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Research Article

Ricci Solitons in α-Sasakian Manifolds

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We study Ricci solitons in α-Sasakian manifolds. It is shown that a symmetric parallel second order-covariant tensor in a α-Sasakian manifold is a constant multiple of the metric tensor. Using this, it is shown that if $\mathcal{L}_V g + 2S$ is parallel where V is a given vector field, then (g, V, λ) is Ricci soliton. Further, by virtue of this result, Ricci solitons for n-dimensional α-Sasakian manifolds are obtained. Next, Ricci solitons for 3-dimensional α-Sasakian manifolds are discussed with an example.

1. Introduction

In 1982, Hamilton [1] introduced the concept of Ricci flow which smooths out the geometry of manifold that is if there are singular points these can be minimized under Ricci flow. Ricci solitons move under the Ricci flow simply by diffeomorphisms of the initial metric that is they are stationary points of the Ricci flow: $\partial g/\partial t = -2\mathrm{Ric}(g)$, (in this paper we use $\mathrm{Ric} = S$) in the space of metrics on M. Hence it is interesting to study Ricci solitons.

Definition 1.1. A Ricci soliton (g, V, λ) on a Riemannian manifold M is defined by

$$\mathcal{L}_V g + 2S + 2\lambda g = 0. \tag{1.1}$$

It is said to be shrinking, steady, or expanding according as $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$.

Note that here the metric $g(t)$ is the pull back of the initial metric $g(0)$ by a 1-parameter family of diffeomorphisms generated by a vector field V on a manifold M. Compact Ricci solitons are the fixed points of the Ricci flow: $\partial g/\partial t = -2\mathrm{Ric}(g)$ projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings and often arise as blow-up limits for the Ricci flow on compact manifolds.

In 1923, Eisenhart [2] proved that if a positive definite Riemannian manifold (M, g) admits a second order parallel symmetric covariant tensor other than a constant multiple of the metric tensor, then it is reducible. In 1925, Levy [3] obtained the necessary and sufficient conditions for the existence of such tensors. In 1989, 1990, and 1991, Sharma [4–6] has generalized Levy's result by showing that a second order parallel (not necessarily symmetric and nonsingular) tensor on an n-dimensional $(n > 2)$ space of constant curvature is a constant multiple of the metric tensor. It is also proved that in a Sasakian manifold there is no nonzero parallel 2-form. In 2007, Das [7] in his paper proved that a second order symmetric parallel tensor on an α-K-contact $(\alpha \in R_0)$ manifold is a constant multiple of the associated metric tensor and also proved that there is no nonzero skew symmetric second order parallel tensor on an α-Sasakian manifold. Note that α-Sasakian manifolds are generalisations of Sasakian manifolds. Hence one can find interest in generalisation, from Sasakian to α-Sasakian manifolds and study Ricci solitons in this manifold.

In 2008, Sharma [8] studied Ricci solitons in K-contact manifolds, where the structure field ξ is killing and he proved that a complete K-contact gradient soliton is compact Einstein and Sasakian. In 2010, Călin and Crasmareanu [9] extended the Eisenhart problem to Ricci solitons in f-Kenmotsu manifolds. They studied the case of f-Kenmotsu manifolds satisfying a special condition called regular and a symmetric parallel tensor field of second order is a constant multiple of the Riemannian metric. Using this result, they obtained the results on Ricci solitons. Recently, Bagewadi and Ingalahalli [10] studied Ricci solitons in Lorentzian α-Sasakian Manifolds.

In this paper, we obtain some results on Ricci solitons.

2. Preliminaries

Let M be an almost contact metric manifold of dimension n, equipped with an almost contact metric structure (ϕ, ξ, η, g) consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g, which satisfy

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \qquad \eta \circ \phi = 0, \qquad \phi\xi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi), \tag{2.2}$$

for all $X, Y \in \mathfrak{X}(M)$. An almost contact metric manifold $M(\phi, \xi, \eta, g)$ is said to be α-Sasakian manifold if the following conditions hold:

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X), \tag{2.3}$$

$$\nabla_X \xi = -\alpha\phi X, \qquad (\nabla_X \eta)Y = \alpha g(X, \phi Y). \tag{2.4}$$

Holds for some smooth function α on M.

