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Let M be a real hypersurface of a complex space form $M^n(c)$, $c \neq 0$, and suppose that the structure vector field ξ is an eigen vector field of the Ricci tensor S, which satisfies $S\xi = \beta\xi$ where β is a function. We show that if $(\nabla_X S)Y$ is proportional to ξ for any vector fields X and Y orthogonal to ξ, then M is a Hopf hypersurface, and if it is perpendicular to ξ, then M is a ruled real hypersurface.

1. Introduction

Takagi [1973] gave a classification of the homogeneous real hypersurface (see also [Takagi 1975a; 1975b]). As a consequence of this result, the structure vector ξ of any homogeneous real hypersurface in $\mathbb{C}P^n$ is principal. If ξ satisfies this property, then M is said to be a *Hopf hypersurface*. When the ambient manifold is a complex hyperbolic space, Lohnherr [1998] (see also [Lohnherr and Reckziegel 1999]) discovered a homogeneous ruled real hypersurface in $\mathbb{C}H^n$ that is not a Hopf hypersurface, and further examples were given (see [Berndt and Brück 2001]). The classification theorem for homogeneous real hypersurfaces in $\mathbb{C}H^n$, $n \geq 2$, was given by Berndt and Tamaru [2007].

When a real hypersurface is Hopf, fundamental formulas are simple. So many classification theorems are given under that assumption (see, for example, [Niebergall and Ryan 1997]). Kimura [1986] has given a classification of Hopf hypersurfaces of $\mathbb{C}P^n$, $n \geq 2$, with constant principal curvatures. He showed that a real hypersurface in $\mathbb{C}P^n$ with constant principal curvatures is a Hopf hypersurface if and only if it is an open part of a homogeneous real hypersurface. A classification theorem for Hopf hypersurfaces with constant principal curvatures in $\mathbb{C}H^n$, $n \geq 2$, was given by Berndt [1989].

On the other hand, the Ricci tensor of the real hypersurfaces is an interesting subject. It is well known that any real hypersurface of $M^n(c)$, $c \neq 0$, is not Einstein. If the Ricci tensor S is of the form $S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y)$, then the real hypersurface is said to be pseudo-Einstein. The classification theorems for pseudo-Einstein real hypersurfaces in a complex space form $M^n(c)$ have been

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completed [Cecil and Ryan 1982; Kim and Ryan 2008; Kon 1979; Montiel 1985]. Ki [1989] showed that there are no real hypersurfaces with parallel Ricci tensor, $\nabla S = 0$, in $M^n(c)$, $n \geq 3$. Several conditions that weaken the condition $\nabla S = 0$ have been studied (see [Ki et al. 1990; Suh 1990]).

We focus on the Ricci tensor S and consider a condition $S\xi = \beta\xi$, where β is a function. We note that this condition contains not only Hopf hypersurfaces, $A\xi = \alpha\xi$, but also some non-Hopf hypersurfaces. For example, ruled hypersurfaces, which are an important example of non-Hopf hypersurfaces, also satisfy $S\xi = \beta\xi$. Under this assumption, we study some Hopf hypersurfaces and ruled real hypersurfaces according to the direction of a covariant differentiation of S.

Our main result is the following theorem:

Theorem 1.1. *Let M be a connected real hypersurface of $M^n(c)$, $c \neq 0$, and suppose that the Ricci tensor S of M satisfies $S\xi = \beta\xi$ for some function β.*

(1) *If $(\nabla_X S)Y$ is proportional to the structure vector field ξ for any vector fields X and Y orthogonal to ξ, then M is a Hopf hypersurface.*

(2) *If $(\nabla_X S)Y$ is perpendicular to the structure vector field ξ for any vector fields X and Y orthogonal to the structure vector field ξ, then M is a ruled real hypersurface.*

When $n = 2$, the author gave a corresponding result in [Kon 2014].

2. Preliminaries

Let $M^n(c)$ denote the complex space form of complex dimension n (real dimension $2n$) with constant holomorphic sectional curvature $4c$. We denote by J the almost complex structure of $M^n(c)$. The Hermitian metric of $M^n(c)$ is denoted by G.

Let M be a real $(2n-1)$-dimensional hypersurface immersed in $M^n(c)$. Throughout this paper, we suppose that M is connected. We denote by g the Riemannian metric induced on M from G. We take the unit normal vector field N of M in $M^n(c)$. For any vector field X tangent to M, we define ϕ, η and ξ by

$$JX = \phi X + \eta(X)N, \quad JN = -\xi,$$

where ϕX is the tangential part of JX, ϕ is a tensor field of type (1,1), η is a 1-form, and ξ is the unit vector field on M. We call ξ the *structure vector field*. Then

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0$$

for any vector field X tangent to M. Moreover, we have

$$g(\phi X, Y) + g(X, \phi Y) = 0, \quad \eta(X) = g(X, \xi), \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

Thus (ϕ, ξ, η, g) defines an almost contact metric structure on M.

We denote by $\tilde{\nabla}$ the operator of covariant differentiation in $M^n(c)$, and by ∇ the operator of covariant differentiation in M determined by the induced metric. Then the *Gauss and Weingarten formulas* are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \tilde{\nabla}_X N = -AX,$$

for any vector fields X and Y tangent to M.

For the contact metric structure on M, we have

$$\nabla_X \xi = \phi AX, \quad (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi.$$

We call A the *shape operator* of M. If the shape operator A of M satisfies $A\xi = \alpha\xi$ for some function α, then M is called a *Hopf hypersurface*. By the Codazzi equation, we have the following result (see [Maeda 1976]).

Proposition A. *Let M be a Hopf hypersurface in $M^n(c)$, $n \geq 2$. If $X \perp \xi$ and $AX = \lambda X$, then $\alpha = g(A\xi, \xi)$ is constant and*

$$(2\lambda - \alpha)A\phi X = (\lambda\alpha + 2c)\phi X.$$

We offer an important example of a non-Hopf hypersurface. Take a regular curve γ in $M^n(c)$ with tangent vector field X. At each point of γ there is a unique complex projective or hyperbolic hyperplane cutting γ so as to be orthogonal to X and JX. The union of these hyperplanes is called a *ruled real hypersurface* (see [Kimura and Maeda 1989; Lohnherr and Reckziegel 1999; Niebergall and Ryan 1997]).

We remark that the shape operator A is η-parallel if it satisfies $g((\nabla_X A)Y, Z) = 0$ for any X, Y and Z orthogonal to ξ.

