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η-Ricci Solitons on Sasakian 3-Manifolds

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Abstract. In this paper we study η-Ricci solitons on Sasakian 3-manifolds. Among others we prove that an η-Ricci soliton on a Sasakian 3-manifold is an η-Einstien manifold. Moreover we consider η-Ricci solitons on Sasakian 3-manifolds with Ricci tensor of Codazzi type and cyclic parallel Ricci tensor. Beside these we study conformally flat and ϕ-Ricci symmetric η-Ricci soliton on Sasakian 3-manifolds. Also η-Ricci soliton on Sasakian 3-manifolds with the curvature condition $Q.R = 0$ have been considered. Finally, we construct an example to prove the non-existence of proper η-Ricci solitons on Sasakian 3-manifolds and verify some results.

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1 Introduction

In 1982, R. S. Hamilton [19] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

$$\frac{\partial}{\partial_t} g_{ij} = -2R_{ij}. \tag{1.1}$$

Ricci solitons are special solutions of the Ricci flow equation (1.1) of the form $g_{ij} = \sigma(t)\psi_t^* g_{ij}$ with the initial condition $g_{ij}(0) = g_{ij}$, where ψ_t are diffeomorphisms of M and $\sigma(t)$ is the scaling function.

A Ricci soliton is a generalization of an Einstein metric. We recall the notion of Ricci soliton according to [6]. On the manifold M, a Ricci soliton is a triple (g, V, λ) with g, a Riemannian metric, V a vector field, called potential vector field and λ a real scalar and S is the Ricci curvature tensor such that

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{1.2}$$

where $\pounds$ is the Lie derivative. Metrics satisfying (1.2) are interesting and useful in physics and are often referred as quasi-Einstein ([7],[8]). Compact Ricci solitons are the fixed points of the Ricci flow $\frac{\partial}{\partial t} g = -2S$ projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise blow-up limits for the Ricci flow on compact manifolds. Theoretical physicists have also been looking into the equation of Ricci soliton in relation with string theory. The initial contribution in this direction is due to Friedan [17] who discusses some aspects of it.

The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive, respectively. Ricci soliton have been studied by several authors such as ([15], [16], [19], [20]) and many others.

As a generalization of Ricci soliton, the notion of η-Ricci soliton was introduced by Cho and Kimura [9]. This notion has also been studied in [6] for Hopf hypersurfaces in complex space forms. An η-Ricci soliton is a tuple (g, V, λ, μ), where V is a vector field on M, λ and μ are constants and g is a Riemannian (or pseudo-Riemannian) metric satisfying the equation

$$\pounds_V g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{1.3}$$

where S is the Ricci tensor associated to g. In this connection we mention the works of Blaga ([4], [5]) and Prakasha et al. [24] on η-Ricci solitons. In particular, if $\mu = 0$ then the notion of η-Ricci soliton (g, V, λ, μ) is reduces to the notion of Ricci soliton (g, V, λ). If $\mu \neq 0$, then the η-Ricci soliton is named proper η-Ricci soliton.

A. Gray [18] introduced the notion of cyclic parallel Ricci tensor and Ricci tensor of Codazzi type. A Riemannian manifold is said to have cyclic parallel Ricci tensor if its Ricci tensor S of type (0,2) is non-zero and satisfies the condition

$$(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0. \tag{1.4}$$

Again a Riemannian manifold is said to have Ricci tensor of Codazzi type if its Ricci tensor S of type (0,2) is non-zero and satisfy the following condition

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z). \tag{1.5}$$

The paper is organized as follows:

After preliminaries in Section 2, we study η-Ricci soliton on a Sasakian 3-manifold. Among others we prove that an η-Ricci soliton on a Sasakian 3-manifold is an η-Einstein manifold. Moreover we consider η-Ricci solitons on Sasakian 3-manifolds with Ricci tensor of Codazzi type and cyclic parallel Ricci tensor. Beside these we study conformally flat and ϕ-Ricci symmetric η-Ricci soliton on Sasakian 3-manifolds. Also η-Ricci soliton on Sasakian 3-manifolds satisfying the curvature condition $Q.R = 0$ has been considered. Finally, we construct an example to prove the non-existence of proper η-Ricci solitons on Sasakian 3-manifolds and verify some results.

