Kähler Einstein metric on any Kähler-Einstein manifolds.*

8 Kähler-Einstein orbifolds

In this section, we will prove that any Kähler-Einstein orbifold such that there is another Kähler metric in the same Kähler class which has strictly positive bisectional curvature must be a global quotient of $\mathbb{C}P^n$ by a finite group. The simplest example of Kähler orbifolds is the global quotient of $\mathbb{C}P^n$ by a finite group. Roughly speaking, a generic Kähler orbifold is the union of a family of open sets, where each open set admits a finite covering from an open smooth Kähler manifold where a finite group acts holomorphically (we will give precise definition later). If it admits a Kähler Einstein metric, then it is called a Kähler-Einstein orbifold. The goal in this section is to show that under our assumption, there exists a global branching covering with a finite group action from $\mathbb{C}P^n$ to the underlying Kähler orbifold. The organization of this section is as follows: In subsection 8.1, we introduce the notion of complex orbifolds and various geometric structures associated with them. In Subsection 8.2, we consider the Kähler Ricci flow on any Kähler Einstein orbifolds. If there is another Kähler metric in the same Kähler class such that the bisectional curvature is positive, then the Kähler Ricci flow converges and the limit metric is a Kähler-Einstein metric with positive bisectional curvature. In Subsection 8.3, we prove that any orbifold which admits a Kähler-Einstein metric of constant bisectional curvature must be a global quotient of $\mathbb{C}P^n$. In subsection 8.4, we re-prove that any Kähler-Einstein metric with positive bisectional curvature must be of constant bisectional curvature (Berger's theorem [3]). We also prove that if a Kähler metric is sufficiently close to a Kähler Einstein metric on the Kähler Ricci flow, then the positivity of bisectional curvature will be preserved when taking limit (Lemma 8.20).

8.1 Kähler orbifolds

Let us begin with the definition of uniformization system over an open connected analytic space [5]:

Definition 8.1. *Let U be a connected analytic space and V a connected $n-$dimensional smooth Kähler manifold and G a finite group acting on V holomorphically. An $n-$dimensional uniformization system of U is a triple (V, G, π), where $\pi : V \to U$ is an analytic map inducing an identification between two analytic spaces V/G and U. Two uniformization systems $(V_i, G_i, \pi_i),\ i = 1, 2$, are isomorphic if there is a bi-holomorphic map $\phi : V_1 \to V_2$ and isomorphism $\lambda : G_1 \to G_2$ such that ϕ is $\lambda-$equivariant, and $\pi_2 \circ \phi = \pi_1$.*

In the above definition, we require that the fixed point set to be real codimension 2 or higher (if the group action preserves orientation, then the fixed point must be codimension 2 or higher.). Then the non-fixed point set (the complement of the fixed point set) is locally connected, which is important for our purpose. The following proposition is immediate:

[5] One reference for orbifolds is Ruan [23].

Proposition 8.2. *Let (V, G, π) be a uniformization system of U. For any connected open subset U' of U, (V, G, π) induces a unique isomorphism class of uniformization systems of U'.*

Proof. We want to clarify what "induces" means in this proposition. For any open subset $U' \subset U$, consider the preimage $\pi^{-1}(U')$ in V. G acts as permutations on the set of connected components of $\pi^{-1}(U')$. Let V' be one of the connected components of $\pi^{-1}(U')$, G' the subgroup of G which fixes the component V' and $\pi' = \pi \mid_{V'}$. Then (V', G', π') is an induced uniformizing system of U'. One can also show that any other induced uniformization system must be isomorphic to this one. We skip this part of the proof and refer interested readers to [23] for details. □

In light of this proposition, we can define equivalence of two uniformization systems at a single point: For any point $p \in U$, let (V_1, G_1, π_1) and (V_2, G_2, π_2) be two uniformization systems of neighborhoods U_1 and U_2 of p. We say that (V_1, G_1, π_1) and (V_2, G_2, π_2) are *equivalent* at p if they induce isomorphic uniformization systems for a smaller neighborhood $U_3 \subset U_1 \bigcap U_2$ of p. Next we define a complex (Kähler) orbifold.

Definition 8.3. *Let M be a connected analytic space. An $n-$dimensional complex orbifold structure on M is given by the following data: for any point $p \in M$, there are neighborhoods U_p and their $n-$dimensional uniformization systems (V_p, G_p, π_p) such that for any $q \in U_p$, (V_p, G_p, π_p) and (V_q, G_q, π_q) are equivalent at q. A point $p \in M$ is called regular if there exists a uniformization system (V_p, G_p, π_p) over $U_p \ni p$ such that G_p is trivial; Otherwise it is called singular. The set of regular points is denoted by M_{reg}. The set of singular points is denoted by M_{sing}, and $M = M_{reg} \bigcup M_{sing}$.*

Next we define orbifold vector bundles over a complex orbifold. As before, we begin with local uniformization systems for orbifold vector bundles. Given an analytic space U which is uniformized by (V, G, π) and a complex analytic space E with a surjective holomorphic map $pr : E \to U$, a uniformization system of rank k complex vector bundle for E over U consists of the following data.