We denote by R the Riemannian curvature tensor field of M. Then the *equation of Gauss* is given by

$$\begin{aligned} R(X, Y)Z &= c\{g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\} \\ &\quad + g(AY, Z)AX - g(AX, Z)AY, \end{aligned}$$

and the *equation of Codazzi* by

$$(\nabla_X A)Y - (\nabla_Y A)X = c\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}.$$

From the equation of Gauss, the Ricci tensor S of M is given by

$$g(SX, Y) = (2n+1)cg(X, Y) - 3c\eta(X)\eta(Y) + \operatorname{tr} A g(AX, Y) - g(AX, AY), \tag{1}$$

where $\operatorname{tr} A$ is the trace of A. Taking a covariant differentiation, we have

$$\begin{aligned} g((\nabla_X S)Y, Z) &= -3cg(Y, \phi AX)\eta(Z) - 3cg(\phi AX, Z)\eta(Y) + (X \operatorname{tr} A)g(AY, Z) \\ &\quad + \operatorname{tr} A g((\nabla_X A)Y, Z) - g((\nabla_X A)AY, Z) - g((\nabla_X A)Y, AZ). \end{aligned} \tag{2}$$

Now we develop some lemmas needed to prove our main theorem. Suppose $n \geq 3$.

Lemma 2.1. *Let M be a real hypersurface in a complex space form $M^n(c)$, $n \geq 3$, $c \neq 0$. If there exists an orthonormal frame $\{\xi, e_1, \ldots, e_{2n-2}\}$ on a sufficiently small neighborhood $\mathcal{N}$ of $x \in M$ such that the shape operator A can be represented as*

$$A = \begin{pmatrix} \alpha & h_1 & 0 & \cdots & 0 \\ h_1 & a_1 & & & \\ 0 & & a_2 & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & & \cdots & 0 & a_{2n-2} \end{pmatrix},$$

then we have

$$(a_j - a_k)g(\nabla_{e_i} e_j, e_k) - (a_i - a_k)g(\nabla_{e_j} e_i, e_k) = 0, \tag{3}$$

$$(a_j - a_1)g(\nabla_{e_i} e_j, e_1) - (a_i - a_1)g(\nabla_{e_j} e_i, e_1) = h_1(a_i + a_j)g(e_i, \phi e_j), \tag{4}$$

$$h_1 g(\nabla_{e_i} e_j, e_1) - h_1 g(\nabla_{e_j} e_i, e_1) = \{2c - 2a_i a_j + \alpha(a_i + a_j)\}g(\phi e_i, e_j), \tag{5}$$

$$(e_j a_i) = (a_j - a_i)g(\nabla_{e_i} e_j, e_i), \tag{6}$$

$$(e_1 a_i) = (a_1 - a_i)g(\nabla_{e_i} e_1, e_i), \tag{7}$$

$$(a_1 - a_j)g(\nabla_{e_i} e_1, e_j) + (a_j - a_i)g(\nabla_{e_1} e_i, e_j) = a_i h_1 g(e_i, \phi e_j), \tag{8}$$

$$(e_i h_1) = \{2c - 2a_1 a_i + \alpha(a_i + a_1)\}g(e_i, \phi e_1) - h_1 g(\nabla_{e_1} e_i, e_1), \tag{9}$$

$$(e_i a_1) = h_1(2a_i + a_1)g(e_i, \phi e_1) + (a_i - a_1)g(\nabla_{e_1} e_i, e_1), \tag{10}$$

$$(\xi a_i) = h_1 g(\nabla_{e_i} e_1, e_i), \tag{11}$$

$$h_1 g(\nabla_{e_i} e_1, e_j) + (a_j - a_i)g(\nabla_\xi e_i, e_j) = (c + a_i \alpha - a_i a_j)g(e_i, \phi e_j), \tag{12}$$

$$(e_i h_1) = (c + a_i \alpha - a_1 a_i + h_1^2)g(e_i, \phi e_1) + (a_i - a_1)g(\nabla_\xi e_i, e_1), \tag{13}$$

$$(e_i \alpha) = h_1(\alpha - 3a_i)g(e_i, \phi e_1) - h_1 g(\nabla_\xi e_i, e_1), \tag{14}$$

$$(e_1 h_1) = (\xi a_1), \tag{15}$$

$$(e_1 \alpha) = (\xi h_1), \tag{16}$$

$$(a_1 - a_i)g(\nabla_\xi e_1, e_i) - h_1 g(\nabla_{e_1} e_1, e_i) = (c + a_1 \alpha - a_1 a_i - h_1^2)g(e_i, \phi e_1), \tag{17}$$

for any $i, j \geq 2$, $i \neq j$.

Proof. By the equation of Codazzi, we have

$$g((\nabla_{e_i} A)e_1 - (\nabla_{e_1} A)e_i, e_j) = 0,$$

where $i, j = 2, \ldots, 2n-2$. On the other hand, we have

$$\begin{aligned} g((\nabla_{e_i} A)e_1 - (\nabla_{e_1} A)e_i, e_j) &= g(\nabla_{e_i}(Ae_1) - A\nabla_{e_i} e_1 - \nabla_{e_1}(Ae_i) + A\nabla_{e_1} e_i, e_j) \\ &= (a_1 - a_j)g(\nabla_{e_i} e_1, e_j) + (a_j - a_i)g(\nabla_{e_1} e_i, e_j) + a_i h_1 g(\phi e_i, e_j). \end{aligned}$$

Thus we obtain (8). We obtain the other results through similar computations. □

We remark that these equations hold in the case that M is a Hopf hypersurface, i.e., $h_1 = 0$. When $n = 2$, we showed the corresponding result in [Kon 2014].

We define the subspace $L_x \subset T_x(M)$ as the smallest subspace that contains ξ and is invariant under the shape operator A. Then M is Hopf if and only if L_x is one-dimensional at each point x.

Lemma 2.2. *Let M be a real hypersurface of $M^n(c)$. If the Ricci tensor S of M satisfies $S\xi = \beta\xi$ for some function β, then* $\dim L_x \le 2$ *at each point x of $M^n(c)$.*

Proof. By (1), we have

$$0 = g(S\xi, Y) = -g(A^2\xi, Y)$$

for any Y orthogonal to ξ and $A\xi$. So $A^2\xi$ is spanned by ξ and $A\xi$. Thus we see that $\dim L_x \le 2$. □

Suppose that M is not a Hopf hypersurface and that $S\xi = \beta\xi$. By Lemma 2.2, we can take an orthonormal frame $\{\xi, e_1, \dots, e_{2n-2}\}$, locally, such that A is of the form

$$A = \begin{pmatrix} \alpha & h_1 & & & 0 \\ h_1 & a_1 & & & \\ & & a_2 & & \\ & & & \ddots & \\ 0 & & & & a_{2n-2} \end{pmatrix},$$

where $h_1 = g(Ae_1, \xi)$, $a_i = g(Ae_i, e_i)$ for $i = 1, \dots, 2n-2$, $g(Ae_i, e_j) = 0$ for $i \ne j$ and $\alpha = g(A\xi, \xi)$. By (1), we obtain

$$\begin{aligned} S\xi &= (2n-2)c\xi + (\operatorname{tr} A)(h_1e_1 + \alpha\xi) - A(h_1e_1 + \alpha\xi) \\ &= (\operatorname{tr} A - \alpha - a_1)h_1e_1 + \{(2n-2)c + (\operatorname{tr} A)\alpha - h_1^2 - \alpha^2\}\xi = \beta\xi. \end{aligned}$$

