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RICCI SOLITONS ON THREE-DIMENSIONAL η-EINSTEIN ALMOST KENMOTSU MANIFOLDS

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Abstract. Let the metric g of a three-dimensional η-Einstein almost Kenmotsu manifold M be a Ricci soliton, we prove that M is a Kenmotsu manifold of constant sectional curvature -1 and the soliton is expanding.

1. INTRODUCTION

In 1982, R. S. Hamilton [11] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

$$\frac{\partial}{\partial_t} g_{ij}(t) = -2R_{ij}. \tag{1.1}$$

On a Riemannian manifold (M, g), a Ricci soliton (see Hamilton [12]) is a generalization of the Einstein metric (that is, the Ricci tensor Ric is a constant multiple of the Riemannian metric g) and is defined by

$$\frac{1}{2}\mathcal{L}_V g + \mathrm{Ric} + \lambda g = 0 \tag{1.2}$$

for a constant λ and a vector field V on M, where $\mathcal{L}$ denotes the Lie-derivative operator. Obviously, a Ricci soliton with V zero or a Killing vector field reduces to an Einstein equation. The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive respectively. Compact Ricci solitons are the fixed points of the Hamilton's Ricci flow projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise as blow-up limits for the Ricci

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flow. If the vector field V is the gradient of a potential function $-f$, then g is called a gradient Ricci soliton and equation (1.2) becomes

$$\mathrm{Hess}f = \mathrm{Ric} + \lambda g, \tag{1.3}$$

where $\mathrm{Hess}f$ denotes the Hession of a smooth function f on M which is defined by $\mathrm{Hess}(f) = \nabla^2 f$. Following G. Perelman [18], we know that a Ricci soliton on a compact Riemannian manifold is always a gradient Ricci soliton.

Recently, Ricci solitons and gradient Ricci solitons on some kinds of almost contact metric manifolds of dimension three were studied by many authors. For instances, De et al. [6] and Turan et al. [20] investigated Ricci solitons and gradient Ricci solitons on three-dimensional normal almost contact metric manifolds and three-dimensional trans-Sasakian manifolds respectively. Moreover, A. Ghosh [9] and J. H. Cho [3] classified Ricci solitons on three-dimensional Kenmotsu manifolds respectively. In addition, Ricci solitons on f-Kenmotsu manifolds and $N(k)$-quasi-Einstein manifolds were also studied by C. Calin and M. Crasmareanu [2] and M. Crasmareanu [4] respectively. With regard to the studies of Ricci solitons on Kenmotsu manifolds of dimension greater than 3, we refer the reader to A. Ghosh [10]. Generalizing some corresponding results of Ricci solitons on three-dimensional Kenmotsu manifolds mentioned above, the present paper is devoted to investigating Ricci solitons on a type of almost Kenmotsu manifolds of dimension three, namely, η-Einstein almost Kenmotsu manifolds, or equivalently, almost Kenmotsu manifolds with the Reeb vector field belonging to the generalized k-nullity distribution. For more results on almost Kenmotsu manifolds with the Reeb vector field belonging to some other nullity distributions, we refer the reader to G. Dileo and A. M. Pastore [8], A. M. Pastore and V. Saltarelli [16], V. Saltarelli [19], Y. Wang and X. Liu [21, 22]. Our main result can be presented as follows:

If the metric g of a three-dimensional η-Einstein almost Kenmotsu manifold (M, g) is a Ricci soliton, then M is a Kenmotsu manifold of constant sectional curvature -1 and the soliton is expanding with $\lambda = 2$.

This paper is organized as follows. In a preliminaries section, we recall some well known basic formulas and properties of almost Kenmotsu manifolds. In section 3, we completely classify Ricci solitons on an almost Kenmotsu manifold of dimension three such that the Reeb vector field belongs to the generalized k-nullity distribution. Moreover, some corollaries of our main theorem can also be seen in the last section.