In an α-Sasakian manifold, the following relations hold:

$$R(X,Y)\xi = \alpha^2\left[\eta(Y)X - \eta(X)Y\right] + (Y\alpha)\phi X - (X\alpha)\phi Y, \tag{2.5}$$

$$R(\xi,X)Y = \alpha^2\left[g(X,Y)\xi - \eta(Y)X\right] + g(X,\phi Y)(\text{grad }\alpha) + (Y\alpha)\phi X, \tag{2.6}$$

$$\eta(R(X,Y)Z) = \alpha^2\left[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\right] + (X\alpha)g(\phi Y,Z) - (Y\alpha)g(\phi X,Z), \tag{2.7}$$

$$S(X,\xi) = \alpha^2(n-1)\eta(X) - ((\phi X)\alpha), \tag{2.8}$$

$$S(\xi,\xi) = \alpha^2(n-1), \tag{2.9}$$

$$Q\xi = \alpha^2(n-1)\xi + \phi(\text{grad }\alpha), \tag{2.10}$$

for all $X, Y, Z \in \mathfrak{X}(M)$, where R is the Riemannian curvature tensor, S is the Ricci tensor and Q is the Ricci operator.

3. Parallel Symmetric Second Order Tensors and Ricci Solitons in α-Sasakian Manifolds

Fix h a symmetric tensor field of $(0,2)$-type which we suppose to be parallel with respect to ∇ that is $\nabla h = 0$. Applying the Ricci identity [4, 11]

$$\nabla^2 h(X,Y;Z,W) - \nabla^2 h(X,Y;W,Z) = 0, \tag{3.1}$$

we obtain the relation

$$h(R(X,Y)Z,W) + h(Z,R(X,Y)W) = 0. \tag{3.2}$$

Replacing $Z = W = \xi$ in (3.2) and by using (2.5) and by the symmetry of h, we have

$$2\left[(Y\alpha)h(\phi X,\xi) - (X\alpha)h(\phi Y,\xi)\right] + 2\alpha^2\left[\eta(Y)h(X,\xi) - \eta(X)h(Y,\xi)\right] = 0. \tag{3.3}$$

Put $X = \xi$ in (3.3) and by virtue of (2.1), we have

$$2\alpha^2\left[\eta(Y)h(\xi,\xi) - h(Y,\xi)\right] - 2(\xi\alpha)h(\phi Y,\xi) = 0. \tag{3.4}$$

Replacing $Y = \phi Y$ in (3.4), we have

$$2(\xi\alpha)\left[h(Y,\xi) - \eta(Y)h(\xi,\xi)\right] - 2\alpha^2 h(\phi Y,\xi) = 0. \tag{3.5}$$

Solving (3.4) and (3.5), we have

$$\left(\alpha^4 + (\xi\alpha)^2\right)\left[\eta(Y)h(\xi,\xi) - h(Y,\xi)\right] = 0. \tag{3.6}$$

Since $\alpha^4 + (\xi\alpha)^2 \neq 0$, it results

$$h(Y,\xi) = \eta(Y)h(\xi,\xi). \tag{3.7}$$

Differentiating (3.7) covariantly with respect to X, we have

$$\begin{aligned}(\nabla_X h)(Y,\xi) + h(\nabla_X Y,\xi) + h(Y,\nabla_X \xi) = \\ \left[(\nabla_X \eta)(Y) + \eta(\nabla_X Y)\right]h(\xi,\xi) + \eta(Y)[(\nabla_X h)(Y,\xi) + 2h(\nabla_X \xi,\xi)].\end{aligned} \tag{3.8}$$

By using the parallel condition $\nabla h = 0$, $\eta(\nabla_X \xi) = 0$ and (3.7) in (3.8), we have

$$h(Y,\nabla_X \xi) = \left(\nabla_X \eta\right)(Y)h(\xi,\xi). \tag{3.9}$$

By using (2.4) in (3.9), we get

$$-\alpha h(Y,\phi X) = \alpha g(X,\phi Y)h(\xi,\xi). \tag{3.10}$$

Replacing $X = \phi X$ in (3.10), we get

$$\alpha\left[h(Y,X) - g(Y,X)h(\xi,\xi)\right] = 0. \tag{3.11}$$

Since α is a nonzero smooth function in α-Sasakian manifold and this implies that

$$h(X,Y) = g(X,Y)h(\xi,\xi), \tag{3.12}$$

the above equation implies that $h(\xi,\xi)$ is a constant, via (3.7). Now by considering the above condition we state the following theorem.