So we see that

$$\operatorname{tr} A = \alpha + a_1, \quad a_2 + \cdots + a_{2n-2} = 0.$$

Moreover, (1) implies that the Ricci tensor S can be represented as

$$S = \begin{pmatrix} \beta & & & 0 \\ & \lambda_1 & & \\ & & \ddots & \\ 0 & & & \lambda_{2n-2} \end{pmatrix},$$

where β and λ_i satisfy

$$\begin{gathered} \beta = (2n-2)c + (\alpha a_1 - h_1^2), \quad \lambda_1 = (2n+1)c + (\alpha a_1 - h_1^2), \\ \lambda_j = (2n+1)c + \operatorname{tr} A \cdot a_j - a_j^2, \quad j = 2, \dots, 2n-2. \end{gathered}$$

3. Real hypersurfaces with η-parallel Ricci tensor

In this section, we consider the additional condition that the Ricci operator S is η-parallel, that is,

$$g((\nabla_X S)Y, Z) = 0$$

for any vector fields X, Y and Z orthogonal to ξ. This is equivalent to the condition that $(\nabla_X S)Y$ is proportional to ξ [Suh 1990].

Theorem 3.1. *Let M be a real hypersurface of $M^n(c)$, $c \neq 0$, with η-parallel Ricci tensor. If the Ricci tensor S of M satisfies $S\xi = \beta\xi$ for some function β, then M is a Hopf hypersurface.*

Before proving Theorem 3.1, we need the following lemma.

Lemma 3.2. *Let M be a real hypersurface of $M^n(c)$, $c \neq 0$, with η-parallel Ricci tensor. If the Ricci tensor S of M satisfies $S\xi = \beta\xi$ for some function β, then we have*

$$\begin{aligned} g((R(W, X)S)Y, Z) = &-g(S\phi AX, Z)g(\phi AW, Y) - g(S\phi AX, Y)g(\phi AW, Z) \\ &+ g(S\phi AW, Z)g(\phi AX, Y) + g(S\phi AW, Y)g(\phi AX, Z) \\ &- g((\nabla_\xi S)Y, Z)g((\phi A + A\phi)X, W) \end{aligned}$$

for any X, Y, Z and W orthogonal to ξ.

Proof. Since S is η-parallel, we have

$$\begin{aligned} &g((R(W, X)S)Y, Z) \\ &\quad = g(R(W, X)SY, Z) - g(R(W, X)Y, SZ) \\ &\quad = g(\nabla_W\nabla_X SY - \nabla_X\nabla_W SY - \nabla_{[W,X]}SY, Z) \\ &\qquad - g(\nabla_W\nabla_X Y - \nabla_X\nabla_W Y - \nabla_{[W,X]}Y, SZ) \\ &\quad = -g((\nabla_X S)Y, \nabla_W Z) + g(\nabla_W(S\nabla_X Y), Z) + g((\nabla_W S)Y, \nabla_X Z) \\ &\qquad - g(\nabla_X(S\nabla_W Y), Z) - g((\nabla_{[W,X]}S)Y, Z) - g(\nabla_W\nabla_Y, SZ) \\ &\qquad + g(\nabla_X\nabla_W Y, SZ) \\ &\quad = -g((\nabla_X S)Y, \xi)g(\xi, \nabla_W Z) + g((\nabla_W S)\nabla_X Y, Z) \\ &\qquad + g((\nabla_W S)Y, \xi)g(\xi, \nabla_X Z) - g((\nabla_X S)\nabla_W Y, Z) \\ &\qquad - g((\nabla_\xi S)Y, Z)g(\xi, [W, X]) \\ &\quad = -g(S\phi AX, Y)g(\phi AW, Z) + g(S\phi AW, Z)g(\phi AX, Y) \\ &\qquad + g(S\phi AW, Y)g(\phi AX, Z) - g(S\phi AX, Z)g(\phi AW, Y) \\ &\qquad - g((\nabla_\xi S)Y, Z)g((\phi A + A\phi)X, W). \end{aligned}$$

□

From Lemma 3.2 we obtain the following:

Lemma 3.3. *Let M be a real hypersurface of $M^n(c)$, $c \neq 0$, with η-parallel Ricci tensor. Suppose that the Ricci tensor S of M satisfies $S\xi = \beta\xi$ for some function β. If $SY = \lambda Y$ and if Y is orthogonal to ξ, then we have*

$$g((\nabla_\xi S)Y, Y)g((\phi A + A\phi)X, W) = 0$$

for any X, Y and W orthogonal to ξ.

Proof of Theorem 3.1.

In the following, we suppose that M is not a Hopf hypersurface. We work in an open set where $h_1 \neq 0$.

Case (I): First we consider the case $g((\nabla_\xi S)Y, Y) = 0$.

Lemma 3.4. *$\beta, \lambda_1, \dots, \lambda_{2n-2}$ are constant.*

Proof. Since the Ricci tensor S is η-parallel and since $g((\nabla_\xi S)Y, Y) = 0$, we have

$$0 = g((\nabla_Z S)Y, Y) = g(\nabla_Z SY, Y) - g(S\nabla_Z Y, Y) = Z\lambda$$

for any tangent vector field Z. So we see that $\lambda_1, \dots, \lambda_{2n-2}$ are constant. On the other hand, since $\beta = \lambda_1 - 3c$, we see that β is also constant. □

Lemma 3.5. *If $\lambda_i \neq \lambda_j$, $i, j = 1, \dots, 2n-2$, then we have $g(\nabla_X e_i, e_j) = 0$ for any X orthogonal to ξ.*

Proof. Since we have $Se_i = \lambda_i e_i$ and $Se_j = \lambda_j e_j$ and since S is η-parallel, we obtain

$$0 = g((\nabla_X S)e_i, e_j) = (\lambda_i - \lambda_j)g(\nabla_X e_i, e_j). \qquad \square$$

If $\lambda_1 = \cdots = \lambda_{2n-2} = \lambda$, then M is pseudo-Einstein, i.e., $SX = \lambda X + (\beta - \lambda)\eta(X)\xi$, and so it is a Hopf hypersurface (see [Kon 1979]).