1. A uniformization system (V, G, π) of U.

2. A unifomization system $(V \times \mathbb{C}^k, G, \tilde{\pi})$ for E. The action of G on $V \times \mathbb{C}^k$ is an extension of the action of G on V given by $g(x, v) = (g \cdot x, \rho(x, g) \cdot v)$, where $\rho : V \times G \to GL(\mathbb{C}^k)$ is a holomorphic map satisfying
$$\rho(g \cdot x, h) \circ \rho(x, g) = \rho(x, h \circ g), \qquad \forall g,\ h \in G,\ x \in V.$$

3. The natural projection map $\tilde{pr} : V \times \mathbb{C}^k \to V$ satisfying
$$\pi \circ \tilde{pr} = pr \circ \tilde{\pi}.$$

We can similarly define isomorphisms between two uniformization systems of orbifold vector bundles for E over U. The only additional requirement is that the diffeomorphism between $V \times \mathbb{C}^k$ are linear on each fiber of $\tilde{pr} : V \times \mathbb{C}^k \to V$. Moreover, we can also define the equivalent relation between two uniformization systems of complex vector bundles at any specific point. Here is the definition of orbifold vector bundles over complex orbifolds:

Definition 8.4. *Let M be a complex orbifold and E a complex vector space with a surjective holomorphic map $pr : E \to M$. A rank k complex orbifold vector bundle structure on E over M consists of the following data: for each point $p \in M$, there is a unformized neighborhood U_p and a uniformization system of a rank k complex vector bundle for $pr^{-1}(U_p)$ over U_p such that for any $q \in U_p$, the rank k complex orbifold vector bundles over U_p and U_q are isomorphic in a smaller open subset $U_p \bigcap U_q$. Two orbifold vector bundles $pr_1 : E_1 \to M$ and $pr_2 : E_2 \to M$ are isomorphic if there is a holomorphic map $\tilde{\psi} : E_1 \to E_2$ given by $\tilde{\psi}_p : (V_{1,p} \times \mathbb{C}^k, G_{1,p}, \tilde{\pi}_{1,p}) \to (V_{2,p} \times \mathbb{C}^k, G_{2,p}, \tilde{\pi}_{2,p})$ which induces an isomorphism between $(V_{1,p}, G_{1,p}, \tilde{\pi}_{1,p})$ and $(V_{2,p}, G_{2,p}, \tilde{\pi}_{2,p})$, and is a linear isomorphism between the fibers of $\tilde{pr}_{1,p}$ and $\tilde{pr}_{2,p}$.*

For a complex orbifold, one can define the tangent bundle, the cotangent bundle, and various exterior or tensor powers of these bundles. All the differential geometric quantities such as cohomology class, connections, metrics, and curvatures can be introduced on the complex orbifold.

Suppose M is a complex orbifold as in Definition 8.3. For any $p \in M$, let $p \in U_p$ be uniformized by (V_p, G_p, π_p). When we say a metric g is defined on U_p, we really mean a metric $\overline{g}$ defined on V_p such that G_p acts on V_p by isometries. For simplicity, we say the metric g is defined on U_p and $\pi_p^* g = \overline{g}$. This simplification makes sense especially when p is a regular point, i.e., when G_p is trivial. One way to define a metric on the entire complex orbifold is first to define it on M_{reg}, then extend it to be a metric on M with possible singularities since M_{sing} is codimension at least 2 or higher. The following gives a definition of what a smooth Kähler metric or a Kähler form on the complex orbifold is:

Definition 8.5. *For any point $p \in M$, let U_p be uniformized by (V_p, G_p, π_p). A Kähler metric g (resp. a Kähler form ω) on a complex orbifold M is a smooth metric on M_{reg} such that for any $p \in M$, ${\pi_p}^* g$ (resp. Kähler form ${\pi_p}^* \omega$)[6] can extends to a smooth Kähler metric (resp. smooth Kähler form) on V_p.*

Definition 8.6. *A function f is called a smooth function on an orbifold M if for any $p \in M, f \circ \pi_p$ is a smooth function on V_p.*

Similarly, one can define any tensor to be smooth on M if its pre-image on each local uniformization system is smooth. Clearly, the curvature tensor and

[6]Note ${\pi_p}^*$ is only defined away from the fixed point set of V_p. Since the fixed point set is at least codimension 2 or higher, any metric defined on non-fixed point set of V_p has a unique smooth extension on V_p if such an extension exists. This definition essentially says that a metric is smooth in the orbifold sense if such an extension always exists in each uniformization system of the underlying Kähler orbifold structure.

the Ricci tensor of any smooth metric on orbifolds, as well as their derivatives, are smooth tensors. A complex orbifold admits a Kähler metric is called a Kähler orbifold.

Definition 8.7. *A curve $c(t)$ on Kähler orbifold M is called geodesic if near any point p on it, $c(t)\bigcap U_p$ can be lifted to a geodesic on V_p and at least one preimage of $c(t)$ is smooth in V_p. Here U_p is any open connected neighborhood of p over which (V_p, G_p, π_p) is a unifomization system.*

Under this definition, we have

Proposition 8.8. *Any minimizing geodesic between two regular points never pass any singular point of the Kähler orbifold.*