Theorem 3.1. *A symmetric parallel second order covariant tensor in an α-Sasakian manifold is a constant multiple of the metric tensor.*

Corollary 3.2. *A locally Ricci symmetric* $(\nabla S = 0)$ *α-Sasakian manifold is an Einstein manifold.*

Remark 3.3. The following statements for α-Sasakian manifold are equivalent:

(1) Einstein,

(2) locally Ricci symmetric,

(3) Ricci semi-symmetric that is $R \cdot S = 0$.

The implication $(1) \to (2) \to (3)$ is trivial. Now, we prove the implication $(3) \to (1)$ and $R \cdot S = 0$ means exactly (3.2) with replaced h by S that is,

$$(R(X,Y)\cdot S)(U,V) = -S(R(X,Y)U,V) - S(U,R(X,Y)V). \tag{3.13}$$

Considering $R \cdot S = 0$ and putting $X = \xi$ in (3.13), we have

$$S(R(\xi, Y)U, V) + S(U, R(\xi, Y)V) = 0. \tag{3.14}$$

By using (2.6) in (3.14), we obtain

$$\begin{aligned}&\left[g(\phi U, Y)S(\text{grad } \alpha, V) + (U\alpha)S(\phi Y, V)\right] + \alpha^2\left[g(Y, U)S(\xi, V) - \eta(U)S(Y, V)\right]\\&\quad + \left[g(\phi V, Y)S(U, \text{grad } \alpha) + (V\alpha)S(U, \phi Y)\right] + \alpha^2\left[g(Y, V)S(U, \xi) - \eta(V)S(U, Y)\right] = 0.\end{aligned} \tag{3.15}$$

Putting $U = \xi$ in (3.15) and by using (2.1), (2.8), and (2.9) on simplification, we obtain

$$\begin{aligned}&(\xi\alpha)S(\phi Y, V) - \alpha^2\eta(Y)((\phi V)\alpha) - \alpha^2 S(Y, V) + g(Y, \phi V)S(\xi, \text{grad } \alpha)\\&\quad + \alpha^4(n-1)g(Y, V) + \alpha^2\eta(V)((\phi Y)\alpha) = 0.\end{aligned} \tag{3.16}$$

Interchanging Y and V in (3.16), we have

$$\begin{aligned}&(\xi\alpha)S(\phi V, Y) - \alpha^2\eta(V)((\phi Y)\alpha) - \alpha^2 S(V, Y) + g(V, \phi Y)S(\xi, \text{grad } \alpha)\\&\quad + \alpha^4(n-1)g(V, Y) + \alpha^2\eta(Y)((\phi V)\alpha) = 0.\end{aligned} \tag{3.17}$$

Adding (3.16) and (3.17), we obtain

$$S(Y, V) = (n-1)\alpha^2 g(Y, V). \tag{3.18}$$

We conclude the following.

Proposition 3.4. *A Ricci semi-symmetric α-Sasakian manifold is an Einstein manifold.*

Corollary 3.5. *Suppose that on a α-Sasakian manifold the $(0,2)$-type field $\mathcal{L}_V g + 2S$ is parallel where V is a given vector field. Then (g, V, λ) yield a Ricci soliton. In particular, if the given α-Sasakian manifold is Ricci semi-symmetric with $\mathcal{L}_V g$ parallel, one has the same conclusion.*

Proof. Follows from Theorem 3.1 and Corollary 3.2.