Suppose that M is non-Hopf and that there exist λ_t and λ_j, $t, j \geq 2$, satisfying $\lambda_1 \neq \lambda_t$ and $\lambda_t \neq \lambda_j$. By Lemma 3.5,

$$\begin{aligned} g(\nabla_j \nabla_t e_t, e_j) &= -g(\nabla_{e_t} e_t, \nabla_{e_j} e_j) \\ &= -g(\nabla_{e_t} e_t, \xi)(\xi, \nabla_{e_j} e_j) - \sum_k g(\nabla_{e_t} e_t, e_k)g(e_k, \nabla_{e_j} e_j) \\ &= -g(e_t, \phi A e_t)g(\phi A e_j, e_j) = 0, \end{aligned}$$

$$\begin{aligned} g(\nabla_t \nabla_j e_t, e_j) &= -g(\nabla_{e_j} e_t, \nabla_{e_t} e_j) = -g(\nabla_{e_j} e_t, \xi)g(\xi, \nabla_{e_t} e_g) \\ &= -g(e_t, \phi A e_j)g(\phi A e_t, e_j) = -a_j a_t g(e_t, \phi e_j)g(\phi e_t, e_j). \end{aligned}$$

On the other hand, from (8),

$$(a_1 - a_t)g(\nabla_{e_j} e_1, e_t) + (a_t - a_j)g(\nabla_{e_1} e_j, e_t) + a_j h_1 g(\phi e_j, e_t) = 0.$$

From Lemma 3.5, we have $g(\nabla_{e_j} e_1, e_t) = 0$, $g(\nabla_{e_1} e_j, e_t) = 0$. Since $h_1 \neq 0$,

$$a_j g(\phi e_j, e_t) = 0,$$

from which we obtain

$$g(\nabla_{e_t}\nabla_{e_j}e_t, e_j) = 0.$$

Moreover, we have

$$\begin{aligned} g(\nabla_{[e_j,e_t]}e_t, e_j) &= g(\nabla_\xi e_t, e_j)g(\xi, [e_j, e_t]) \\ &= g(\nabla_\xi e_t, e_j)(-g(\phi Ae_j, e_t) + g(\phi Ae_t, e_j)) \\ &= g(\nabla_\xi e_t, e_j)(a_t - a_j)g(\phi e_t, e_j) \\ &= g(\nabla_\xi e_t, e_j)a_t g(\phi e_t, e_j). \end{aligned}$$

Using (12), we see that

$$(c + a_j\alpha - a_j a_t)g(\phi e_j, e_t) + h_1 g(\nabla_{e_j}e_1, e_t) + (a_t - a_j)g(\nabla_\xi e_j, e_t) = 0.$$

From these equations, we obtain

$$cg(\phi e_j, e_t)^2 + a_t g(\phi e_j, e_t)g(\nabla_\xi e_j, e_t) = 0.$$

Hence we have

$$g(\nabla_{[e_j,e_t]}e_t, e_j) = -cg(\phi e_j, e_t)^2.$$

Therefore,

$$g(R(e_j, e_t)e_t, e_j) = cg(\phi e_j, e_t)^2.$$

On the other hand, the equation of Gauss implies

$$g(R(e_j, e_t)e_t, e_j) = c + 3cg(\phi e_j, e_t)^2 + a_t a_j.$$

From these equations, we have

$$c(1 + 2g(\phi e_j, e_t)^2) + a_t a_j = 0.$$

Sine $c \neq 0$, we see that $a_t \neq 0$ and $a_j \neq 0$. Thus $g(\phi e_j, e_t) = 0$ and $c + a_t a_j = 0$. So we can represent A as

$$A = \begin{pmatrix} \alpha & h_1 & & & & & & \\ h_1 & a_1 & & & & & & \\ & & a & & & & & \\ & & & \ddots & & & & \\ & & & & a & & & \\ & & & & & b & & \\ & & & & & & \ddots & \\ & & & & & & & b \end{pmatrix}$$

by setting $a = a_j,\ b = a_t$ and taking a suitable permutation of $\{e_2, \ldots, e_{2n-2}\}$.

Suppose there exist j and t such that $g(\phi e_j, e_1) \neq 0$ and $g(\phi e_t, e_1) \neq 0$. Then ϕe_j and ϕe_t satisfy

$$\phi e_j = \sum_k g(\phi e_j, e_k)e_k + g(\phi e_j, e_1)e_1, \quad Ae_k = ae_k,$$
$$\phi e_t = \sum_l g(\phi e_t, e_l)e_l + g(\phi e_t, e_1)e_1, \quad Ae_l = be_l.$$

So we have

$$0 = g(\phi e_j, \phi e_t) = g(\phi e_j, e_1)g(\phi e_t, e_1),$$

from which we see that $g(\phi e_j, e_1) = 0$ or $g(\phi e_t, e_1) = 0$, and hence $A\phi e_1 = a\phi e_1$ or $A\phi e_1 = b\phi e_1$.

When $A\phi e_1 = a\phi e_1$, we have $A\phi e_t = b\phi e_t$. By (4),

$$(b - a_1)g(\nabla_{e_t}\phi e_t, e_1) - (b - a_1)g(\nabla_{\phi e_t} e_t, e_1) + 2h_1 b g(\phi e_t, \phi e_t) = 0.$$

Thus we obtain $b = 0$, which contradicts $c + ab = 0$ and $c \neq 0$. By a similar computation, the case $A\phi e_1 = b\phi e_1$ does not occur.

Next we consider the case $\lambda_2 = \cdots = \lambda_{2n-2} \neq \lambda_1$. We set $\lambda = \lambda_j,\ j = 2, \ldots, 2n-2$. From Lemma 3.5, we have $g(\nabla_X e_1, e_i) = 0,\ i \geq 2$, for any X orthogonal to ξ.

By (4) and (5),

$$h_1(a_i + a_j)g(\phi e_i, e_j) = 0, \quad (2c - 2a_i a_j + \alpha(a_i + a_j))g(\phi e_i, e_j) = 0.$$

Since a_j satisfies

$$\lambda = (2n+1)c + \operatorname{tr} A \cdot a_j - a_j^2,$$

we can represent A as

$$A = \begin{pmatrix} \alpha & h_1 & & & & & & \\ h_1 & a_1 & & & & & & \\ & & a & & & & & \\ & & & \ddots & & & & \\ & & & & a & & & \\ & & & & & b & & \\ & & & & & & \ddots & \\ & & & & & & & b \end{pmatrix}$$

by taking a suitable permutation of $\{e_2, \ldots, e_{2n-2}\}$.

There exist i and j satisfying $g(\phi e_i, e_j) \neq 0$. Therefore, using $h_1 \neq 0$,

$$a_i + a_j = 0, \quad 2c - 2a_i a_j + \alpha(a_i + a_j) = 0.$$

We notice that $\operatorname{tr} A = a_1 + \alpha$ and $\sum_{j=2}^{2n-2} a_j = ka + lb = 0$, where k and l are the multiplicities of a and b, respectively.

When $a_i = a_j = a$, then we have $a_i + a_j = 2a = 0$. Combining this with the above equations, we obtain $b = 0$ and $c = 0$. This is a contradiction. Similarly, the case $a_i = a_j = b$ does not occur.

Next, when $a_i = a,\ a_j = b$ and $a = b$, we have $a = b = 0$ and $c = 0$. This is a contradiction.

Finally we consider the case $a_i = a,\ a_j = b$ and $a \neq b$. Then we have $a = -b \neq 0$. Since $ka + lb = 0$, we obtain $k = l$. This contradicts the fact that M is an odd-dimensional real hypersurface.