2 Preliminaries

An odd dimensional smooth manifold M^{2n+1} $(n \geq 1)$ is said to admit an almost contact structure, sometimes called a (ϕ, ξ, η)-structure, if it admits a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η satisfying ([1], [2])

$$\phi^2 = -I + \eta \otimes \xi, \;\; \eta(\xi) = 1, \;\; \phi\xi = 0, \;\; \eta \circ \phi = 0. \tag{2.1}$$

The first and one of the remaining three relations in (2.1) imply the other two relations in (2.1). An almost contact structure is said to be normal if the induced almost complex structure J on $M^{2n+1} \times \mathbb{R}$ defined by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt}) \tag{2.2}$$

is integrable, where X is tangent to M, t is the coordinate of $\mathbb{R}$ and f is a smooth function on $M^{2n+1} \times \mathbb{R}$. Let g be a compatible Riemannian metric with (ϕ, ξ, η), structure, that is,

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{2.3}$$

or equivalently,

$$g(X, \phi Y) = -g(\phi X, Y) \tag{2.4}$$

$$g(X, \xi) = \eta(X),$$

for all vector fields X, Y tangent to M. Then M becomes an almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g).

An almost contact metric structure becomes a contact metric structure if

$$g(X, \phi Y) = d\eta(X, Y), \tag{2.5}$$

for all X, Y tangent to M. The 1-form η is then a contact form and ξ is its characteristic vector field.

Given the contact metric manifold (M, η, ξ, ϕ, g), we define a symmetric (1,1)-tensor field h as $h = \frac{1}{2}L_\xi \phi$, where $L_\xi \phi$ denotes Lie differentiation in the direction of ξ. We have the following identities ([1], [2]): and

$$h\xi = 0, \quad h\phi + \phi h = 0, \tag{2.6}$$

$$\nabla_X \xi = -\phi X - \phi h X, \tag{2.7}$$

$$\nabla_\xi \phi = 0, \tag{2.8}$$

$$R(\xi, X)\xi - \phi R(\xi, \phi X)\xi = 2(h^2 + \phi^2)X, \tag{2.9}$$

$$(\nabla_\xi h)X = \phi X - h^2 \phi X + \phi R(\xi, X)\xi, \tag{2.10}$$

$$S(\xi, \xi) = 2n - trh^2, \tag{2.11}$$

$$R(X, Y)\xi = -(\nabla_X \phi)Y + (\nabla_Y \phi)X - (\nabla_X \phi h)Y + (\nabla_Y \phi h)X. \tag{2.12}$$

Here, ∇ is the Levi-Civita connection and R is the Riemannian curvature tensor of (M, g) with the sign convention defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for vector fields X, Y, Z on M.

If the characteristic vector field ξ is a Killing vector field, the contact metric manifold (M, η, ξ, ϕ, g) is called K-contact manifold. This is the case if and only if $h = 0$. The contact structure on M is said to be normal if the almost complex structure on $M \times \mathbb{R}$ defined by $J(X, f\frac{d}{dt}) = (\phi X -$

$f\xi, \eta(X)\frac{d}{dt})$, where f is a real function on $M \times \mathbb{R}$, is integrable. A normal contact metric manifold is called a Sasakian manifold. Sasakian metrics are K-contact and K-contact 3-metrics are Sasakian. For a Sasakian manifold, the following hold ([1], [2]):

$$\nabla_X \xi = -\phi X, \tag{2.13}$$

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.14}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.15}$$

$$Q\xi = 2n\xi, \tag{2.16}$$

where ∇, R and Q denote respectively, the Riemannian connection, curvature tensor and the $(1,1)$-tensor metrically equivalent to the Ricci tensor of g. The curvature tensor of a 3-dimensional Riemannian manifold is given by

$$\begin{aligned} R(X,Y)Z &= [S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ &\quad -\frac{r}{2}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{2.17}$$

where S and r are the Ricci tensor and scalar curvature respectively and Q is the Ricci operator defined by $g(QX,Y) = S(X,Y)$.

It is known that the Ricci tensor of a Sasakian 3-manifold is given by [3]

$$S(X,Y) = \frac{1}{2}[(r-2)g(X,Y) + (6-r)\eta(X)\eta(Y)], \tag{2.18}$$

where r is the scalar curvature which need not be constant, in general. So, g is Einstein (hence has constant curvature 1) if and only if $r = 6$.