Proof. Otherwise, we can argue that the geodesic is not minimizing. Suppose that p is a singular point and $p \in U_p$ is a small open set which is uniformized by (V_p, G_p, π_p) with an equivariant metric g on V_p. Suppose that a portion of geodesic lies inside of U_p is $c(t) : [-\epsilon, \epsilon]$ such that $A = c(-\epsilon), B = c(\epsilon) \in U_p$ and $p = c(0)$. Assume that this geodesic is parameterized by arc length. Thus the distance between A and B is 2ϵ, while the distance between A (or B) and O is ϵ. Without loss of generality, we may assume that A and B are regular points. Suppose that $\pi^{-1}(p) = O$; $\pi^{-1}(A) = \{A_1, A_2, \cdots, A_{l_1}\}$ and $\pi^{-1}(B) = \{B_1, B_2, \cdots, B_{l_2}\}$. Note that $\{A_1, A_2, \cdots, A_{l_1}\}$ and $\{B_1, B_2 \cdots, B_{l_2}\}$ are on the sphere of radius ϵ which centers at O. If G_p is non-trivial (then the preimages of A and B are not unique, i.e., $l_1 > 1$ and $l_2 > 1$.), then there is at least one pair of $A_i, B_j (1 \leq i \leq l_1,\ 1 \leq j \leq l_2)$ such that the distance between the two points is shorter than 2ϵ on V_p[7]. Suppose this geodesic is $\tilde{C}$. Then $\pi_p(\tilde{C})$ is a geodesic (which connects A and B) whose length is shorter than 2ϵ. Thus $c(t)$ is not a minimizing geodesic between A and B since $\pi_p(A_i) = A$ and $\pi_p(B_j) = B$. ☐

8.2 Kähler Ricci flow on Kähler-Einstein orbifolds

A Kähler-Einstein orbifold metric is a metric on orbifold such that the Ricci curvature is proportional to the metric. A Kähler orbifold with a Kähler-Einstein metric is called a Kähler-Einstein orbifold.

Theorem 8.9. *Let M be any Kähler Einstein orbifold. If there is another Kähler metric in the same cohomology class which has non-negative bisectional curvature and positive at least at one point, then the Kähler-Ricci flow converges to a Kähler-Einstein metric with positive bisetcional curvature.*

We want to generalize our proof of Theorem 1.1 to the orbifold case. Note that the analysis for Kähler orbifolds is exactly the same as that for Kähler manifolds (cf. [11]). We want to show that this theorem can be proved exactly

[7] In any ball of radius 1 on any metric space, the maximum distance between any two points in the ball is 2 which is the diameter of the unit ball. Fixed a point in the ball, then the minimal distance from that point to any set of points in the ball is strictly less than 2, if that set contains two or more points.

like Theorem 1.1. First, we need to set up some notations. Following Section 2.1, we use the Kähler form ω as a smooth Kähler form on the orbifold M. Locally on M_{reg}, it can be written as

$$\omega = \sqrt{-1} \sum_{i,j=1}^{n} g_{i\bar{j}} d\, z^i \wedge d\, z^{\bar{j}},$$

where $\{g_{i\bar{j}}\}$ is a positive definite Hermitian matrix function. Denote by $\mathcal{B}$ the set of all real valued smooth functions on M in the orbifold sense (cf. Definition 8.6). Then the Kähler class $[\omega]$ consists of all Kähler form which can be expressed as

$$\omega_\varphi = \omega + i\partial\overline{\partial}\varphi > 0$$

on M for some $\varphi \in \mathcal{B}$. In other words, the space of all Kähler potentials in this Kähler class is

$$\mathcal{H} = \{\varphi \in \mathcal{B} \mid \omega_\varphi = \omega + i\partial\overline{\partial}\varphi > 0\}.$$

The Ricci form for ω is:

$$Ric(\omega) = -\sqrt{-1}\partial\overline{\partial} \log \omega^n.$$

As in the case of smooth manifolds, $[\omega]$ is the canonical Kähler class if ω and the Ricci form $Ric(\omega)$ is in the same cohomology class after proper rescalling. In the canonical Kähler class, consider the Kähler Ricci flow

$$\frac{\partial \varphi}{\partial t} = \log \frac{{\omega_\varphi}^n}{\omega^n} + \varphi - h_\omega,$$

where h_ω is defined as in Section 2.2. Clearly, this flow preserves the structure of Kähler orbifold, in particular, preserves the Kähler class $[\omega]$. Examining our proof of Theorem 7.2, the following three parts are crucial

1. The preservation of positive bisectional curvature under the Kähler Ricci flow.

2. The introduction of a set of new functionals E_k and new invariants $\Im_k (k = 0, 1, \cdots, n)$.

3. The uniform estimate on the diameter; consequently, the uniform control on the Sobolev constant and the Poincare constant.

To extend these to the case of Kähler orbifolds, we really need to make sure that the following tools for geometric analysis hold in the orbifold case:

1. Maximum principle for smooth functions and tensors on Kähler orbifold(cf. Definition 8.7).

2. Integration by parts for smooth functions/tensors in the orbifold case.

3. The second variation formula for any smooth geodesics(cf. Proposition 8.8).

By our definition of Kähler orbifolds, it is not difficult to see that the maximum principle holds on orbifolds. Thus, Theorem 2.1 still holds in the orbifold case. In other words, the bisectional curvature of the evolved metric is strictly positive after the initial time, if the initial metric has non-negative bisectional curvature and positive at least at one point. Moreover, the integration by parts on orbifold holds for any smooth function on M with smooth metrics in the orbifold sense. Thus, our definitions of new functionals $E_0, E_1, \cdots E_n$ can be carried over to this Kähler orbifold setting without any change. Moreover, the formula for their derivatives still holds. In particular, E_0 and E_1 are decreasing strictly under the Kähler Ricci flow. Furthermore, the set of invariants $\Im_0, \Im_1, \cdots, \Im_n$ are well defined and vanish on any Kähler-Einstein orbifold. Since Tian's inequality holds on any Kähler Einstein orbifold, then Prop. 2.14, Corollary 2.16 hold as well. Finally, the second variation formula for minimizing geodesic between any two regular points on Kähler orbifolds is exactly the same as the formula on smooth manifold(cf. Prop. 8.8). Thus we can use the same set of ideas in Section 3 to estimate diameter [8]; consequently, the Sobolev constant and the Poincare constant can be uniformly controlled as well. The rest of arguments in our proof of Theorem 7.2 can be extended to the orbifold case directly. Thus we can prove Theorem 8.9 for Kähler-Einstein orbifolds.

8.3 Kähler Einstein orbifolds with constant positive bisectional curvature

In this subsection, we want to prove the following

Theorem 8.10. *Let M be any Kähler orbifold. If there is a Kähler-Einstein metric with constant positive bisectional curvature, then it is a global quotient of $\mathbb{C}P^n$.*

Suppose $\overline{g}$ is the standard Fubini-Study metric on $\mathbb{C}P^n$ with constant bisectional curvature. Suppose g is a Kähler-Einstein metric on M with constant bisectional curvature. Normalize the bisectional curvature of g on M and of $\overline{g}$ on $\mathbb{C}P^n$ so that both bisectional curvature is 1. Consequently, the conjugate radius of $\mathbb{C}P^n$ is π. Let p be any regular point in M. By definition, let U_p be a small neighborhood of p and (V_p, G_p, π_p) be the uniformization system. Since $p \in M_{reg}$, then G_p is trivial group. Consider $g' = {\pi_p}^* g$ as a Kähler metric with constant bisectional curvature on V_p. If we choose U_p sufficiently small, then (V_p, g') is an open subset of $(\mathbb{C}P^n, \overline{g})$ with the induced metric from $(\mathbb{C}P^n, \overline{g})$. In the following, we will drop notation g' and use $\overline{g}$ only. Our goal is to extend π_p into a local isometric map from $\mathbb{C}P^n$ to M.

[8] In the proof of Lemma 3.5, without loss of generality, may assume $p, q \in M_{reg}$ where the diameter $D = d(p,q)$. Furthermore, we may assume $A_1 = B_{p,r} \subset M_{reg}$ and $A_2 = B_{q,r} \subset M_{reg}$. According to Lemma 8.8, any minimizing geodesic between A_1 and A_2 belong to M_{reg}. Consequently, we can use Lemma 3.1 of J. Cheeger and T. Coldings to conclude the diameter bounds as in the smooth case.

Next we set up some notations first. Denote by q the pre-image of p. Consider

$$\begin{array}{llll} & \mathbb{C}P^n & & M \\ & \bigcup & & \bigcup \\ \pi_p: & V_p & \rightarrowtail & U_p. \end{array}$$

Now, we want to lift this map π_p to a map from $\mathbb{C}P^n$ to M. First, we need to rewrite this map in a different way:

$$\begin{array}{llll} exp_q: & T_q(\mathbb{C}P^n) & \to & \mathbb{C}P^n \\ & & & \bigcup \\ & & & V_p \\ & \downarrow id & & \downarrow \pi_p \\ & & & U_p \\ & & & \bigcap \\ exp_p: & T_p(M) & \to & M. \end{array}$$

Set

$$\Pi = exp_p \circ id \circ \exp_q^{-1}.$$

Then at least Π is defined in V_p, and

$$\Pi = \pi_p = exp_p \circ id \circ \exp_q^{-1}, \qquad \text{in } V_p. \tag{8.1}$$

Consider the open ball of radius π in $T_q(\mathbb{C}P^n)$ which we will denote it by B_π. Then $\exp_q^{-1}$ is well defined on $exp_q(B_\pi) \subset \mathbb{C}P^n$. The image of ∂B_π under the exponential map is a projective subspace of codimension 1, which will be denoted as $\mathbb{C}P^{n-1}_\infty$. Then

$$exp_q(B_\pi) = \mathbb{C}P^n \setminus \mathbb{C}P^{n-1}(\infty).$$

We claim that we can extend the map Π in this way to $\mathbb{C}P^n \setminus \mathbb{C}P^{n-1}(\infty)$ via Formula (8.1). The key step is the following lemma (in the following arguments, we abuse notation by using letters p and q for generic points on M.).

Lemma 8.11. *Any smooth geodesic on M can be extended uniquely and indefinitely. In particular, it can be extended uniquely (before the length π[9]).*

Proof. Suppose $c(t) : [0, a]$ is a geodesic defined on M with length $a > 0$. If $c(a) \in M_{reg}$, then it can easily be extended as usual. If $c(a) \in U_p$ for some $p \in M_{sing}$, in particular, if $c(a) \in M_{sing}$, we want to extend the geodesic uniquely as well. Consider the part of geodesic $c(t) \bigcap U_p$; And still denotes it as $c(t)$. Suppose that U_p is uniformized by (V_p, G_p, π_p). For convenience, the pull backed metric $g'_p = \pi^* g$ is a smooth metric on V_p and G_p acts isometrically on (V_p, g'_p). Consider its pre-images $\tilde{c}(t)$ in V_p under π_p (note that π_p is a local isometric map from (V_p, g'_p) to (U_p, g), in particular, if we restrict the map to $M_{reg} \bigcap U_p$.). Although the preimages are not unique in V_p, each preimage $\tilde{c}(t)$