Case (II): Next we consider the case

$$g((\phi A + A\phi)X, W) = 0 \tag{18}$$

for any X and W orthogonal to ξ.

Since $\{\xi, \phi e_1, \dots, \phi e_{2n-2}\}$ is an orthonormal basis of the tangent space, we have

$$\begin{aligned} \mathrm{tr}A &= g(A\xi, \xi) + \sum_{i=1}^{2n-2} g(A\phi e_i, \phi e_i) \\ &= \alpha - \sum_{i=1}^{2n-2} g(\phi A e_i, \phi e_i) = \alpha - \sum_{i=1}^{2n-2} g(Ae_i, e_i). \end{aligned}$$

Since $\mathrm{tr}\, A = \alpha + \sum_{i=1}^{2n-2} g(Ae_i, e_i)$, we obtain $\sum_{i=1}^{2n-2} g(Ae_i, e_i) = 0$ and $\mathrm{tr}\, A = \alpha$. On the other hand, from $\mathrm{tr}A = a_1 + \alpha$, we have $a_1 = 0$. Substituting $X = e_1$ in (18), we see that $g(A\phi e_1, W) = 0$ for any W orthogonal to ξ. Since

$$g(A\phi e_1, \xi) = g(\phi e_1, A\xi) = 0,$$

we have $A\phi e_1 = 0$. Without loss of generality, we can set $\phi e_1 = e_2$. From (13) and (17), we obtain

$$(e_2 h_1) = c + h_1^2, \tag{19}$$

$$(c - h_1^2) + h_1 g(\nabla_{e_1} e_2, e_1) = 0. \tag{20}$$

On the other hand, since S is η-parallel, putting $X = Y = e_1$ and $Z = e_2$ into (2), we have

$$0 = \mathrm{tr}A g((\nabla_{e_1} A)e_1, e_2) - g((\nabla_{e_1} A)Ae_1, e_2) = h_1^2 g(e_1, \nabla_{e_1} e_2).$$

Since $h_1 \neq 0$, we have $g(\nabla_{e_1} e_2, e_1) = 0$. Combining this with (20), we see that $h_1^2 = c$. This contradicts (19), finishing the proof. □

We remark that Suh [1990] and Maeda [2013] classified Hopf hypersurfaces of nonflat complex space forms with η-parallel Ricci tensor.

4. Ruled real hypersurfaces

In the previous sections, under the condition that the Ricci tensor S of M satisfies $S\xi = \beta\xi$, we gave sufficient conditions for M to be a Hopf hypersurface with respect to the covariant derivative of the Ricci tensor of S. The purpose of this section is to give a condition on the Ricci tensor for M to be a ruled real hypersurface.

Theorem 4.1. *Let M be a real hypersurface of $M^n(c)$, $c \neq 0$. If the Ricci tensor S of M satisfies $S\xi = \beta\xi$ for some function β and if $g((\nabla_X S)Y, \xi) = 0$ for any vector fields X and Y orthogonal to ξ, then M is a ruled real hypersurface.*

Proof. To prove Theorem 4.1, we need the following proposition:

Proposition 4.2. *Let M be a real hypersurface of $M^n(c)$, $c \neq 0$. If the Ricci tensor S of M satisfies $S\xi = \beta\xi$ for some function β and if $g((\nabla_X S)Y, \xi) = 0$ for any vector fields X and Y orthogonal to ξ, then M is not Hopf.*

Proof. Suppose that M is a Hopf hypersurface. Then we have $A\xi = \alpha\xi$, and hence $S\xi = \beta\xi$. We note that α is constant. Therefore, we have

$$\begin{aligned} g((\nabla_X S)Y, \xi) &= g((\nabla_X S)\xi, Y) \\ &= g(\nabla_X S\xi, Y) - g(S\phi AX, Y) \\ &= \beta g(\phi AX, Y) - g(\phi AX, SY) \end{aligned}$$

for any X and Y orthogonal to ξ. We take an orthonormal basis $\{\xi, e_1, \dots, e_{2n-2}\}$ that satisfies $e_{2i} = \phi e_{2i-1}$, $i = 1, \dots, n-1$, and set $Ae_t = a_t e_t$, $t = 1, \dots, 2n-2$. Then we have $A\phi e_t = \overline{a}_t \phi e_t$ since M is Hopf. Then the Ricci operator S satisfies $S\xi = \beta\xi$ and $Se_t = \lambda_t e_t$, $t = 1, \dots, 2n-2$, where

$$\beta = (2n-2)c + \operatorname{tr} A \cdot \alpha - \alpha^2, \quad \lambda_t = (2n+1)c + \operatorname{tr} A \cdot a_t - a_t^2.$$

Thus we obtain

$$0 = (\beta - \lambda_t) g(\phi AX, e_t) = -(\beta - \lambda_t) g(X, A\phi e_t)$$

for any X orthogonal to ξ. Since $A\xi = \alpha\xi$, we have $g(A\phi e_t, \xi) = 0$. From these equations, we have:

Lemma 4.3. *If $\beta \neq \lambda_t$, then $A\phi e_t = 0$, that is, $\bar{a}_t = 0$.*

We suppose $\beta \neq \lambda_t$. Then, from (1), we have

$$\overline{\lambda_t} = g(S\phi e_t, \phi e_t) = (2n+1)c.$$

Using Proposition A and $c \neq 0$, we have $\alpha \neq 0$ and

$$a_t = -\frac{2c}{\alpha}.$$

If $\beta \neq \lambda_t$ and $\beta \neq \overline{\lambda_t} = g(S\phi e_t, \phi e_t)$, then we have $a_t = \overline{a_t} = 0$. This is a contradiction. Thus we obtain:

Lemma 4.4. *If $\beta \neq \lambda_t$, then $\beta = \overline{\lambda_t} = (2n+1)c$.*

Since M is not Einstein, there exists a t such that $\beta \neq \lambda_t$. So we see that λ_t satisfies $\beta = \lambda_t = \overline{\lambda_t}$ or $\beta = \overline{\lambda_t} \neq \lambda_t$.

When $\beta = \lambda_t = \bar{\lambda}_t$, since $\beta = (2n+1)c$, we have

$$0 = a_t(\operatorname{tr} A - a_t).$$

So we obtain $a_t = 0$ or $a_t = \operatorname{tr} A$. If $a_t = 0$, then $\bar{a}_t = -2c/\alpha$. There exists an s that satisfies $\lambda_s \neq \beta$, and hence $a_s = -2c/\alpha$. Thus we have

$$\beta \neq \lambda_s = (2n+1)c + \operatorname{tr} A\Big(\frac{-2c}{\alpha}\Big) - \Big(-\frac{2c}{\alpha}\Big)^2.$$

Thus $\bar{\lambda}_t = \lambda_s \neq \beta$. This is a contradiction. So we see that $a_t = \operatorname{tr} A \neq 0$. In the following, we set $a = a_t = \operatorname{tr} A$. Since $a_t = \bar{a}_t = \operatorname{tr} A$, we have

$$(2a - \alpha)a = (\alpha a + 2c).$$

Thus a satisfies $a^2 - \alpha a - c = 0$, and hence a turns to be constant. In the following, we set $a_1 = -2c/\alpha$ and $\bar{a}_1 = a_2 = 0$.