Contact metric manifolds have also been studied by several authors such as ([3], [10], [11], [12], [14], [21], [22], [23]) and many others.

Proposition 2.1. *An η-Ricci soliton on a Sasakian 3-manifold is an η-Einstien manifold.*

Proof. Assume that the Sasakian 3-manifold admits a proper η-Ricci soliton (g, ξ, λ, μ). Then the relation (1.3) yields

$$(\pounds_\xi g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0. \tag{2.19}$$

It follows that

$$2S(X,Y) = -(\pounds_{\xi}g)(X,Y) - 2\lambda g(X,Y) - 2\mu\eta(X)\eta(Y), \tag{2.20}$$

for all smooth vector fields $X, Y \in \Gamma(TM)$, since ξ is Killing in Sasakian 3-manifolds, that is,

$$(\pounds_{\xi}g)(X,Y) = 0. \tag{2.21}$$

Making use of (2.21) in (2.20) we get

$$S(X,Y) = -\lambda g(X,Y) - \mu\eta(X)\eta(Y). \tag{2.22}$$

Thus we conclude that η-Ricci soliton (g, ξ, λ, μ) on Sasakian 3-manifolds is an η-Einstien manifold. This completes the proof.

□

Proposition 2.2. *For an η-Ricci soliton on a Sasakian 3-manifold we have $\lambda + \mu = -2$.*

Proof. The Ricci tensor of a Sasakian 3-manifold is given by [3]

$$S(X,Y) = \frac{1}{2}\{(r-2)g(X,Y) + (6-r)\eta(X)\eta(Y)\}, \tag{2.23}$$

where r is the scalar curvature. Comparing the above equation with $S(X,Y) = -\lambda g(X,Y) - \mu\eta(X)\eta(Y)$, we infer $\lambda = -\frac{1}{2}(r-2)$ and $\mu = -\frac{1}{2}(6-r)$. From which it follows that $\lambda + \mu = -2$. This completes the proof. □

3 η-Ricci Soliton on Sasakian 3-manifolds with Ricci tensor of Coddazi type

In this section we consider proper η-Ricci soliton on Sasakian 3-manifold with Ricci tensor of Coddazi type. Therefore taking the covariant differentiation of (2.22) with respect to Z we obtain

$$\begin{aligned}(\nabla_Z S)(X,Y) &= -\mu[(\nabla_Z\eta)(X)\eta(Y) + \eta(X)(\nabla_Z\eta)(Y)] \\ &= -\mu[g(Z,\phi X)\eta(Y) + \eta(X)g(Z,\phi Y)] \\ &= \mu[g(X,\phi Z)\eta(Y) + g(Y,\phi Z)\eta(X)]. \end{aligned} \tag{3.1}$$

By hypothesis the Ricci tensor S is of Coddazi type. Then

$$(\nabla_Z S)(X,Y) = (\nabla_Y S)(Z,X). \tag{3.2}$$

Using (3.1) in (3.2) we have

$$\mu[g(X,\phi Z)\eta(Y) + g(Y,\phi Z)\eta(X)] = \mu[g(Z,\phi Y)\eta(X) + g(X,\phi Y)\eta(Z)]. \tag{3.3}$$

Putting $Z = \xi$ in (3.3), we get $\mu = 0$, which is a contradiction. Hence a Sasakian 3-manifold with Ricci tensor of Coddazi type does not admit proper η-Ricci soliton. Thus we conclude the following:

Theorem 3.1. *A Sasakian 3-manifold with Ricci tensor of Coddazi type does not admit a proper η-Ricci soliton.*

4 η-Ricci soliton on Sasakian 3-manifolds with cyclic parallel Ricci tensor

This section is devoted to study proper η-Ricci soliton on Sasakian 3-manifolds with cyclic parallel Ricci tensor. Therefore

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0, \tag{4.1}$$

for all smooth vector fields $X, Y, Z \in \Gamma(TM)$. Using (2.22) in (4.1), we have

$$\begin{aligned} &\mu[g(Y,\phi X)\eta(Z) + g(Z,\phi X)\eta(Y) + g(Z,\phi Y)\eta(X) \\ &+g(X,\phi Y)\eta(Z) + g(X,\phi Z)\eta(Y) + g(Y,\phi Z)\eta(X) = 0. \end{aligned} \tag{4.2}$$

