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Ricci Solitons on Para-Sasakian Manifolds

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Abstract: *The present paper aims at studying a Ricci solitons on para-Sasakian manifolds. We give the results on Ricci solitons in para-Sasakian manifold satisfying the conditions* $S(\xi, X)\cdot R = 0$, $R(\xi, X)\cdot \overline{P} = 0$ *and* $R(\xi, X)\cdot \overline{H} = 0$ *,where* $\overline{P}$, $\overline{H}$ *are pseudo-projective and quasi-conharmonic curvature tensors, respectively.*
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I. Introduction

A *Ricci soliton* is a natural generalization of Einstein metric. A Ricci soliton (g,V,λ) is defined on a pseudo-Riemannian manifold (M,g) by

$$(\pounds_V g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0, \tag{1.1}$$

where $\pounds_V g$ denotes the Lie derivative of Riemannian metric g along a vector feild V , λ is a constant, and X , Y are arbitrary vector fields on M . A Ricci soliton is said to be *shrinking*, *steady*, and *expanding* according as λ is *negative*, *zero*, and *positive*, respectively. Theoretical physicists have also been taking interest in the equation of Ricci soliton in relation with string theory, and the fact that equation (1.1) is a special case of Einstein field equations. The of Ricci soliton in Riemannian Geometry was introduced [1] as self-similar solutions of the Ricci flow. Recent progress on Riemannian Ricci solitons may be found in [2]. Also, Ricci solitons have been studied extensively in the context of pseudo-Riemannian Geometry; we may refer to [3, 4, 5, 6, 7, 8] and references therein.

In 1976, Sato [9] introduced the notion of almost paracontact structure (ϕ,ξ,η) on a differentiable manifolds. This structure is an analogue of the almost contact structure. An almost contact manifold is always odd-dimensional but an almost paracontact manifold could be of even dimension as well. Takahashi [10] defined almost contact manifolds (in particular, Sasakian manifolds) equipped with an associated Pseudo-Riemannian metric.

In 1977, Adati and Matsumoto defined Para-Sasakian and special Para-Sasakian manifolds [11], which are special classes of an almost paracontact manifold. Also, The geometry of these manifolds is extensively studied by [12, 13, 14, 15, 16, 17] and many others. In 1985, Kaneyuki and Williams [18] defined the notion of almost para contact structure on pseudo-Riemannian manifold of dimension $(2n+1)$. Later, Zamkovoy [19] showed that any almost para contact structure admits a pseudo-Riemannian metric with signature $(n+1,n)$.

In the present paper, we study Ricci solitons on para-Sasakian manifolds. The paper is organised as follows: section 2 is devoted to preliminaries on para-Sasakian manifolds. In section 3, 4 and 5 we study para-Sasakian Ricci solitons satisfying $S(\xi, X)\cdot R = 0$, $R(\xi, X)\cdot \overline{P} = 0$ and $R(\xi, X)\cdot \overline{H} = 0$, where $\overline{P}$ and $\overline{H}$ are pseudo-projective and quasi-conharmonic curvature tensors, respectively.

II. Preliminaries

An *almost paracontact structure* on a manifold M of dimension n is a triplet (ϕ,ξ,η) consisting of a (1,1)-tensor field ϕ , a vector field ξ , a 1-form η satisfying:

$$\phi^2 = I - \eta\otimes\xi,\ \ \eta(\xi) = 1,\ \ \phi\xi = 0, \tag{2.1}$$

$$\eta\cdot\phi = 0,\ \ rank(\phi) = n-1, \tag{2.2}$$

where I denotes the identity transformation. A pseudo-Riemannian metric g on M is compatible with the almost paracontact structure (ϕ,ξ,η) if

$$g(\phi X,\phi Y) = -g(X,Y) + \eta(X)\eta(Y). \tag{2.3}$$

In such case, (ϕ,ξ,η,g) is called an *almost paracontact metric structure*. By (2.1)-(2.3), it is clear that, $g(X,\xi)=\eta(X)$ for any compatible metric. Any almost paracontact structure admits compatible metrics. The fundamental 2-form Φ of an almost paracontact structure (ϕ,ξ,η,g) is defined by $\Phi = g(X,\phi Y)$, for all tangent vector fields X, Y. If $\Phi = d\eta$, then the manifold (M,ϕ,ξ,η,g) is called a *paracontact metric manifold* associated to the metric g.