Next we compute $g(R(e_1, e_2)e_2, e_1)$. By the equation of Gauss,

$$g(R(e_1, e_2)e_2, e_1) = g(R(e_1, \phi e_1)\phi e_1, e_1) = 4c.$$

Using (7), $a_1 g(\nabla_{e_2} e_1, e_2) = 0$. Since $a_1 \neq 0$, we have $g(\nabla_{e_2} e_2, e_1) = 0$. Moreover,

$$g(\nabla_{e_2} e_2, e_2) = 0, \quad g(\nabla_{e_2} e_2, \xi) = -g(e_2, \phi A e_2) = 0.$$

When $k \geq 3$, by (6),

$$a_k g(\nabla_{e_2} e_2, e_k) = 0.$$

When $a_k \neq 0$, we have $g(\nabla_{e_2} e_2, e_k) = 0$. By (10), $g(\nabla_{e_1} e_1, e_2) = 0$. Moreover,

$$g(\nabla_{e_1} e_1, e_1) = 0, \quad g(\nabla_{e_1} e_1, \xi) = 0.$$

Since $k \geq 3$, by (10) and the fact that a_1 is constant,

$$(a_1 - a_k) g(\nabla_{e_1} e_k, e_1) = 0.$$

By $a_1 \neq 0$, if $a_k = 0$, then $g(\nabla_{e_1} e_1, e_k) = 0$. Thus we have

$$\sum_{k=1}^{2n-2} g(\nabla_{e_1} e_1, e_k) g(e_k, \nabla_{e_2} e_2) = 0.$$

So we have

$$\begin{aligned} g(\nabla_{e_1}\nabla_{e_2}e_2, e_1) &= e_1 g(\nabla_{e_2}e_2, e_1) - g(\nabla_{e_2}e_2, \nabla_{e_1}e_1) \\ &= -\sum_k g(\nabla_{e_2}e_2, e_k) g(e_k, \nabla_{e_1}e_1) = 0, \end{aligned}$$

$$\begin{aligned} g(\nabla_{e_2}\nabla_{e_1}e_2, e_1) &= e_2 g(\nabla_{e_1}e_2, e_1) - g(\nabla_{e_1}e_2, \nabla_{e_2}e_1) = -g(\nabla_{e_1}\phi e_1, \nabla_{e_2}e_1) \\ &= g(\nabla_{e_1}e_1, \phi\nabla_{e_2}e_1) = g(\nabla_{e_1}e_1, \nabla_{e_2}e_2) = 0, \end{aligned}$$

and

$$\begin{aligned} &g(\nabla_{[e_1,e_2]}e_2, e_1) \\ &\quad = g(\nabla_\xi e_2, e_1) g(\xi, [e_1, e_2]) + \sum_{k\geq 3} g(\nabla_k e_2, e_1) g(e_k, [e_1, e_2]) \\ &\quad = -a_1 g(\nabla_\xi e_2, e_1) + \sum_{k\geq 3} g(\nabla_{e_k}e_2, e_1) g(e_k, \nabla_{e_1}e_2) - \sum_{k\geq 3} g(\nabla_{e_k}e_2, e_1) g(e_k, \nabla_{e_2}e_1). \end{aligned}$$

By (13),

$$a_1 g(\nabla_\xi e_2, e_1) = c.$$

Using (4), we have

$$g(\nabla_{e_k}e_2, e_1) = \frac{a_k - a_1}{a_1} g(\nabla_{e_2}e_1, e_k).$$

On the other hand, by (8),

$$g(\nabla_{e_k}e_2, e_1) = \frac{a_k}{a_1} g(\nabla_{e_1}e_2, e_k).$$

So we obtain

$$\begin{aligned} &\sum_{k\geq 3} g(\nabla_{e_k}e_2, e_1)(e_k, \nabla_{e_1}e_2) - \sum_{k\geq 3} g(\nabla_{e_k}e_2, e_1) g(e_k, \nabla_{e_2}e_1) \\ &\qquad = \sum \frac{(a_k - a_1)}{a_1} g(\nabla_{e_2}e_1, e_k) g(e_k, \nabla_{e_1}e_2) - \sum \frac{a_k}{a_1} g(\nabla_{e_1}e_2, e_k)(e_k, \nabla_{e_2}e_1) \\ &\qquad = -\sum g(\nabla_{e_2}e_1, e_k) g(e_k, \nabla_{e_1}e_2) \\ &\qquad = -\sum g(\nabla_{e_2}e_1, \phi e_k) g(\phi e_k, \nabla_{e_1}e_2) \\ &\qquad = \sum g(\nabla_{e_2}e_2, e_k) g(e_k, \nabla_{e_1}e_1) = 0. \end{aligned}$$

Thus we have

$$g(R(e_1, e_2)e_2, e_1) = c,$$

from which we obtain $c = 0$. This is a contradiction. Hence we see that M is not Hopf. Thus we have proven Proposition 4.2. □

From Proposition 4.2, if $g((\nabla_X S)Y, \xi) = 0$ for $X, Y \in H$, then M is not Hopf. In the following, we suppose that M is not Hopf, that is, $h_1 \neq 0$. Then, by Lemma 2.2, we can take an orthonormal basis $\{\xi, e_1, \ldots, e_{2n-2}\}$ such that

$$A\xi = \alpha\xi + h_1 e_1, \quad Ae_1 = a_1 e_1 + h_1 \xi, \quad Ae_j = a_j e_j, \quad j = 2, \ldots, 2n-2, \tag{21}$$
$$\mathrm{tr}A = \alpha + a_1, \quad a_2 + \cdots + a_{2n-2} = 0.$$

Then we have

$$\beta = g(S\xi, \xi) = (2n-2)c + (a_1\alpha - h_1^2),$$
$$\lambda_1 = g(Se_1, e_1) = (2n+1)c + (a_1\alpha - h_1^2),$$
$$\lambda_j = g(Se_j, e_j) = (2n+1)c + \mathrm{tr}\, A \cdot a_j - a_j^2, \quad j \geq 2.$$

By the assumption, for any X and Y orthogonal to ξ,

$$0 = g((\nabla_X S)\xi, Y) = g(\nabla_X S\xi, Y) - g(S\phi AX, Y).$$

We set $SY = \lambda Y$. Then we have

$$0 = (\beta - \lambda) g(\phi AX, Y).$$

Since $\beta \neq \lambda_1$, we see that

$$g(\phi AX, e_1) = -g(AX, \phi e_1) = -g(X, A\phi e_1) = 0$$

for any $X \in H$. We also have $g(\xi, A\phi e_1) = 0$. Thus we have $A\phi e_1 = 0$. In the following, we set $\phi e_1 = e_2$. Then we have

$$0 = (\beta - \lambda_2) g(\phi A e_1, e_2) = (-3c + a_1\alpha - h_1^2)a_1.$$

Lemma 4.5. *If $h_1 \neq 0$, then $a_2 = 0$. Moreover, $a_1 = 0$ or $a_1\alpha - h_1^2 = 3c$.*

Case (I): Suppose $a_1 = 0$.