A Ricci soliton in α-Sasakian manifold defined by (1.1). Thus $\mathcal{L}_V g + 2S$ is parallel. In Theorem 3.1 we proved that if an α-Sasakian manifold admits a symmetric parallel $(0,2)$ tensor, then the tensor is a constant multiple of the metric tensor. Hence $\mathcal{L}_V g + 2S$ is a constant multiple of the metric tensor g that is $(\mathcal{L}_V g + 2S)(X, Y) = g(X, Y)h(\xi, \xi)$, where $h(\xi, \xi)$ is a nonzero constant. We close this section with applications of our Theorem 3.1 to Ricci solitons. □

Corollary 3.6. *If a metric g in an α-Sasakian manifold is a Ricci soliton with $V = \xi$ then it is Einstein.*

Proof. Putting $V = \xi$ in (1.1), then we have

$$\left(\mathcal{L}_{\xi} g + 2S + 2\lambda g\right)(X, Y) = 0, \tag{3.19}$$

where

$$\left(\mathcal{L}_{\xi} g\right)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 0. \tag{3.20}$$

Substituting (3.20) in (3.19), then we get the result. □

Hence we state the following result.

Corollary 3.7. *A Ricci soliton (g, ξ, λ) in an n-dimensional α-Sasakian manifold cannot be steady but is shrinking.*

Proof. From Linear Algebra either the vector field $V \in \text{Span } \xi$ or $V \perp \xi$. However the second case seems to be complex to analyse in practice. For this reason we investigate for the case $V = \xi$.

A simple computation of $\mathcal{L}_{\xi} g + 2S$ gives

$$\left(\mathcal{L}_{\xi} g\right)(X, Y) = 0. \tag{3.21}$$

From (1.1), we have $h(X, Y) = -2\lambda g(X, Y)$ and then putting $X = Y = \xi$, we have

$$h(\xi, \xi) = -2\lambda, \tag{3.22}$$

where

$$h(\xi, \xi) = \left(\mathcal{L}_{\xi} g\right)(\xi, \xi) + 2S(\xi, \xi), \tag{3.23}$$

by using (2.9) and (3.21) in the above equation, we have

$$h(\xi, \xi) = 2\alpha^2(n-1). \tag{3.24}$$

Equating (3.22) and (3.24), we have

$$\lambda = -(n-1)\alpha^2. \tag{3.25}$$

Since α is some nonzero function, we have $\lambda \neq 0$, that is Ricci soliton in an n-dimensional α-Sasakian manifold cannot be steady but is shrinking because $\lambda < 0$. □

Corollary 3.8. *If an n-dimensional α-Sasakian manifold is η-Einstein then the Ricci solitons in α-Sasakian manifold that is (g, ξ, λ), where $\lambda = -(n-1)\alpha^2$ with varying scalar curvature cannot be steady but it is shrinking.*

Proof. The proof consists of three parts.

(i) We prove α-Sasakian manifold is η-Einstein.

(ii) We prove the Ricci soliton in α-Sasakian manifold is consisting of varying scalar curvature.

(iii) We find that the Ricci soliton in α-Sasakian manifold is shrinking.

First we prove that the α-Sasakian manifold is η-Einstein: the metric g is called η-Einstein if there exists two real functions a and b such that the Ricci tensor of g is given by the general equation

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y). \tag{3.26}$$

Now by simple calculations we find the values of a and b. Let e_i, $\{i = 1,2,\ldots n\}$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = Y = e_i$ in (3.26) and taking summation over i, we get

$$r = na + b. \tag{3.27}$$

Again putting $X = Y = \xi$ in (3.26) then by using (2.9), we have

$$a + b = (n-1)\alpha^2. \tag{3.28}$$

Then from (3.27) and (3.28), we have

$$a = \left[\frac{r}{(n-1)} - \alpha^2\right], \qquad b = \left[n\alpha^2 - \frac{r}{(n-1)}\right]. \tag{3.29}$$

Substituting the values of a and b in (3.26), we have

$$S(X,Y) = \left[\frac{r}{(n-1)} - \alpha^2\right]g(X,Y) + \left[n\alpha^2 - \frac{r}{(n-1)}\right]\eta(X)\eta(Y), \tag{3.30}$$

the above equation is an η-Einstein α-Sasakian manifold.