Putting $X = \xi$ in (4.2), we get

$$\mu g(\phi Y, Z) = 0.$$

It follows that

$$\mu = 0, \tag{4.3}$$

which is a contradiction. Thus we are in a position to state the following:

Theorem 4.1. *A Sasakian 3-manifold with cyclic parallel Ricci tensor does not admit a proper η-Ricci soliton.*

5 ϕ-Ricci Symmetric η-Ricci soliton on Sasakian 3-manifolds

In this section we study ϕ-Ricci symmetric proper η-Ricci soliton on Sasakian 3-manifolds. A Sasakian manifold is said to be ϕ-Ricci symmetric [13] if

$$\phi^2(\nabla_X Q)Y = 0, \tag{5.1}$$

for all smooth vector fields X, Y.
The Ricci tensor for an η-Ricci soliton on Sasakian 3-manifold is given by

$$S(X,Y) = -\lambda g(X,Y) - \mu\eta(X)\eta(Y). \tag{5.2}$$

Then it follows that

$$QX = -\lambda X - \mu\eta(X)\xi, \tag{5.3}$$

for all smooth vector fields X. Taking covariant derivative of (5.3) we get

$$(\nabla_X Q)Y = \mu g(\phi X, Y)\xi + \mu\eta(Y)\phi X. \tag{5.4}$$

Applying ϕ^2 on both sides of the above equation we get

$$\phi^2(\nabla_X Q)Y = \mu\eta(Y)\phi^3 X. \tag{5.5}$$

From (5.1) and (5.5) we have

$$\mu = 0, \tag{5.6}$$

which is a contradiction. Therefore a ϕ-Ricci symmetric Sasakian 3-manifold does not admit a proper η-Ricci soliton. Thus we can state the following:

Theorem 5.1. *A ϕ-Ricci symmetric Sasakian 3-manifold does not admit a proper η-Ricci soliton.*

6 Conformally flat η-Ricci soliton on Sasakian 3-manifolds

In this section we study conformally flat proper η-Ricci soliton in Sasakian 3-manifolds. Therefore [25]

$$(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = \frac{1}{4}[g(Y,Z)dr(X) - g(X,Z)dr(Y)]. \tag{6.1}$$

Using (2.22) in (6.1), we have

$$\mu[g(Y,\phi X)\eta(Z) + g(Z,\phi X)\eta(Y) - g(X,\phi(Y))\eta(Z) - g(Z,\phi Y)\eta(X)] \\ = \frac{1}{4}[g(Y,Z)dr(X) - g(X,Z)dr(Y)]. \quad (6.2)$$

Putting $X = \xi$ in (6.2), we get

$$4\mu g(\phi Y, Z) = -\eta(Z)dr(Y). \quad (6.3)$$

It follows that

$$4\mu\phi Y = -dr(Y)\xi. \quad (6.4)$$

This implies

$$\mu\phi^2 Y = 0, \quad (6.5)$$

and hence

$$\mu = 0, \quad (6.6)$$

which is a contradiction. Therefore a conformally flat Sasakian 3-manifold does not admit a proper η-Ricci soliton. Thus we can state the following:

Theorem 6.1. *A conformally flat Sasakian 3-manifold does not admit a proper η-Ricci soliton.*

7 η-Ricci soliton on Sasakian 3-manifold satisfying the curvature condition $Q.R = 0$

This section is devoted to study proper η-Ricci soliton on Sasakian 3-manifold satisfying the curvature condition $Q.R = 0$. Therefore

$$(Q.R)(X,Y)Z = 0, \quad (7.1)$$

for all smooth vector fields X, Y, Z. The explicit form of the above equation is

$$Q(R(X,Y)Z) - R(QX,Y)Z - R(X,QY)Z - R(X,Y)QZ = 0. \quad (7.2)$$

Using (2.22) in (7.2) we have

$$-\lambda R(X,Y)Z - \mu\eta(R(X,Y)Z)\xi + \lambda R(X,Y)Z + \mu\eta(X)R(\xi,Y)Z$$
$$+\lambda R(X,Y)Z + \mu\eta(Y)R(X,\xi)Z + \lambda R(X,Y)Z$$
$$+\mu\eta(Z)R(X,Y)\xi = 0. \quad (7.3)$$