[9]We are interested in the unique extension up to length π since it is the conjugate radius of any Kähler metric with constant bisectional curvature 1.

has a unique extension on V_p. More importantly, the images of these geodesic extensions on V_p under π_p are unique in U_p. Therefore, the geodesic $c(t)$ is also extendable uniquely in this setting. □

In fact, we have the following

Corollary 8.12. *Any geodesic in a Kähler orbifold with constant bisectional curvature can be extended long enough to become a closed geodesic. If the bisectional curvature is* 1*, then the length of each closed geodesic is either* 2π *or* $\frac{2\pi}{l}$ *for some integers* l*. Moreover, there exists a maximum integer of all such integers* l*, denoted by* l_{max}*. Then the conjugate radius of the Kähler orbifold with constant bisectional curvature* 1 *is* $\frac{\pi}{l_{max}}$*.*

Lemma 8.13. Π *can be extended to be a global map from* $\mathbb{C}P^n$ *to* M*. Moreover,* Π *is a local isometry from an open dense set* $(\Pi^{-1}(M_{reg}), \overline{g})$ *in* $\mathbb{C}P^n$ *to* (M_{reg}, g)*.*

Proof. Consider the open ball $B_\pi \subset T_q(\mathbb{C}P^n)$ of radius π. Then the closure of $exp_q(B_\pi)$ is just the whole $\mathbb{C}P^n$. By the preceding lemma, Π can be defined in the open set $exp_p(B_\pi)$. Next taking the closure of this map, we define a map from $\mathbb{C}P^n$ to M. Moreover, Π is a local isometry from $(\Pi^{-1}(M_{reg}), \overline{g})$ to (M, g). □

Now we want to prove the following lemma

Lemma 8.14. *For any point* A *on* M*, there exists only a finite number of preimages on* $\mathbb{C}P^n$*.*

Proof. Otherwise, there exists an infinite number of pre-images of the point A on $\mathbb{C}P^n$. This set of infinite number points must have a concentration point on $\mathbb{C}P^n$. In particular, for any small $\epsilon > 0$, there are at least two preimages of A such that the distance between these two points in $\mathbb{C}P^n$ is less than ϵ. Consider the image of the minimal geodesic which connects these two pre-image points on $\mathbb{C}P^n$ under Π. We obtain a geodesic loop centered at point A whose length is less than ϵ. This violates the fact that the conjugate radius of (M, g) is at least $\frac{\pi}{l_{max}}$ (cf. Corollary 8.12). Thus the lemma holds. □

Lemma 8.15. *For any* $p \in M$*, let* U_p *be a small neighborhood of* p *and* (V_p, G_p, π_p) *be the uniformization system over* U_p*. Let* W_p *be any connected component of* $\Pi^{-1}(U_p)$*. Then there exists a finite group* G'_p *acting isometrically on* $(W_p, \overline{g})$ *such that* $(W_p, G'_p, \Pi\mid_{W_p})$ *is a uniformization system over* U_p*, which is equivalent to* (V_p, G_p, π_p) *(In particular, if we choose a different connected components of* $\Pi^{-1}(U_p)$*, then the induced uniformization system* $(W_p, G'_p, \Pi\mid_{W_p})$ *is invariant up to isometries.).*

Proof. Set $f = \Pi\mid_W$ and $g' = \pi^* g$. Then g' is a smooth Kähler metric with constant bisectional curvature 1 on V_p, where G_p acts isometrically on V_p with respect to this metric g'. There exists a smooth lifting of f to $\tilde{f} : W_p \to V_p$ such that $\pi_p \circ \tilde{f} = f$. It is easy to verify that $\tilde{f}$ is an isometric map from $(W_p, \overline{g})$

to (V_p, g'). Moreover, it can be proved that $\tilde{f}$ is an one-to-one map from W_p to V_p. Now consider the pull back group $G' = \tilde{f}^{-1}G_p$; and define the group action via $\tilde{f}$. Since G_p acts isometrically on (V_p, g'), then G'_p acts isometrically on $(W_p, \overline{g})$. By definition, (W_p, G'_p, f) is a uniformization system over U_p which is equivalent to the original uniformization system (V_p, G_p, π_p). □

Lemma 8.16. *For any $p \in M$, let U_p be a small neighborhood with $(W_p, G_p, \Pi\mid_{W_p})$ a uniformization system where $(W_p, \overline{g}) \subset (\mathbb{C}P^n, \overline{g})$. Then the fixed point set of G_p is a totally geodesic Kähler submanifold of $(W_p, \overline{g}) \subset (\mathbb{C}P^n, \overline{g})$.*

Proof. Following properties of isometric group actions on Kähler manifold. □

This lemma has an immediate

Corollary 8.17. *Consider $W_{sing} = \Pi^{-1}(M_{sing})$. Then W_{sing} is a union of CP^k for some $k \leq n-1$. When $k = 0$, it is the preimage of isolated singular points on M.*

Proof. The only totally geodesic Kähler submanifold in $(\mathbb{C}P^n, \overline{g})$ is $\mathbb{C}P^k$ for some $k \leq n-1$. □