In case the paracontact metric structure is normal. The structure is called para-Sasakian. Equivalently, a paracontact metric structure (ϕ,ξ,η,g) is para-Sasakian if An almost paracontact metric structure becomes a paracontact metric structure if

$$(\nabla_X\phi)Y = -g(X,Y)\xi + \eta(Y)X \tag{2.4}$$

for any vector fields $X,Y\in\Gamma(TM)$, where ∇ is Levi-Civita connection of g.

From (2.4), it follows that

$$\nabla_X\xi = -\phi X. \tag{2.5}$$

Also, in an n-dimensional para-Sasakian manifold, the following relations hold:

$$\eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \tag{2.6}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.7}$$

$$R(\xi,X)Y = -g(X,Y)\xi + \eta(Y)X, \tag{2.8}$$

$$S(X,\xi) = -(n-1)\eta(X), \tag{2.9}$$

$\forall X,Y,Z\in\Gamma(TM)$. Here R is Riemannian curvature tensor and S is Ricci tensor defined by $S(X,Y)=g(QX,Y)$, where Q is Ricci operator.

Let $M(\phi,\xi,\eta,g)$ be an n-dimension para-Sasakian manifold and let (g,ξ,λ) be a Ricci soliton on M. Then the relation (1.1) implies

$$(£_\xi g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0$$

or

$$2S(X,Y) = -(£_\xi g)(X,Y) - 2\lambda g(X,Y) \tag{2.10}$$

for any $X, Y \in\Gamma(M)$.

On a para-Sasakian manifold M, from (2.5) and the skew-symmetric property of ϕ, we obtain

$$(£_\xi g)(X,Y) = g(\nabla_X\xi,Y) + g(X,\nabla_Y\xi) = 0. \tag{2.11}$$

By plugging (2.11) in (2.10), we have

$$S(X,Y) = -\lambda g(X,Y). \tag{2.12}$$

Also, By Putting $Y=\xi$ in (2.12) we have

$$S(X,\xi) = -\lambda\eta(X). \tag{2.13}$$

By virtue of (2.9), we obtain from (2.13) that

$$\lambda = (n-1). \tag{2.14}$$

Thus, we can state the following:

Theorem 1. *A para-Sasakian Ricci soliton in an n-dimentsional (n>1) para-Sasakian manifold is expanding.*

III. Ricci Soliton in a Para-Sasakian Manifold Satisfying $S(\xi,X)\cdot R = 0.$

Consider a para-Sasakian manifold (M^n,g), satifying the condition

$$S(\xi,X)\cdot R = 0. \tag{3.1}$$

By definition we have

$$\begin{aligned} S((X,\xi)\cdot R)(U,V)W &= ((X\wedge_S\xi)R(U,V)W \\ &= (X\wedge_\xi)R(U,V)W + R((X\wedge_S)U,V)W \\ &+ R(U,(X\wedge_S\xi)V)W + R(U,V)(X\wedge_S\xi)W, \end{aligned} \tag{3.2}$$

where the endomorphisim $X\wedge_S Y$ is defined by

$$(X \wedge_S Y)Z = S(Y,Z)X - S(X,Z)Y. \tag{3.3}$$

In the view of (3.3) in (3.2), we get

$$S((X,\xi)\cdot R)(U,V)W$$
$$= S(\xi,R(U,V)W)X - S(X,R((U,V)W)\xi + S(\xi,U)R(X,V)W - S(X,U)R(\xi,V)W$$
$$+ S(\xi,V)R(U,X)W - S(X,V)R(U,\xi)W + S(\xi,W)R(U,V)X - S(X,W)R(U,V)\xi. \tag{3.4}$$

Using (2.12) and (2.13), we obatin

$$-\lambda[\eta(R(U,V)W)X - g(X,R(U,V)W)\xi + \eta(U)R(X,V)W - g(X,U)R(\xi,V)W$$
$$+\eta(V)R(U,X)W - g(X,V)R(U,\xi)W + \eta(W)R(U,V)X - g(X,W)R(U,V)\xi] = 0. \tag{3.5}$$