Now, we have to show that the scalar curvature r is not a constant and it is varying. For an n-dimensional α-Sasakian manifolds the symmetric parallel covariant tensor $h(X,Y)$ of type $(0,2)$ is given by

$$h(X,Y) = (\mathcal{L}_\xi g)(X,Y) + 2S(X,Y). \tag{3.31}$$

By using (3.21) and (3.30) in (3.31), we have

$$h(X,Y) = \left[\frac{2r}{(n-1)} - 2\alpha^2\right]g(X,Y) + \left[2n\alpha^2 - \frac{2r}{(n-1)}\right]\eta(X)\eta(Y). \tag{3.32}$$

Differentiating (3.32) covariantly with respect to Z, we have

$$(\nabla_Z h)(X,Y) = \left[\frac{2(\nabla_Z r)}{(n-1)} - 4\alpha(Z\alpha)\right]g(X,Y) + \left[4n\alpha(Z\alpha) - \frac{2(\nabla_Z r)}{(n-1)}\right]\eta(X)\eta(Y) + \left[2n\alpha^2 - \frac{2r}{(n-1)}\right]\left[g(X,\nabla_Z\xi)\eta(Y) + g(Y,\nabla_Z\xi)\eta(X)\right]. \tag{3.33}$$

By substituting $Z = \xi$ and $X = Y \in (\text{Span } \xi)^{\perp}$ in (3.33) and by using $\nabla h = 0$, we have

$$\nabla_\xi r = 2(n-1)\alpha(\xi\alpha) \Longrightarrow \nabla_\xi r = (n-1)\nabla_\xi \alpha^2. \tag{3.34}$$

On integrating (3.34), we have

$$r = (n-1)\alpha^2 + c, \tag{3.35}$$

where c is some integral constant. Thus from (3.35), we have r is a varying scalar curvature.

Finally, we have to check the nature of the soliton that is Ricci soliton in α-Sasakian manifold:

From (1.1), we have $h(X,Y) = -2\lambda g(X,Y)$ then putting $X = Y = \xi$, we have

$$h(\xi,\xi) = -2\lambda. \tag{3.36}$$

If we put $X = Y = \xi$ in (3.32), that is

$$h(\xi,\xi) = \left[\frac{2r}{(n-1)} - 2\alpha^2\right]g(\xi,\xi) + \left[2n\alpha^2 - \frac{2r}{(n-1)}\right]\eta(\xi)\eta(\xi), \tag{3.37}$$

Above equation reduced as,

$$h(\xi,\xi) = 2(n-1)\alpha^2. \tag{3.38}$$

Equating (3.36) and (3.38), we have

$$\lambda = -(n-1)\alpha^2. \tag{3.39}$$

Since, $\lambda \neq 0$ because α is some smooth function and $\lambda < 0$, that is the Ricci soliton in an α-Sasakian manifold is shrinking. □

4. Ricci Solitons in 3-Dimensional α-Sasakian Manifold

In this section we restrict our study to 3-dimensional α-Sasakian manifold, that is Ricci solitons in 3-dimensional α-Sasakian manifold.

Corollary 4.1. *If a Ricci soliton (g, ξ, λ) where $\lambda = -2\alpha^2$ of 3-dimensional α-Sasakian manifold with varying scalar curvature cannot be steady but it is shrinking.*

Proof. The proof consists of three parts.

(i) We prove that the Riemannian curvature tensor of 3-dimensional α-Sasakian manifold is η-Einstein.

(ii) We prove that the Ricci soliton in 3-dimensional α-Sasakian manifold is consisting of varying scalar curvature.

(iii) We find that the Ricci soliton in a 3-dimensional α-Sasakian manifold is shrinking.

First we consider: the Riemannian curvature tensor of 3-dimensional α-Sasakian manifold and it is given by

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X \\ &\quad - S(X,Z)Y - \frac{r}{2}\left[g(Y,Z)X - g(X,Z)Y\right]. \end{aligned} \tag{4.1}$$

Put $Z = \xi$ in (4.1) and by using (2.5) and (2.8), we have

$$\begin{aligned} &\left[(Y\alpha)\phi X - (X\alpha)\phi Y\right] + \alpha^2\left[\eta(Y)X - \eta(X)Y\right] \\ &= \eta(Y)QX - \eta(X)QY + 2\alpha^2\left[\eta(Y)X - \eta(X)Y\right] - ((\phi Y)\alpha)X + ((\phi X)\alpha) - \frac{r}{2}\left[\eta(Y)X - \eta(X)Y\right]. \end{aligned} \tag{4.2}$$