Denote by $Aut(\mathbb{C}P^n)$ the holomorphic transformation group of $\mathbb{C}P^n$. Then we have

Lemma 8.18. *For any $p \in M$, let U_p be a small neighborhood and $(W_p, G_p, \Pi\mid_{W_p})$ be a uniformization system over U_p, where $(W_p, \overline{g}) \subset (\mathbb{C}P^n, \overline{g})$. For any $\sigma \in G_p$, it can be extended to be a group element in $Aut(\mathbb{C}P^n)$, and we still denote it as σ. Moreover, $\Pi \circ \sigma = \Pi$ on $\mathbb{C}P^n$.*

Proof. It is easy to see that σ can be extended uniquely to an element in $Aut(\mathbb{C}P^n)$ which acts isometrically on $(\mathbb{C}P^n, \overline{g})$. Consider two local isometries from $(\mathbb{C}P^n, \overline{g})$ to (M, g) : Π and $\Pi \circ \sigma$. Since the two maps agree on an open set $W_p \subset \mathbb{C}P^n$, they must agree on all $\mathbb{C}P^n$. □

From now on, we may view G_p as a subgroup of $Aut(\mathbb{C}P^n)$ directly. Now we are ready to give a proof of Theorem 8.10.

Proof. For any $p \in M$, let U_p be a small neighborhood with a uniformization system $(W_p, G_p, \Pi\mid_{W_p})$, then the preceding lemma implies that G_p is a subgroup of $Aut(\mathbb{C}P^n)$. If $p \in M_{reg}$, then G_p is trivial. If $p_1, p_2 \in M_{sing}$ and is near to each other, then $G_{p_1} = G_{p_2}$ by continuity. Consequently, for any $p_1, p_2 \in M_{sing}$ such that the fixed point sets of W_{p_1} and W_{p_2} belong to the same connected component of $\Pi^{-1}(M_{sing})$, then $G_{p_1} = G_{p_2} \subset Aut(\mathbb{C}P^n)$. Consider $G \subset Aut(\mathbb{C}P^n)$ to be the subgroup generated by all such G_p's. Then G acts isometrically on $(\mathbb{C}P^n, \overline{g})$ and

$$\Pi \circ \sigma = \Pi, \qquad \forall \sigma \in G \subset Aut(\mathbb{C}P^n). \tag{8.2}$$

This induces a covering map

$$\mathbb{C}P^n/G \to M.$$

By this explicit construction, one can verify directly that this is an orbifold isomorphism. Consequently, M is a global quotient of $\mathbb{C}P^n$ by this group G. The only thing left is to show that G is of finite order, which follows directly from equation (8.2) and Lemma 8.14. □

8.4 Pinching theorem for bisectional curvature

In this subsection, we want to prove the following lemma

Lemma 8.19. *If g is a Kähler-Einstein metric with strictly positive bisectional curvature on a Kähler orbifold, then g has constant bisectional curvature.*

This lemma was first proved by Berger [3] on Kähler manifolds. We note that his proof can be easily modified for Kähler orbifolds. For reader's convenience, we include a proof here.

Proof. We begin with a simple observation: In any Kähler orbifold, any Kähler-Einstein metric satisfies the following elliptic equation :

$$\triangle R_{i\bar{j}k\bar{l}} + R_{i\bar{j}p\overline{q}}R_{q\overline{p}k\bar{l}} - R_{i\overline{p}k\overline{q}}R_{p\bar{j}q\bar{l}} + R_{i\bar{l}p\overline{q}}R_{q\overline{p}k\bar{j}} - R_{i\bar{j}k\bar{l}} = 0.$$

Define a new symmetric tensor $T_{i\bar{j}k\bar{l}}$ as

$$T_{i\bar{j}k\bar{l}} = g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}\, g_{k\bar{j}}.$$

And for any fixed $\epsilon \in (0, \frac{1}{n+1})$, we define

$$Q_{i\bar{j}k\bar{l}} = R_{i\bar{j}k\bar{l}} - \epsilon T_{i\bar{j}k\bar{l}}.$$

Note that $T_{i\bar{j}k\bar{l}}$ is parallel in the manifold. By a direct but tedious calculation, we arrive at the following

$$\begin{aligned} & -\triangle Q_{i\bar{j}k\bar{l}} \\ = \quad & Q_{i\bar{j}p\overline{q}}Q_{q\overline{p}k\bar{l}} + Q_{i\bar{l}p\overline{q}}Q_{q\overline{p}k\bar{j}} - Q_{i\overline{p}k\overline{q}}Q_{p\bar{j}q\bar{l}} \\ & \qquad + \epsilon(1-(n+1)\epsilon)T_{i\bar{j}k\bar{l}} - Q_{i\bar{j}k\bar{l}}. \end{aligned} \tag{8.3}$$