Making use of (2.17) in (7.3) we get

$$-\mu g(Y,Z)\eta(QX)\xi + \mu g(X,Z)\eta(QY)\xi - \mu\eta(X)\eta(QZ)Y + \mu\eta(X)g(Y,Z)Q\xi$$
$$-2\mu\eta(X)\eta(Z)QY + \mu r\eta(X)\eta(Z)Y + \mu\eta(Y)\eta(QZ)X + 2\mu\eta(Y)\eta(Z)QX$$
$$-\mu\eta(Y)g(X,Z)Q\xi - \mu r\eta(Y)\eta(Z)X + \mu\eta(Z)\eta(QY)X - \mu\eta(Z)\eta(QX)Y$$
$$+2\lambda[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY - \frac{r}{2}\{g(Y,Z)X$$
$$-g(X,Z)Y\}] = 0, \quad (7.4)$$

Again using (2.22) in (7.4) we obtain

$$\lambda\mu g(Y,Z)\eta(X)\xi + \mu^2 g(Y,Z)\eta(X)\xi - \mu g(Y,Z)\eta(-\lambda X - \mu\eta(X)\xi)\xi$$
$$+\mu g(X,Z)\eta(-\lambda Y - \mu\eta(Y)\xi)\xi - \mu\eta(X)\eta(-\lambda Z - \mu\eta(Z)\xi)Y$$
$$+\mu\eta(X)g(Y,Z)(-\lambda\xi - \mu\xi) - 2\mu\eta(X)\eta(Z)(-\lambda Y - \mu\eta(Y)\xi) + \mu r\eta(X)\eta(Z)Y$$
$$+\mu\eta(Y)\eta(-\lambda Z - \mu\eta(Z)\xi)X + 2\mu\eta(Y)\eta(Z)(-\lambda X - \mu\eta(X)\xi)$$
$$-\mu\eta(Y)g(X,Z)(-\lambda\xi - \mu\xi) - \mu r\eta(Y)\eta(Z)X + \mu\eta(Z)\eta(-\lambda Y - \mu\eta(Y)\xi)X$$
$$-\mu\eta(Z)\eta(-\lambda X - \mu\eta(X)\xi)Y + 2\lambda[\{-\lambda g(Y,Z) - \mu\eta(Y)\eta(Z)\}X$$
$$-\{-\lambda g(X,Z) - \mu\eta(X)\eta(Z)\}Y + g(Y,Z)(-\lambda X - \mu\eta(X)\xi\}$$
$$-g(X,Z)(-\lambda Y - \mu\eta(Y)\xi\} - \frac{r}{2}\{g(Y,Z)X - g(X,Z)Y\}] = 0, \quad (7.5)$$

from which it follows that

$$4\lambda\mu\eta(X)\eta(Z)Y + 2\mu^2\eta(X)\eta(Z)Y - 4\lambda\mu\eta(Y)\eta(Z)X + \mu r\eta(X)\eta(Z)Y$$
$$-\mu r\eta(Y)\eta(Z)X + 2\lambda[\{-\lambda g(Y,Z) - \mu\eta(Y)\eta(Z)\}X$$
$$-\{-\lambda g(X,Z) - \mu\eta(X)\eta(Z)\}Y + g(Y,Z)(-\lambda X - \mu\eta(X)\xi\}$$
$$-g(X,Z)(-\lambda Y - \mu\eta(Y)\xi\} - \frac{r}{2}\{g(Y,Z)X - g(X,Z)Y\}]$$
$$-2\mu^2\eta(Y)\eta(Z)X + \lambda\mu g(Y,Z)\eta(X)\xi + \mu^2 g(Y,Z)\eta(X)\xi = 0. \quad (7.6)$$

Putting $X = Z = \xi$ in (7.6) yields

$$4\lambda\mu Y + 2\mu^2 Y - 4\lambda\mu\eta(Y)\xi + \mu rY - \mu r\eta(Y)\xi + 2\lambda[-\lambda\eta(Y)\xi$$
$$-\mu\eta(Y)\xi + \lambda Y + \mu Y - \lambda\eta(Y)\xi - \mu\eta(Y)\xi + \lambda Y + \mu\eta(Y)\xi$$
$$-\frac{1}{2}\eta(Y)\xi + \frac{1}{2}Y] - 2\mu^2\eta(Y)\xi + \lambda\mu\eta(Y)\xi + \mu^2\eta(Y)\xi = 0. \quad (7.7)$$