Again put $Y = \xi$ in (4.2) and by using (2.1) and (2.10), on simplification we get

$$QX = \left[\frac{r}{2} - \alpha^2\right]X + \left[3\alpha^2 - \frac{r}{2}\right]\eta(X)\xi + (\xi\alpha)\phi X + \eta(X)(\phi(\operatorname{grad}\alpha)) + ((\phi X)\alpha)\xi. \tag{4.3}$$

By taking an inner product Y in (4.3), we have

$$\begin{aligned} S(X,Y) &= \left[\frac{r}{2} - \alpha^2\right]g(X,Y) + \left[3\alpha^2 - \frac{r}{2}\right]\eta(X)\eta(Y) + (\xi\alpha)g(\phi X, Y) \\ &\quad - \eta(X)((\phi Y)\alpha) + \eta(Y)((\phi X)\alpha). \end{aligned} \tag{4.4}$$

Interchanging X and Y in (4.4), we have

$$\begin{aligned} S(Y,X) &= \left[\frac{r}{2} - \alpha^2\right](Y,X) + \left[3\alpha^2 - \frac{r}{2}\right]\eta(X)\eta(Y) + (\xi\alpha)g(\phi Y, X) \\ &\quad - \eta(Y)((\phi X)\alpha) + \eta(X)((\phi Y)\alpha). \end{aligned} \tag{4.5}$$

Adding (4.4) and (4.5), we have

$$S(X,Y) = \left[\frac{r}{2} - \alpha^2\right] g(X,Y) + \left[3\alpha^2 - \frac{r}{2}\right] \eta(X)\eta(Y). \tag{4.6}$$

Equation (4.6) shows that a 3-dimensional α-Sasakian manifold is η-Einstein.

Now, we have to show that the scalar curvature r is not a constant that is r is varying. Now,

$$h(X,Y) = (\mathcal{L}_\xi g)(X,Y) + 2S(X,Y). \tag{4.7}$$

By using (3.21) and (4.6) in (4.7), we have

$$h(X,Y) = \left[r - 2\alpha^2\right] g(X,Y) + \left[6\alpha^2 - r\right] \eta(X)\eta(Y). \tag{4.8}$$

Differentiating the above equation covariantly with respect to Z, we have

$$\begin{aligned}(\nabla_Z h)(X,Y) = {} & [\nabla_Z r - 4\alpha(Z\alpha)] g(X,Y) + [12\alpha(Z\alpha) - \nabla_Z r]\eta(X)\eta(Y) \\ & + \left[6\alpha^2 - r\right]\left[g(X, \nabla_Z \xi)\eta(Y) + g(Y, \nabla_Z \xi)\eta(X)\right].\end{aligned} \tag{4.9}$$

Substituting $Z = \xi$, $X = Y \in (\text{Span }\ \xi)^{\perp}$ in (4.9) and by virtue of $\nabla h = 0$, we have

$$\nabla_\xi r = 4\alpha(\xi\alpha) \Longrightarrow \nabla_\xi r = \nabla_\xi\left(2\alpha^2\right). \tag{4.10}$$

On integrating (4.10), we have

$$r = 2\alpha^2 + c, \tag{4.11}$$

where c is some integral constant. Thus from (4.11), we have r a varying scalar curvature.

Finally, we have to check the nature of the soliton that is Ricci soliton (g, ξ, λ) in 3-dimensional α-Sasakian manifold.