2. Preliminaries

Following D. E. Blair [1], an almost contact structure on a $(2n+1)$-dimensional smooth manifold M^{2n+1} is a triplet (ϕ, ξ, η), where ϕ is a $(1,1)$-type tensor field, ξ a global vector field and η a 1-form, such that

$$\phi^2 = -\mathrm{id} + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{2.1}$$

where id denotes the identity mapping, which imply that $\phi(\xi) = 0$, $\eta \circ \phi = 0$ and $\mathrm{rank}(\phi) = 2n$. Generally, ξ is called the Reeb vector field or the characteristic vector field. A Riemannian metric g on M^{2n+1} is said to be compatible with the almost contact structure (ϕ, ξ, η) if

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.2}$$

for any vector fields $X, Y \in \Gamma(TM)$, where $\Gamma(TM)$ denotes the Lie algebra of all differentiable vector fields on M^{2n+1}. An almost contact structure endowed with a compatible Riemannian metric is said to be an almost contact metric structure. The fundamental 2-form Φ of an almost contact metric manifold M^{2n+1} is defined by $\Phi(X, Y) = g(X, \phi Y)$ for any vector fields X, Y on M^{2n+1}. We may define on the product manifold $M^{2n+1} \times \mathbb{R}$ an almost complex structure J by

$$J\left(X, \delta\frac{\partial}{\partial_t}\right) = \left(\phi X - \delta\xi, \eta(X)\frac{d}{dt}\right),$$

where X denotes the vector field tangent to M^{2n+1}, t is the coordinate of $\mathbb{R}$ and δ is a $\mathcal{C}^\infty$-function on product $M^{2n+1} \times \mathbb{R}$. It is well known [1] that the normality of an almost contact structure is expressed by the vanishing of the tensor $N_\phi = [\phi, \phi] + 2d\eta \otimes \xi$, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ. According to D. Janssens and L. Vanhecke [13], an almost Kenmotsu manifold is defined as an almost contact metric manifold such that $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$. Consequently, the normality of the almost contact metric structure of an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is expressed by

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$$

for any vector fields X, Y. A normal almost Kenmotsu manifold is said to be a Kenmotsu manifold (see [13]). It is known [14] that a Kenmotsu manifold M^{2n+1} is locally isometric to a warped product $I \times_\theta M^{2n}$, where M^{2n} is a Kählerian manifold, I is an open interval with coordinate t and the warping function $\theta = ce^t$ for some positive constant c.

We consider two tensor fields $l = R(\cdot\,, \xi)\xi$ and $h = \frac{1}{2}\mathcal{L}_\xi \phi$ on an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, where R is the Riemannian curvature tensor of g and $\mathcal{L}$ is the Lie differentiation. Following Kim and Pak [15], the two $(1,1)$-type tensor fields l and h are symmetric and satisfy

$$h\xi = 0, \quad l\xi = 0, \quad \mathrm{tr}h = 0, \quad \mathrm{tr}(h\phi) = 0, \quad h\phi + \phi h = 0. \tag{2.3}$$

We also have the following formulas (see Dileo and Pastore [7, 8]):

$$\nabla_X \xi = X - \eta(X)\xi + h'X, \tag{2.4}$$

$$\phi l\phi - l = 2(h^2 - \phi^2), \tag{2.5}$$

$$\mathrm{tr}(l) = S(\xi,\xi) = g(Q\xi,\xi) = -2n - \mathrm{tr}h^2, \tag{2.6}$$

$$R(X,Y)\xi = \eta(X)(Y - \phi hY) - \eta(Y)(X - \phi hX) + (\nabla_Y \phi h)X - (\nabla_X \phi h)Y, \tag{2.7}$$

for any $X, Y \in \Gamma(TM)$, where $h' = h \circ \phi$, S, Q and ∇ denote the Ricci curvature tensor, the Ricci operator and the Levi-Civita connection of g, respectively. On an almost contact metric manifold M, if the Ricci operator satisfies

$$Q = \alpha \mathrm{id} + \beta\eta \otimes \xi, \tag{2.8}$$

where both α and β are smooth functions on M, then M is said to be an η-Einstein manifold. Obviously, an η-Einstein manifold with vanishing β and α a constant is an Einstein manifold. An η-Einstein manifold is said to be proper η-Einstein if $\beta \neq 0$.