Taking inner product of (7.7) with ξ we get

$$\mu(\lambda + \mu)\eta(Y) = 0. \tag{7.8}$$

By hypothesis $\mu \neq 0$, therefore from the above equation we get $\lambda + \mu = 0$. Moreover for an η-Ricci soliton on Sasakian 3-manifold $\lambda + \mu = -2$. Thus in this case $\lambda = -1$ and $\mu = -1$. Therefore we have the following:

Theorem 7.1. *An proper η-Ricci soliton on a Sasakian 3-manifold satisfying the curvature condition $Q.R = 0$ is of the type $(g, V, -1, -1)$.*

8 Example of non-existence of proper η-Ricci Solitons on Sasakian 3-manifolds

We consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3, (x, y, z) \neq (0, 0, 0)\}$, where (x, y, z) are the standard coordinate in $\mathbb{R}^3$. Then $e_1 = \frac{\partial}{\partial y}$, $e_2 = \frac{\partial}{\partial z}$, $e_3 = 2\frac{\partial}{\partial x} - y\frac{\partial}{\partial z} + z\frac{\partial}{\partial y}$ are linearly independent at each point of M and

$$[e_1, e_2] = 0, [e_1, e_3] = -e_2, [e_2, e_3] = e_1.$$

Let g be the Riemannian metric defined by

$$g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0, \quad g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

Let η be the 1-form defined by

$$\eta(U) = g(U, e_3)$$

for any $U \in \Gamma(TM)$ and ϕ be the $(1, 1)$-tensor field defined by

$$\phi e_1 = -e_2, \phi e_2 = e_1, \phi e_3 = 0.$$

Using the linearity of ϕ and g we have

$$\eta(e_3) = 1$$

$$\phi^2(U) = -U + \eta(U)e_3$$

and

$$g(\phi U, \phi W) = g(U, W) - \eta(U)\eta(W)$$

for any $U, W \in \Gamma(TM)$. Therefore for $e_3 = \xi$ the structure (M, η, ξ, ϕ, g) defines an almost contact metric manifold.
The Riemannian connection ∇ of the metric tensor g is given by Koszul's formula which is

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &\quad -g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]). \end{aligned}$$

Using Koszul's formula we get the following:

$$\begin{array}{lll} \nabla_{e_1} e_3 = e_2, & \nabla_{e_1} e_2 = -e_3, & \nabla_{e_1} e_1 = -e_3, \\ \nabla_{e_2} e_3 = -e_1, & \nabla_{e_2} e_2 = 0, & \nabla_{e_2} e_1 = -e_3, \\ \nabla_{e_3} e_3 = 0, & \nabla_{e_3} e_2 = 0, & \nabla_{e_3} e_1 = 0. \end{array}$$

In view of the above relations we have

$$\nabla_X \xi = -\phi X, \quad \text{for} \quad \xi = e_3$$

Therefore the structure $M(\phi, \xi, \eta, g)$ is a Sasakian 3-manifold.
Now, we find the curvature tensor as follows:

$$\begin{array}{lll} R(e_1,e_2)e_2 = e_1, & R(e_1,e_3)e_3 = e_1, & R(e_2,e_1)e_1 = -e_1 + e_2, \\ R(e_2,e_3)e_3 = e_2, & R(e_3,e_1)e_1 = e_3, & R(e_3,e_2)e_2 = e_3, \\ R(e_1,e_2)e_3 = e_3, & R(e_2,e_3)e_1 = -e_3, & R(e_3,e_1)e_2 = 0. \end{array}$$

Using the expressions of the curvature tensor we find the values of the Ricci tensor as follows:

$$\begin{array}{lll} S(e_1,e_1) = 2, & S(e_2,e_2) = 2, & S(e_3,e_3) = 2, \\ S(e_1,e_2) = 0, & S(e_1,e_3) = 0, & S(e_2,e_3) = 0. \end{array}$$

In this case from (2.22), we have $\lambda = -2$ and $\mu = 0$. Thus the manifold (g, ξ, λ, μ) does not admit proper η-Ricci soliton. Also the Ricci tensor is of Codazzi type and cyclic parallel. Hence the Theorem 3.1 and Theorem 4.1 are verified. Again the manifold is ϕ-Ricci symmetric. Thus the Theorem 5.1 is verified.

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