Suppose that the bisectional curvature of g is not constant. Note that for $\epsilon = \frac{1}{n+1}$, we have

$$g^{i\bar{j}}Q_{i\bar{j}k\bar{l}} = g^{k\bar{l}}Q_{i\bar{j}k\bar{l}} = 0.$$

Thus if $R_{i\bar{j}k\bar{l}} > 0$, there exists a small positive $\epsilon \in (0, \frac{1}{n+1})$, such that $Q_{i\bar{j}k\bar{l}} \geq 0$ in the whole manifold and vanishes in some direction at some points. In other words, there exists a point $x_0 \in M$ and two vectors $v_0, w_0 \in T_{x_0}M$ such that

$$Q_{i\bar{j}k\bar{l}}(x_0)v_0^{\bar{i}}v_0^{j}w_0^{\bar{k}}w_0^{l} = 0$$

and for any other point $x \in M$ and any other pair of vectors $v, w \in T_xM$, we have

$$Q_{i\bar{j}k\bar{l}}(x)v^{\bar{i}}v^{j}w^{\bar{k}}w^{l} \geq 0.$$

Now consider a pair of parallel vector fields v, w in a small neighborhood of x_0 such that

$$v^i_{,j} = v^i_{,\bar{j}} = w^i_{,j} = w^i_{,\bar{j}} = 0,$$

where

$$v = \sum_{i=1}^{n} v^i \frac{\partial}{\partial z^i}, \qquad \text{and} \qquad w = \sum_{i=1}^{n} w^i \frac{\partial}{\partial z^i}.$$

Furthermore, $v(x_0) = v_0$ and $w(x_0) = w_0$. Consider the scalar function

$$Q = Q_{i\bar{j}k\bar{l}}(x) v^{\bar{i}} v^j w^{\bar{k}} w^l$$

in a neighborhood of x_0. Clearly, Q achieves minimum in x_0. Thus the maximum principle implies that

$$\begin{aligned} -\triangle Q &= \left(\triangle Q_{i\bar{j}k\bar{l}} v^{\bar{i}} v^j w^{\bar{k}} w^l\right)_{x=x_0} \\ &= \left(\triangle Q_{i\bar{j}k\bar{l}}\right)|_{x=x_0} v^{\bar{i}} v^j w^{\bar{k}} w^l \leq 0 \end{aligned}.$$

Plugging this into the equation (8.3), we obtain

$$\begin{aligned} & -\triangle Q_{i\bar{j}k\bar{l}} v^{\bar{i}} v^j w^{\bar{k}} w^l \\ = \;& Q_{i\bar{j}p\bar{q}} Q_{q\bar{p}k\bar{l}} v^{\bar{i}} v^j w^{\bar{k}} w^l + Q_{i\bar{l}p\bar{q}} Q_{q\bar{p}k\bar{j}} v^{\bar{i}} v^j w^{\bar{k}} w^l - Q_{i\bar{p}k\bar{q}} Q_{p\bar{j}q\bar{l}} v^{\bar{i}} v^j w^{\bar{k}} w^l \\ & + \epsilon(1-(n+1)\epsilon) T_{i\bar{j}k\bar{l}} v^{\bar{i}} v^j w^{\bar{k}} w^l - Q_{i\bar{j}k\bar{l}} v^{\bar{i}} v^j w^{\bar{k}} w^l. \end{aligned}$$

Define the following linear operator at point x_0

$$A_{i\bar{j}} = R_{i\bar{j}k\bar{l}} v^{\bar{k}} v^l, \qquad \text{and} \qquad C_{i\bar{j}} = R_{k\bar{l}i\bar{j}} w^{\bar{k}} w^l$$

and

$$M_{i\bar{j}} = R_{i\bar{p}q\bar{j}} v^p w^{\bar{q}}, \qquad \text{and} \qquad M_{ij} = R_{i\bar{p}j\bar{q}} v^p w^q.$$

Plugging these into the above equation and evaluate at x_0, we have

$$\begin{aligned} 0 &\geq -\triangle Q_{i\bar{j}k\bar{l}}(x) v^{\bar{i}} v^j w^{\bar{k}} w^l \,|_{x=x_0} \\ &= A_{p\bar{q}} C_{q\bar{p}} + M_{p\bar{q}} M_{q\bar{p}} - M_{pq} M_{\bar{p}\bar{q}} + \epsilon(1-(n+1)\epsilon)|v|^2|w|^2. \end{aligned}$$

By a calculation of N. Mok, we have

$$A_{p\bar{q}} C_{q\bar{p}} \geq M_{p\bar{q}} M_{q\bar{p}} + M_{pq} M_{\bar{p}\bar{q}}.$$

Since $0 < \epsilon < \frac{1}{n+1}$, then

$$0 \geq -\triangle Q_{i\bar{j}k\bar{l}}(x_0) v^{\bar{i}} v^j w^{\bar{k}} w^l \geq \epsilon(1-(n+1)\epsilon)|v|^2|w|^2 > 0.$$

This is a contradiction! Thus, $\epsilon = \frac{1}{n+1}$. In this case,

$$g^{i\bar{j}} Q_{i\bar{j}k\bar{l}} = g^{k\bar{l}} Q_{i\bar{j}k\bar{l}} = 0$$

and

$$Q_{i\bar{j}k\bar{l}} \geq 0.$$

Thus

$$Q_{i\bar{j}k\bar{l}}(x_0) = 0.$$

Since x_0 is the minimum for $R_{i\bar{j}k\bar{l}}$, we obtain that

$$R_{i\bar{j}k\bar{l}} \equiv \frac{1}{n+1} T_{i\bar{j}k\bar{l}} \equiv \frac{1}{n+1} \left(g_{i\bar{j}} g_{k\bar{l}} + g_{i\bar{l}} g_{k\bar{j}} \right).$$

□

From the proof, we can actually prove slightly more:

Lemma 8.20. *If the Kähler Ricci flow converges to a unique Kähler Einstein metric exponentially fast (Prop. 8.3), and if the bisectional curvature remains positive before taking limit, then the limit Kähler Einstein metric has positive bisectional curvature.*