3. Main Results

In what follows, let (M, g) be a three-dimensional almost Kenmotsu manifold. If the Reeb vector field ξ of M belongs to the generalized k-nullity distribution defined by

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] \tag{3.1}$$

for arbitrary vector fields X, Y on M and a smooth function k on M, then by Proposition 3.1 of [19] we see that relation (3.1) is equivalent to relation (2.8), i.e., M is η-Einstein. Moreover, the function k in (3.1) can be expressed by $k = \frac{\alpha+\beta}{2}$.

Let (M, g) be a three-dimensional almost Kenmotsu manifold with ξ belonging to the generalized k-nullity distribution. Making use of relations (2.4)–(2.7), A. M. Pastore and V. Saltarelli in [16] obtained the following two equations:

$$h^2 = h'^2 = (k+1)\phi^2, \quad Q\xi = 2k\xi. \tag{3.2}$$

Then it follows that $k \leq -1$ everywhere on M. Moreover, $k = -1$ identically if and only if $h = 0$, this is also equivalent to $h' = 0$. If $k < -1$, we denote the two nonzero eigenvalues of h by ν and $-\nu$ respectively, where we assume $\nu = \sqrt{-1-k} > 0$. Furthermore, by Proposition 3.1 of [16] we also have

$$\nabla_\xi h' = -2h'. \tag{3.3}$$

The following result directly follows from Lemma 3.3 of [19].

Lemma 3.1. *Let (M,g) be a three-dimensional almost Kenmotsu manifold such that the Reeb vector field belongs to the generalized k-nullity distribution, then we have*

$$\mathrm{D}k = -4(k+1)\xi, \tag{3.4}$$

where D *denotes the gradient operator with respect to g.*

Applying the above lemma we have $\mathrm{d}k = -4(k+1)\eta$, this implies $\mathrm{d}k \wedge \eta = 0$. Then, making use of Lemma 4.1 of [19], we immediately obtain the following result.

Lemma 3.2. *Let (M,g) be a three-dimensional almost Kenmotsu manifold such that the Reeb vector field belongs to the generalized k-nullity distribution, then either $k=-1$ identically or $k<-1$ everywhere on M.*

Taking into account the above two lemmas, now we may present our main theorem with its proof as follows.

Theorem 3.1. *Let the metric g of a three-dimensional η-Einstein almost Kenmotsu manifold (M,g) be a Ricci soliton, then M is a Kenmotsu manifold of constant sectional curvature -1 and the soliton is expanding with $\lambda = 2$.*

Proof. Applying Lemma 3.2 and Proposition 3.1 of [19] we may divide our discussions into two cases: case 1: $k=-1$ identically and case 2: $k<-1$ everywhere on M. Being $k=-1$ we get from the first term of (3.2) that $h=0$ and hence M is a Kenmotsu manifold (see Proposition 3 of [7]). In this case, the proof follows from Theorem 1 of Ghosh [9].

Next we consider the case $k<-1$ everywhere on M, which is also equivalent to $h \neq 0$. Putting relation (2.8) into (1.2) we obtain

$$(\mathcal{L}_V g)(Y,Z) = -2(\alpha+\lambda)g(Y,Z) - 2\beta\eta(Y)\eta(Z) \tag{3.5}$$

for any vector fields Y, Z. Taking the covariant differentiation of (3.5) along an arbitrary vector field X we obtain

$$\begin{aligned} &(\nabla_X \mathcal{L}_V g)(Y,Z) \\ &= -2X(\alpha)g(Y,Z) - 2X(\beta)\eta(Y)\eta(Z) - 2\beta g(X+h'X,Y)\eta(Z) \\ &\quad - 2\beta g(X+h'X,Z)\eta(Y) + 4\beta\eta(X)\eta(Y)\eta(Z) \end{aligned} \tag{3.6}$$

for any vector fields X, Y, Z. Following Yano [23], the following formula

$$(\mathcal{L}_V \nabla_X g - \nabla_X \mathcal{L}_V g - \nabla_{[V,X]} g)(Y,Z) = -g((\mathcal{L}_V \nabla)(X,Y),Z) - g((\mathcal{L}_V \nabla)(X,Z),Y)$$

is well known for any vector fields X, Y, Z on M. As g is parallel with respect to the Levi-Civita connection ∇, then the above relation becomes