From (1.1), we have $h(X,Y) = -2\lambda g(X,Y)$ and then putting $X = Y = \xi$, we have

$$h(\xi,\xi) = -2\lambda. \tag{4.12}$$

If $X = Y = \xi$ in (4.8), that is

$$h(\xi,\xi) = \left[r - 2\alpha^2\right] g(\xi,\xi) + \left[6\alpha^2 - r\right] \eta(\xi)\eta(\xi). \tag{4.13}$$

Above equation reduced as

$$h(\xi,\xi) = 4\alpha^2. \tag{4.14}$$

Equating (4.12) and (4.14), we have

$$\lambda = -2\alpha^2. \tag{4.15}$$

Since from (4.15), we have $\lambda \neq 0$. Therefore Ricci soliton (g, ξ, λ) in 3-dimensional α-Sasakian manifold is shrinking. □

Example 4.2. Let $M = \{(x, y, z) \in R^3\}$. Let (E_1, E_2, E_3) be linearly independent vector fields given by

$$E_1 = e^x \frac{\partial}{\partial y}, \qquad E_2 = e^x \left[\frac{\partial}{\partial x} + 2y \frac{\partial}{\partial z}\right], \qquad E_3 = \frac{\partial}{\partial z}. \tag{4.16}$$

Let g be the Riemannian metric defined by $g(E_1, E_2) = g(E_2, E_3) = g(E_1, E_3) = 0$, $g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1$, where g is given by

$$g = \frac{1}{e^{2x}} \left[\left(1 - 4e^{2x} y^2\right) dx \otimes dx + dy \otimes dy + e^{2x} dz \otimes dz\right]. \tag{4.17}$$

Let η be the 1-form defined by $\eta(U) = g(U, E_3)$ for any $U \in \mathfrak{X}(M)$. Let ϕ be the $(1,1)$ tensor field defined by $\phi E_1 = E_2$, $\phi E_2 = -E_1$, $\phi E_3 = 0$. Then using the linearity of ϕ and g yields that $\eta(E_3) = 1$, $\phi^2 U = -U + \eta(U) E_3$ and $g(\phi U, \phi W) = g(U, W) - \eta(U)\eta(W)$ for any vector fields $U, W \in \mathfrak{X}(M)$. Thus for $E_3 = \xi$, (ϕ, ξ, η, g) defines a Sasakian structure on M. By definition of Lie bracket, we have

$$[E_1, E_2] = -e^x E_1 + 2e^{2x} E_3, \qquad [E_1, E_3] = [E_2, E_3] = 0. \tag{4.18}$$

Let ∇ be the Levi-Civita connection with respect to above metric g Koszula formula is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) = {} & X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ & - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned} \tag{4.19}$$

Then

$$\begin{gathered} \nabla_{E_1} E_1 = e^x E_2, \qquad \nabla_{E_2} E_2 = 0, \quad \nabla_{E_3} E_3 = 0, \\ \nabla_{E_1} E_2 = -e^x E_1 + e^{2x} E_3, \quad \nabla_{E_2} E_1 = -e^{2x} E_3, \quad \nabla_{E_2} E_3 = e^{2x} E_1, \\ \nabla_{E_1} E_3 = -e^{2x} E_2, \quad \nabla_{E_3} E_1 = -e^{2x} E_2, \quad \nabla_{E_3} E_2 = e^{2x} E_1. \end{gathered} \tag{4.20}$$

Clearly (ϕ, ξ, η, g) structure is an α-Sasakian structure and satisfy,

$$(\nabla_X \phi) Y = \alpha\left(g(X, Y)\xi - \eta(Y) X\right), \qquad \nabla_X \xi = -\alpha \phi X, \tag{4.21}$$

where $\alpha = e^{2x} \neq 0$. Hence (ϕ, ξ, η, g) structure defines α-Sasakian structure. Thus M equipped with α-Sasakian structure is a α-Sasakian manifold. The tangent vectors X and Y to M are expressed as linear combination of E_1, E_2, E_3, that is $X = \sum_{i=1}^{3} a_i E_i$ and $Y = \sum_{i=1}^{3} b_i E_i$, where a_i and b_i $(i = 1, 2, 3)$ are scalars.

Using $\alpha = e^{2x}$ in (4.11), we have

$$r = 2e^{4x} + c \neq 0, \tag{4.22}$$

and it shows that the scalar curvature is not constant.

Using $\alpha = e^{2x}$ in (4.15), we have

$$\lambda = -2e^{4x} \neq 0. \tag{4.23}$$

In this example $\alpha = e^{2x} \neq 0$, this implies that $\lambda < 0$, that is the Ricci soliton in 3-dimensional α-Sasakian manifold is shrinking.