By taking an inner product with ξ and by Using (2.6), (2.7) and (2.8) in (3.5), we get

$$\lambda[g(X,R(U,V)W) - g(X,U)g(V,W) + g(X,V)g(U,W)$$
$$+2\eta(X)\{g(V,W)\eta(U) - g(U,W)\eta(V)\}] = 0. \tag{3.6}$$

Putting $X = U = e_i$ in (3.6) and summing over $i = 1,2,\ldots,n$, we get

$$\lambda[S(V,W) - (n-3)g(V,W) - 2\eta(V)\eta(W)] = 0. \tag{3.7}$$

Putting $V = W = \xi$ in (3.7) and by virtue of (2.12) and (2.13), we have

$$\lambda\{\lambda + (n-1)\} = 0. \tag{3.8}$$

Implies either $\lambda = 0$ or $\lambda = -(n-1)$.

Therefore, we can state the following theorem.

Theorem 2. *A Ricci soliton in a para-Sasakian manifold* $(n > 1)$ *satisfying* $S(\xi, X)\cdot R = 0$ *is either steady or shrinking.*

IV. Ricci Soliton in a Para-Sasakian Manifolds Satisfying $R(\xi, X)\cdot \overline{P} = 0$.

Pseudo-projective curvature tensor $\overline{P}$ is defined by B. Prasad [20].

$$\overline{P}(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] - \frac{r}{n}\left(\frac{a}{n-1} + b\right)[g(Y,Z)X - g(X,Z)Y] \tag{4.1}$$

where $a, b \neq 0$ are constants.

Taking $Z = \xi$ in (4.1) and by using (2.7) and (2.12) in (4.1), we have

$$\overline{P}(X,Y)\xi = \left[a + b\lambda + \frac{r}{n}\left(\frac{a}{n-1} + b\right)\right][\eta(X)Y - \eta(Y)X]. \tag{4.2}$$

Similarly using (2.6), (2.12) in (4.1), we get

$$\eta(\overline{P}(X,Y)Z) = \left[a + b\lambda + \frac{r}{n}\left(\frac{a}{n-1} + b\right)\right][g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]. \tag{4.3}$$

By using the condition $R(\xi, X)\cdot \overline{P} = 0$, we have

$$R(\xi,X)\overline{P}(U,V)W - \overline{P}(R(\xi,X)U,V)W - \overline{P}(U,R(\xi,X)V)W - \overline{P}(U,V)R(\xi,X)W = 0. \tag{4.4}$$

In the view of (2.8) in (4.4), we obtain

$$\eta(\overline{P}(U,V)W)X - g(X,\overline{P}(U,V)W)\xi - \eta(U)\overline{P}(X,V)W + g(X,U)\overline{P}(\xi,V)W$$

$$-\eta(V)\overline{P}(U,X)W + g(X,V)\overline{P}(U,\xi)W - \eta(W)\overline{P}(U,V)X + g(X,W)\overline{P}(U,V)\xi = 0. \tag{4.5}$$

Taking inner product with ξ in (4.5), we get

$$\eta(\overline{P}(U,V)W)\eta(X) - g(X,\overline{P}(U,V)W) - \eta(U)\eta(\overline{P}(X,V)W) + g(X,U)\eta(\overline{P}(\xi,V)W)$$

$$-\eta(V)\eta(\overline{P}(U,X)W + g(X,V)\eta(\overline{P}(U,\xi)W) - \eta(W)\eta(\overline{P}(U,V)X) + g(X,W)\eta(\overline{P}(U,V)\xi) = 0. \tag{4.6}$$

By virtue of (4.2), (4.3) in (4.6), we have

$$-g(X,\bar{P}(U,V)W)+\left[a+b\lambda+\frac{r}{n}\left(\frac{a}{n-1}+b\right)\right][g(U,W)g(V,X)-g(X,U)g(V,W)]=0. \tag{4.7}$$

Using (4.1) in (4.7), we obtain

$$-ag(X,R(U,V)W)+a[g(U,W)g(X,V)-g(X,U)g(V,W)]=0. \tag{4.8}$$

Putting $X=U=e_i$ in (4.8) and summing over $i=1,2,\ldots,n,$ we get

$$S(V,W)+(n-1)g(V,W)=0. \tag{4.9}$$

Putting $V=W=\xi$ in (4.9) and by virtue of (2.12), we obtain

$$\lambda=(n-1). \tag{4.10}$$

It implies λ is positive for every $(n>1)$. Hence we state the following theorem.