$$(\nabla_X \mathcal{L}_V g)(Y,Z) = g((\mathcal{L}_V\nabla)(X,Y),Z) + g((\mathcal{L}_V\nabla)(X,Z),Y) \tag{3.7}$$

for any vector fields X, Y, Z. Since $\mathcal{L}_V\nabla$ is a symmetric tensor of type (1.2), i.e., $(\mathcal{L}_V\nabla)(X,Y) = (\mathcal{L}_V\nabla)(Y,X)$, then it follows from (3.7) that

$$\begin{aligned} &g((\mathcal{L}_V\nabla)(X,Y),Z) \\ &= \frac{1}{2}(\nabla_X\mathcal{L}_V g)(Y,Z) + \frac{1}{2}(\nabla_Y\mathcal{L}_V g)(Z,X) - \frac{1}{2}(\nabla_Z\mathcal{L}_V g)(X,Y) \end{aligned} \tag{3.8}$$

for any vector fields X, Y, Z. Making use of (3.6) in (3.8) we have

$$\begin{aligned} &(\mathcal{L}_V\nabla)(X,Y) \\ &= -X(\alpha)Y - Y(\alpha)X + g(X,Y)\mathrm{D}\alpha + \eta(X)\eta(Y)\mathrm{D}\beta \\ &\quad - [X(\beta)\eta(Y) + 2\beta g(X+h'X,Y) - 2\beta\eta(X)\eta(Y) + Y(\beta)\eta(X)]\xi \end{aligned} \tag{3.9}$$

for any vector fields X, Y, Z. Now we consider a local field of orthonormal frame $\{e_i : 1 \leq i \leq 3\}$ on M, replacing $X = Y$ by e_i in relation (3.9) and summing over $i = 1, 2, 3$, we obtain

$$\sum_{i=1}^{3}(\mathcal{L}_V\nabla)(e_i,e_i) = \mathrm{D}\alpha + \mathrm{D}\beta - 2[\xi(\beta) + 2\beta]\xi. \tag{3.10}$$

On the other hand, taking the covariant differentiation of the Ricci soliton equation (1.2) along an arbitrary vector field X we obtain $\nabla_X\mathcal{L}_V g = -2\nabla_X S$, putting this relation into (3.8) we have

$$g((\mathcal{L}_V\nabla)(X,Y),Z) = (\nabla_Z S)(X,Y) - (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) \tag{3.11}$$

for any vector fields X, Y, Z. Hence, setting $X = Y = e_i$ in (3.11) and summing over $i = 1, 2, 3$ we obtain $\sum_{i=1}^{3}(\mathcal{L}_V\nabla)(e_i,e_i) = 0$, comparing this relation with (3.10) we have

$$\mathrm{D}\alpha + \mathrm{D}\beta - 2[\xi(\beta) + 2\beta]\xi = 0. \tag{3.12}$$

Making use of relation (3.9) and taking the covariant differentiation of $(\mathcal{L}_V\nabla)(Y,Z)$ along an arbitrary vector field X, we may obtain

$$
\begin{aligned}
&(\nabla_X \mathcal{L}_V \nabla)(Y,Z) \\
=&-g(Y,\nabla_X \mathrm{D}\alpha)Z-g(Z,\nabla_X \mathrm{D}\alpha)Y+\eta(Y)\eta(Z)\nabla_X \mathrm{D}\beta \\
&+g(Y,Z)\nabla_X \mathrm{D}\alpha+[g(X+h'X,Z)\eta(Y)+g(X+h'X,Y)\eta(Z) \\
&-2\eta(X)\eta(Y)\eta(Z)]\mathrm{D}\beta-[Y(\beta)\eta(Z)+2\beta g(Y+h'Y,Z) \\
&-2\beta\eta(Y)\eta(Z)+\eta(Y)Z(\beta)](X+h'X)-g(Y,\nabla_X \mathrm{D}\beta)\eta(Z)\xi \\
&-g(Z,\nabla_X \mathrm{D}\beta)\eta(Y)\xi-2\beta g((\nabla_X h')Y,Z)\xi+2\beta[g(X+h'X,Y)\eta(Z) \\
&+g(X+h'X,Z)\eta(Y)+g(Y+h'Y,Z)\eta(X)]\xi-Y(\beta)[g(X+h'X,Z) \\
&-2\eta(X)\eta(Z)]\xi-Z(\beta)[g(X+h'X,Y)-2\eta(X)\eta(Y)]\xi \\
&-2X(\beta)[g(Y+h'Y,Z)-\eta(Y)\eta(Z)]\xi-6\beta\eta(X)\eta(Y)\eta(Z)\xi
\end{aligned} \tag{3.13}
$$