Since $a_1 = a_2 = 0$, (13) implies

$$(e_2 h_1) = c + h_1^2.$$

If $\beta = (2n+1)c = \lambda_2$, then $h_1^2 = -3c$ and $e_2 h_1 = 0$. Then we have $h_1^2 = -c$ and $c = 0$. This is a contradiction. So we have:

Lemma 4.6. *If $a_1 = 0$, then $\beta \neq (2n+1)c = \lambda_2$.*

For any $X \in H$, we see that

$$(\beta - \lambda_k) g(\phi AX, e_k) = 0, \quad k \geq 3.$$

If $\beta \neq \lambda_k$, then $g(A\phi e_k, X) = 0$, and moreover $g(A\phi e_k, \xi) = 0$. This shows that $A\phi e_k = 0$ and that ϕe_k is a principal vector of A. We set

$$\bar{\lambda}_k = g(S\phi e_k, \phi e_k).$$

Since $a_1\alpha - h_1^2 \neq 3c$, we have $\bar{\lambda}_k = (2n+1)c \neq \beta$. Then, from

$$(\beta - \bar{\lambda}_k)g(\phi AX, \phi e_k) = 0,$$

we have $g(Ae_k, X) = 0$. We also have $g(Ae_k, \xi) = 0$ since $k \geq 3$. Hence we obtain $Ae_k = 0$ for e_k satisfying $\beta \neq \lambda_k$.

We next consider the case $\beta = \lambda_j$ for some $j \geq 3$. If $\beta = \lambda_j = \lambda_i$, then

$$\beta = (2n+1)c + \operatorname{tr} A \cdot a_j - a_j^2 = (2n+1)c + \operatorname{tr} A \cdot a_i - a_i^2.$$

Therefore, at most two a_j are different. By this equation, we have

$$0 = (a_j - a_i)(\operatorname{tr} A - (a_j + a_i)).$$

If $a_j = a_i = a$ for all j and i, then (21) implies $\sum a_j = 0$. Thus we have all $a_j = 0$, $j = 2, \ldots, 2n-2$. Since $a_1 = 0$, M is a ruled real hypersurface.

Let us suppose that two a_j are different. We set

$$T_a = \{X \mid AX = aX,\ X \in H_x\}, \quad T_b = \{X \mid AX = bX,\ X \in H_x\},$$

where $\beta = \lambda_a = \lambda_b$, $a \neq b$. We notice $\operatorname{tr} A = a + b$. If $a = 0$ or $b = 0$, then, by (21), $a = b = 0$. This contradicts the assumption that $a \neq b$. So we obtain $a \neq 0$ and $b \neq 0$. We notice that $\dim T_a + \dim T_b$ is even number.

Let $e_i, e_j \in T_a$. By (8) and (12),

$$-ag(\nabla_{e_i} e_1, e_j) + ah_1 g(\phi e_i, e_j) = 0,$$
$$(c + a\alpha - a^2)g(\phi e_i, e_j) + h_1 g(\nabla_{e_i} e_1, e_j) = 0.$$

From these, we obtain

$$(c + a\alpha - a^2 + h_1^2)g(\phi e_i, e_j) = 0.$$

If there exist e_i and e_j such that $g(\phi e_i, e_j) \neq 0$, then

$$c + a\alpha - a^2 + h_1^2 = 0.$$

On the other hand, we have

$$\beta = (2n-2)c - h_1^2 = (2n+1)c + \operatorname{tr} A \cdot a - a^2.$$

Since $\operatorname{tr} A = \alpha + a_1 = \alpha$, we have

$$3c + \alpha a - a^2 + h_1^2 = 0.$$

Therefore, we have $2c = 0$. This contradicts $c \neq 0$. Hence $g(\phi e_i, e_j) = 0$ for all e_i and e_j of T_a. So we have $\phi T_a \subset T_b$. Similarly, we also have $\phi T_b \subset T_a$. Consequently, we see that

$$\phi T_a = T_b, \quad \phi T_b = T_a.$$

If $\dim T_a = \dim T_b = 1$, then $\phi T_a = T_b$. We see that if $Ae_j = ae_j$, then $A\phi e_j = b\phi e_j$ and $a+b = \operatorname{tr} A$. From (21), we have $a+b=0$ and $\operatorname{tr} A = 0$. Therefore, we obtain $\operatorname{tr} A = \alpha = 0$.

We will prove that there is no real hypersurface that satisfies

$$a+b=0, \quad \alpha=0, \quad a_1=0, \quad a_2=0, \quad \operatorname{tr}A=0,$$

and also

$$a^2 - h_1^2 = 3c.$$

By (5),

$$(2c+2a^2)g(\phi e_i,\phi e_i) - h_1 g(\nabla_{e_i}\phi e_i, e_1) + h_1 g(\nabla_{\phi e_i} e_i, e_1) = 0. \tag{22}$$

On the other hand, we have

$$g(\nabla_{e_i}\phi e_i, e_1) = g(\phi\nabla_{e_i} e_i, e_1) = -g(\nabla_{e_i} e_i, e_2).$$

By (6),

$$(a_2 - a_i)g(\nabla_{e_i} e_2, e_i) - (e_2 a_i) = 0.$$

Using $a_2 = 0$ and $a_i = a$, we obtain

$$ag(\nabla_{e_i} e_i, e_2) = (e_2 a).$$

From this equation and $a \neq 0$, we have

$$g(\nabla_{e_i} e_i, e_2) = \frac{(e_2 a)}{a}.$$

On the other hand,

$$g(\nabla_{\phi e_i} e_i, e_1) = g(\phi\nabla_{\phi e_i} e_i, \phi e_1) = g(\nabla_{\phi e_i}\phi e_i, e_2).$$

By (6), we obtain

$$(a_2 + a)g(\nabla_{\phi e_i} e_2, \phi e_i) + (e_2 a) = 0,$$

and hence

$$g(\nabla_{\phi e_i}\phi e_i, e_2) = \frac{(e_2 a)}{a}.$$

Substituting these equations into (22), we get

$$2(c+a^2) + h_1\frac{(e_2 a)}{a} + h_1\frac{(e_2 a)}{a} = 0.$$

Thus we have

$$(c+a^2)a = -h_1(e_2 a). \tag{23}$$

On the other hand, since $a^2 - h_1^2 = 3c$,

$$a(e_2 a) = h_1(e_2 h_1).$$

Since $a_1 = a_2 = 0$, by (13), we have

$$e_2 h_1 = c + h_1^2,$$

from which we obtain

$$e_2 a = \frac{h_1}{a}(c + h_1^2).$$

Substituting this into (23), we get

$$(c + a^2)a = -\frac{h_1^2}{a}(c + h_1^2) = -\frac{1}{a}(a^2 - 3c)(a^2 - 2c).$$

Thus we obtain

$$(a^2 - c)^2 + 2c^2 = 0.$$

So we have $c = 0$. This is a contradiction. Consequently, if $a_1 = 0$, then M is a ruled real hypersurface.