Theorem 3. *A Ricci soliton in a para-Sasakian manifold* $(n>1)$ *satisfying* $R(\xi,X)\cdot\bar{P}=0$ *is expanding.*

V. Ricci Soliton in a Para-Sasakian Manifold Satisfying $R(\xi,X)\cdot\bar{H}=0$.

Quasi-conharmonic curvature tensor $\bar{H}$ $(n>3)$ of the type $(1,3)$ is defined [21].

$$\bar{H}(X,Y)Z=aR(X,Y)Z+b[S(Y,Z)X-S(X,Z)Y]+c[g(Y,Z)QX-g(X,Z)QY]$$
$$-\frac{r}{n}\left(\frac{2a}{n-2}+b+c\right)[g(Y,Z)X-g(X,Z)Y], \tag{5.1}$$

where a, b and c are constants such that $a,b,c\neq 0$.

Taking $Z=\xi$ in (5.1) and by using (2.7) and (2.12) in (5.1), we have

$$\bar{H}(X,Y)\xi=\left[a+(b+c)\lambda+\frac{r}{n}\left(\frac{2a}{n-2}+b+c\right)\right][\eta(X)Y-\eta(Y)X]. \tag{5.2}$$

Similarly using (2.6), (2.12) in (5.1), we get

$$\eta(\bar{H}(X,Y)Z)=\left[a+(b+c)\lambda+\frac{r}{n}\left(\frac{2a}{n-2}+b+c\right)\right][g(X,Z)\eta(Y)-g(Y,Z)\eta(X)]. \tag{5.3}$$

We assume the condition $R(\xi,X)\cdot\bar{H}=0$, then we have

$$R(\xi,X)\bar{H}(U,V)W-\bar{H}(R(\xi,X)U,V)W-\bar{H}(U,R(\xi,X)V)W-\bar{H}(U,V)R(\xi,X)W=0. \tag{5.4}$$

In the view of (2.8) in (5.4), we obtain

$$\eta(\bar{H}(U,V)W)X-g(X,\bar{H}(U,V)W)\xi-\eta(U)\bar{H}(X,V)W+g(X,U)\bar{H}(\xi,V)W$$
$$-\eta(V)\bar{H}(U,X)W+g(X,V)\bar{H}(U,\xi)W-\eta(W)\bar{H}(U,V)X+g(X,W)\bar{H}(U,V)\xi=0. \tag{5.5}$$

Taking inner product with ξ in (5.5), we get

$$\eta(\bar{H}(U,V)W)\eta(X)-g(X,\bar{H}(U,V)W)-\eta(U)\eta(\bar{H}(X,V)W)+g(X,U)\eta(\bar{H}(\xi,V)W)$$
$$-\eta(V)\eta H(U,X)W)+g(X,V)\eta(\bar{H}(U,\xi)W)-\eta(W)\eta(\bar{H}(U,V)X)+g(X,W)\eta(\bar{H}(U,V)\xi)=0. \tag{5.6}$$

By virtue of (5.2), (5.3) in (5.6), we have

$$-g(X,\bar{H}(U,V)W)+\left[a+(b+c)\lambda+\frac{r}{n}\left(\frac{2a}{n-2}+b+c\right)\right][g(U,W)g(X,V)-g(V,W)g(X,U)]=0. \tag{5.7}$$

Using (5.1) in (5.7), we obtain

$$-ag(X,R(U,V)W)+a[g(U,W)g(X,V)-g(V,W)g(X,U)]=0. \qquad (5.8)$$

Putting $X=U=e_i$ in (5.8) and summing over $i=1,2,\ldots,n,$ we get

$$S(V,W)+(n-1)g(V,W)=0. \qquad (5.9)$$

Putting $V=W=\xi$ in (4.9) and by virtue of (2.12), we obtain

$$\lambda=(n-1). \qquad (5.10)$$

It implies λ is positive for every $(n>1)$. Hence we state the following theorem.

Theorem 4 . *A Ricci soliton in a para-Sasakian manifold* $(n>1)$ *satisfying* $R(\xi,X)\cdot\overline{H}=0$ *is expanding.*

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