for any vector fields X, Y, Z. It is well known [23] that the following relation

$$
(\mathcal{L}_V R)(X,Y)Z=(\nabla_X \mathcal{L}_V \nabla)(Y,Z)-(\nabla_Y \mathcal{L}_V \nabla)(X,Z) \tag{3.14}
$$

holds for any vector fields X, Y, Z. Furthermore, applying the well known Poincare lemma: $\mathrm{d}^2=0$, i.e., $g(\nabla_X \mathrm{D}\zeta, Y)=g(\nabla_Y \mathrm{D}\zeta, X)$ for any smooth function ζ on M (see Ghosh [9]) and making use of relations (3.13), (2.3) in (3.14), we may obtain

$$
\begin{aligned}
&(\mathcal{L}_V R)(X,Y)Z \\
=&g(Z,\nabla_Y \mathrm{D}\alpha)X-g(Z,\nabla_X \mathrm{D}\alpha)Y+[g(X+h'X,Z)\eta(Y) \\
&-g(Y+h'Y,Z)\eta(X)]\mathrm{D}\beta+\eta(Z)[\eta(Y)\nabla_X \mathrm{D}\beta-\eta(X)\nabla_Y \mathrm{D}\beta] \\
&+g(Y,Z)\nabla_X \mathrm{D}\alpha-g(X,Z)\nabla_Y \mathrm{D}\alpha+[X(\beta)\eta(Z)+2\beta g(X+h'X,Z) \\
&-2\beta\eta(X)\eta(Z)+\eta(X)Z(\beta)](Y+h'Y)-[Y(\beta)\eta(Z)+2\beta g(Y+h'Y,Z) \\
&-2\beta\eta(Y)\eta(Z)+\eta(Y)Z(\beta)](X+h'X)-X(\beta)g(Y+h'Y,Z)\xi \\
&+Y(\beta)g(X+h'X,Z)\xi-2\beta g((\nabla_X h')Y,Z)\xi-[g(Z,\nabla_X \mathrm{D}\beta)\eta(Y) \\
&-g(Z,\nabla_Y \mathrm{D}\beta)\eta(X)]\xi+2\beta(g(\nabla_Y h')X,Z)\xi
\end{aligned} \tag{3.15}
$$

for any vector fields X, Y, Z. Consider again the local orthonormal frame $\{e_i : 1 \leq i \leq 3\}$, hence the Laplacian operator Δ can be defined by $\Delta\rho=-\mathrm{tr}(\mathrm{Hess}\rho)=-\sum_{i=1}^{3} g(\nabla_{e_i}\mathrm{D}\rho, e_i)$ for any smooth function ρ on M. Thus, contracting (3.15) over X and making use of the Poincare lemma, (2.3) and (3.3), we obtain

$$
\begin{aligned}
&(\mathcal{L}_V S)(Y,Z) \\
=&-g(Y,Z)\Delta\alpha-\eta(Y)\eta(Z)\Delta\beta+2g(Z,\nabla_Y \mathrm{D}\alpha)-2\xi(\beta)g(Y+h'Y,Z) \\
&+\eta(Y)g(Z+h'Z,\mathrm{D}\beta)+\eta(Z)g(Y+h'Y,\mathrm{D}\beta)-\eta(Y)\eta(\nabla_Z \mathrm{D}\beta) \\
&-\eta(Z)\eta(\nabla_Y \mathrm{D}\beta)-4\beta g(Y,Z)-2\beta g(h'Y,Z)+4\beta\eta(Y)\eta(Z) \\
&-2Z(\beta)\eta(Y)-2Y(\beta)\eta(Z)
\end{aligned} \tag{3.16}
$$