Combining this lemma, Lemma 8.19 and Theorem 7.3, we can prove

Theorem 8.21. *For any integer $l > 0$, $\frac{\partial \varphi}{\partial t}$ converges exponentially fast to 0 in any C^l norm. Furthermore, the Kähler Ricci flow converges exponentially fast to a unique Kähler Einstein metric with constant bisectional curvature on any Kähler-Einstein manifolds.*

9 Concluding Remarks

In this section, we want to prove our main Theorem 1.1, Corollary 1.3 and Theorem 1.4. Note that the proof of Theorem 1.1 and Corollary 1.3 is similar to the proof we gave in the final section of our earlier paper [8] except in the final step, where we need to use Berger's theorem (Lemma 8.19) and Theorem 8.21 to show that any Kähler Einstein metric with positive bisectional curvature must be a space form. We will skip this part and just give a proof for Theorem 1.4.

Proof. For any Kähler metric in the canonical Kähler class such that it has non-negative bisectional curvature on M(Kähler-Einstein orbifold) but positive bisectional curvature at least at one point, we apply the Kähler Ricci flow to this metric on M. Theorem 2.1 still holds in the orbifold case. In other words, the bisectional curvature of the evolved metric is strictly positive over all the time. By our Theorem 8.9 and Lemma 8.20, the Kähler Ricci flow converges exponentially to a unique Kähler-Einstein metric of positive bisectional curvature. According to Lemma 8.19, any Kähler-Einstein metric of positive bisectional curvature on a Kähler orbifold must have a constant positive bisectional curvature. Moreover, using Theorem 8.10, we arrive at the conclusion that M must be a global quotient of $\mathbb{C}P^n$ by a finite group action.

Furthermore, this also proves that any Kähler metric with nonnegative bisectional curvature on M and positive at least at one point is path connected to a Kähler-Einstein metric of constant positive bisectional curvature. Note that all the Kähler-Einstein metrics are path connected by automorphisms [2]. Therefore, the space of all Kähler metrics with nonnegative bisectional curvature on M and positive at least at one point, is path connected. Similarly, using Theorem 2.2 and our Theorem 1.4, we can show that all of Kähler metrics with nonnegative curvature operator on M and positive at least at one point is path connected. Note that the nonnegative curvature operator implies the nonnegative bisectional curvature. □

Remark 9.1. *Combining our main Theorem 1.1 and Theorems 2.1, 2.2, we can easily generalize Corollary 1.2 to the case that the bisectional curvature (or curvature operator) is only assumed to be non-negative. We can show the flow converges exponentially fast to a unique Kähler-Einstein metric with non-negative bisectional curvature. Then, following an earlier work of Zhong and Mok, the underlying manifold must be compact symmetric homogeneous manifold.*

Next we want to propose some future problems. Some of them may not be hard to solve.

Question 9.2. *It is clear that E_1 plays a critically important role in proving the convergence theorem of this paper. Note that the Ricci flow is a gradient-like of the functional E_1. It will be interesting to study the gradient flow of E_1.*

Consider the expansion formula in t:

$$(\omega_\varphi + t\,\mathrm{Ric}(\omega_\varphi))^n = \left(\sum_{k=0}^{n} \sigma_k(\omega_\varphi) t^k\right) \omega_\varphi^n.$$

Clearly, $\sigma_0(\omega_\varphi) = 1$, $\sigma_1(\omega_\varphi) = R(\omega_\varphi)$, the scalar curvature of ω_φ. The equation for the gradient flow of E_1 is

$$\frac{\partial \varphi(t)}{\partial t} = 2\Delta_\varphi R(\omega_\varphi) - (n-1)\sigma_2(\omega_\varphi) - c_1. \tag{9.1}$$

Here c_1 is some constant which depends only on the Kähler class. Clearly, this is a 6 order parabolic equation.

Question 9.3. *As Remark 1.5 indicates, what we really need is the positivity of Ricci curvature along the Kähler Ricci flow. However, it is not expected that the positivity of Ricci curvature is preserved under the Kähler Ricci flow except on Riemann surfaces. The positivity of bisectional curvature is a technical assumption to assure the positivity of Ricci curvature. It is very interesting to extend Theorem 1.1 to metrics without the assumption on the bisectional curvature.*

Conjecture 9.4. *(Hamilton-Tian) On a Kähler-Einstein manifold, the Kähler-Ricci flow converges to a Kähler-Einstein metric. On a general Kähler manifold with positive first Chern class, the Kähler Ricci flow will converge, at least by taking sequences, to a Kähler Ricci soliton modulo diffeomorphism. Note that the complex structure may change in the limit and the limit may have mild singularities.*

Question 9.5. *Is the positivity of the sectional curvature preserved under the Kähler-Ricci flow?*

Question 9.6. *For any holomorphic vector field and Kähler class $[\omega]$, are the invariants $\mathcal{I}_k(X, [\omega])$ independent? In the non-canonical class, we expect these invariants to be different. Note that one can derive localization formulas for these invariants as what are done in [27].*

Question 9.7. *According to our Theorem 1.4, any Kähler-Einstein orbifold with positive bisectional curvature is necessarily biholomorphic to a global quotient of $\mathbb{C}P^n$. What happens if we drop the Kähler-Einstein condition?*

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