5. Conclusion

In this paper we have shown that the Ricci soliton in an α-Sasakian manifold cannot be steady but it is shrinking accordingly because λ is negative.

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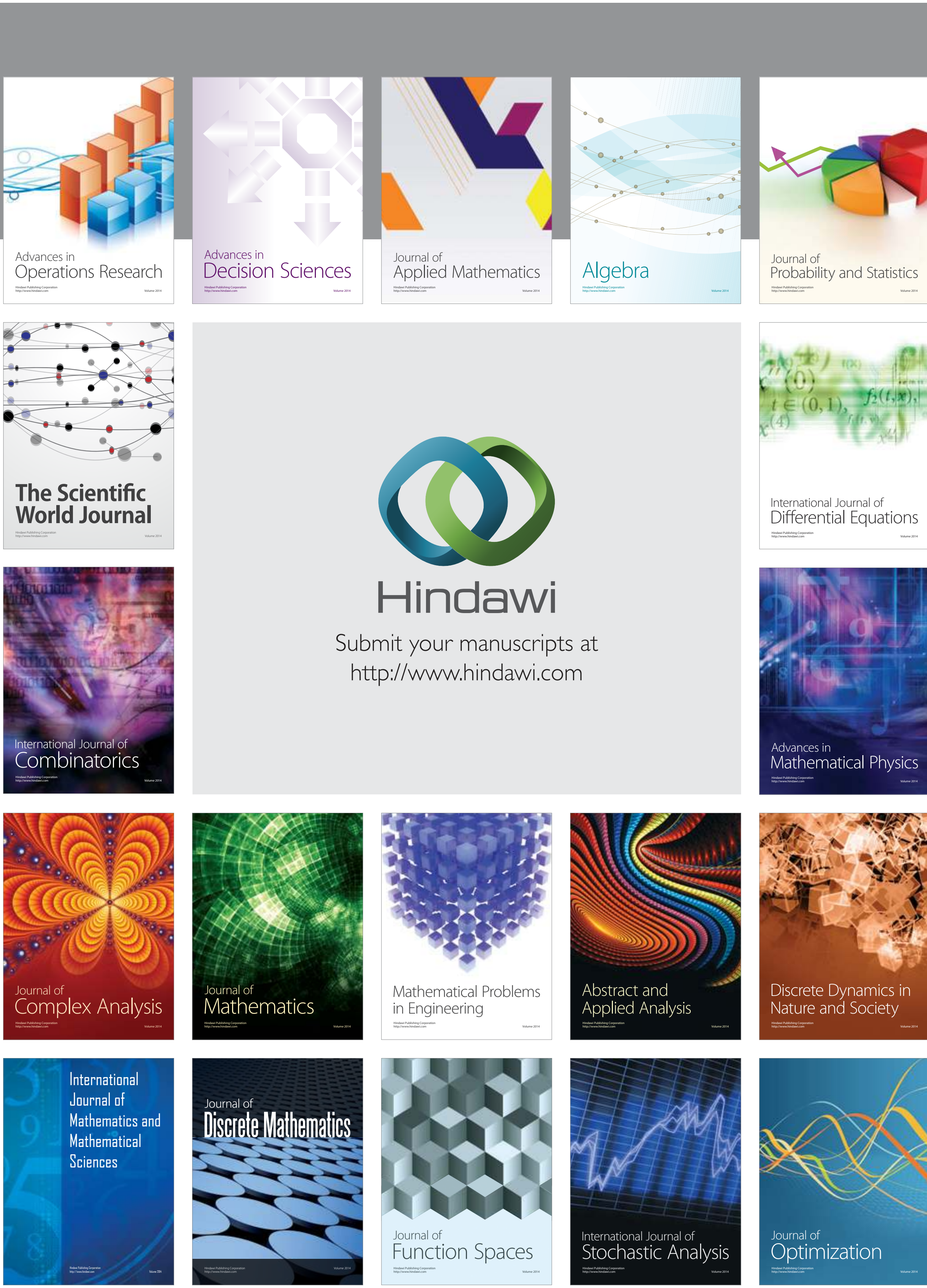
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