for any vector fields Y, Z. Moreover, notice that M is an η-Einstein manifold, by (2.8) and a straightforward calculation we obtain that

$$\begin{aligned}&(\mathcal{L}_V S)(Y,Z)\\ =\ &[V(\alpha)-2(\lambda+\alpha)]g(Y,Z)+[V(\beta)-2\alpha\beta-2\beta\eta(V)]\eta(Y)\eta(Z)\\ &+\beta[g(Z+h'Z,V)+\eta(\nabla_Z V)]\eta(Y)+\beta[g(Y+h'Y,V)+\eta(\nabla_Y V)]\eta(Z)\end{aligned} \tag{3.17}$$

for any vector fields Y, Z on M.

Subtracting (3.16) from (3.17) gives an equation, substituting Y and Z with ϕY and ϕZ respectively in the resulting equation we obtain

$$\begin{aligned}&g(\phi Y,\phi Z)\Delta\alpha-2g(\phi Z,\nabla_{\phi Y}\mathrm{D}\alpha)+2\xi(\beta)g(\phi Y,Z)\\ &+[V(\alpha)-2(\lambda+\alpha)+4\beta]g(\phi Y,\phi Z)-2\beta g(h'Y,Z)+2\xi(\beta)g(hY,Z)=0\end{aligned} \tag{3.18}$$

for any vector fields Y, Z. Interchanging Y, Z of relation (3.18) yields an equation, subtracting the resulting equation from (3.18) and applying the Poincare lemma again we may obtain $\xi(\beta)g(\phi Y, Z) = 0$ for any vector fields Y, Z, then it follows that

$$\xi(\beta)=0. \tag{3.19}$$

Using (3.19) in relation (3.12) we get $\mathrm{D}\alpha+\mathrm{D}\beta-4\beta\xi = 0$, taking the inner product of this relation with ξ we obtain $\xi(\alpha) = 4\beta$. Recall that M is an η-Einstein almost Kenmotsu manifold of dimension 3 if and only if ξ belongs to the generalized k-nullity distribution with $k = \frac{\alpha+\beta}{2}$, then by applying Lemma 3.1 we obtain $\xi(k) = -4(k+1)$, making use of $k = \frac{\alpha+\beta}{2}$, $\xi(\beta) = 0$ and $\xi(\alpha) = 4\beta$ in this relation we obtain

$$\alpha+2\beta+2=0. \tag{3.20}$$

It is easy to see from (3.19) and (3.20) that $\xi(\alpha) = -2\xi(\beta) = 0$, then we get $\beta = 0$ and hence $\alpha = -2$. However, in fact, in this context we have $k = \frac{\alpha+\beta}{2} = -1$, this contradicts the assumption $k < -1$ everywhere on M. Thus we complete the proof. ■

By Theorem 3.1, the following corollary (see also Theorem 1 of [9]) follows immediately.

Corollary 3.1. *If the metric of a three-dimensional Kenmotsu manifold is a Ricci soliton, then such a Kenmotsu manifold is of constant curvature* -1 *and the soliton is expanding.*

By Theorem 5.3 of [17] (see also Lemma 1 of [9]), we see that a three-dimensional Kenmotsu manifold is an η-Einstein manifold, then our Theorem 3.1 extends the corresponding result shown in Ghosh [9].

Let ω be a complete unit vector field orthogonal to the Reeb vector field ξ on a three-dimensional Kenmotsu manifold, we consider the following Ricci soliton:

$$\frac{1}{2}\mathcal{L}_\omega g+\mathrm{Ric}+\lambda g=0 \tag{3.21}$$

for certain constant λ. Applying Theorem 3.1 we get immediately the following corollary (see also Theorem 3 of Cho [3]).

Corollary 3.2. *A Kenmotsu* 3*-manifold* M *admitting a Ricci soliton* (g,ω) *is of constant curvature* -1*, where the potential vector field* ω *is orthogonal to* ξ*.*

Remark 3.1. By considering the warped product of a real line and a 2-dimensional Kählerian manifold, Ghosh [9] constructed an example of Ricci soliton on a three-dimensional Kenmotsu manifold. For more details, we refer the reader to section 4 of [9].

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