Case (II): Suppose $a_1 \neq 0$.

We notice that $a_2 = 0$ and $\alpha a_1 h_1^2 = 3c$ by Lemma 4.5. So we have

$$(Xa_1)\alpha + a_1(X\alpha) - 2h_1(Xh_1) = 0 \tag{24}$$

for any tangent vector field X.

Lemma 4.7. $\nabla_{e_1} e_1$ *and* $\nabla_{e_2} e_2$ *are perpendicular to* ξ, e_1 *and* e_2.

Proof. By (14),

$$(e_2\alpha) = \alpha h_1 + h_1 g(\nabla_\xi e_1, e_2).$$

By (10),

$$(e_2 a_1) = a_1 h_1 + a_1 g(\nabla_{e_1} e_1, e_2).$$

Substituting these into (24), we get

$$2a_1\alpha h_1 + \alpha a_1 g(\nabla_{e_1} e_1, e_2) + a_1 h_1 g(\nabla_\xi e_1, e_2) - 2h_1(e_2 h_1) = 0.$$

By (9) and (13),

$$\begin{aligned}
(e_2 h_1) &= (2c + \alpha a_1) + h_1 g(\nabla_{e_1} e_1, e_2) = (5c + h_1^2) + h_1 g(\nabla_{e_1} e_1, e_2),\\
(e_2 h_1) &= (c + h_1^2) + a_1 g(\nabla_\xi e_1, e_2).
\end{aligned}$$

From these equations and (24), we have

$$2h_1(a_1\alpha - h_1^2 - 3c) + (a_1\alpha - h_1^2) g(\nabla_{e_1} e_1, e_2) = 0.$$

Since $a_1\alpha - h_1^2 = 3c$, we have

$$g(\nabla_{e_1} e_1, e_2) = 0.$$

By (7), $a_1 \neq 0$ and $a_2 = 0$,

$$g(\nabla_{e_2} e_2, e_1) = 0.$$

Moreover, we have

$$g(\nabla_{e_2} e_2, \xi) = -g(e_2, \phi A e_2) = 0, \quad g(\nabla_{e_1} e_1, \xi) = -g(e_1, \phi A e_1) = 0.$$

These equations prove our lemma. □

Lemma 4.8. *Suppose $j \geq 3$. If $a_j = 0$, then $g(\nabla_{e_1} e_1, e_j) = 0$. If $a_j \neq 0$, then $g(\nabla_{e_2} e_2, e_j) = 0$.*

Proof. By (6), we have

$$a_j g(\nabla_{e_2} e_2, e_j) = 0, \quad j \geq 3.$$

If $a_j \neq 0$, then $g(\nabla_{e_2} e_2, e_j) = 0$ for $j \geq 3$. Suppose $a_j = 0,\ j \geq 3$. Then, by (10), (14), (9) and (13),

$$(e_j a_1) = a_1 g(\nabla_{e_1} e_1, e_j), \quad (e_j \alpha) = h_1 g(\nabla_\xi e_1, e_j),$$
$$(e_j h_1) = h_1 g(\nabla_{e_1} e_1, e_j), \quad (e_j h_1) = a_1 g(\nabla_\xi e_1, e_j).$$

Substituting these into (24), we get

$$\begin{aligned} 0 &= (e_j a_1)\alpha + a_1(e_j \alpha) - 2h_1(e_j h_1) \\ &= \alpha a_1 g(\nabla_{e_1} e_1, e_j) + a_1 h_1 g(\nabla_\xi e_1, e_j) - h_1^2 g(\nabla_{e_1} e_1, e_j) - h_1 a_1 g(\nabla_\xi e_1, e_j) \\ &= (\alpha a_1 - h_1^2) g(\nabla_{e_1} e_1, e_j). \end{aligned}$$

Since $a_1\alpha - h_1^2 = 3c$, we have our lemma. □

Using these lemmas, we compute $g(R(e_1, e_2)e_2, e_1)$. We note that $e_2 = \phi e_1$ and $a_2 = 0$. First, we have

$$\begin{aligned} g(\nabla_{e_1}\nabla_{e_2} e_2, e_1) &= e_1 g(\nabla_{e_2} e_2, e_1) - g(\nabla_{e_2} e_2, \nabla_{e_1} e_1) \\ &= -g(\nabla_{e_2} e_2, \xi) g(\xi, \nabla_{e_1} e_1) - g(\nabla_{e_2} e_2, e_1) g(e_1, \nabla_{e_1} e_1) \\ &\quad - g(\nabla_{e_2} e_2, e_2) g(e_2, \nabla_{e_1} e_1) - \sum_{k \geq 3} g(\nabla_{e_2} e_2, e_j) g(e_j, \nabla_{e_1} e_1) = 0. \end{aligned}$$

Next, we have

$$
\begin{aligned}
g(&\nabla_{e_2}\nabla_{e_1}e_2, e_1)\\
&= e_2 g(\nabla_{e_1}e_2, e_1) - g(\nabla_{e_1}e_2, \nabla_{e_2}e_1)\\
&= -g(\nabla_{e_1}e_2, \xi)g(\xi, \nabla_{e_2}e_1) - g(\nabla_{e_1}e_2, e_1)g(e_1, \nabla_{e_2}e_1)\\
&\qquad - g(\nabla_{e_1}e_2, \xi)g(\xi, \nabla_{e_2}e_1) - \sum_{k\geq 3} g(\nabla_{e_1}e_2, e_k)g(e_k, \nabla_{e_2}e_1)\\
&= -\sum_{k\geq 3} g(\nabla_{e_1}e_2, e_k)g(e_k, \nabla_{e_2}e_1) = -\sum_{k\geq 3} g(\nabla_{e_1}\phi e_1, e_k)g(\phi e_k, \phi\nabla_{e_2}e_1)\\
&= \sum_{k\geq 3} g(\nabla_{e_1}e_1, \phi e_k)g(\phi e_k, \nabla_{e_2}e_2) = \sum_{l\geq 3} g(\nabla_{e_1}e_1, e_l)g(e_l, \nabla_{e_2}e_2) = 0.
\end{aligned}
$$

Moreover, we obtain

$$
\begin{aligned}
g(\nabla_{[e_1,e_2]}e_2, e_1) &= g(\nabla_\xi e_2, e_1)g(\xi, [e_1, e_2]) + g(\nabla_{e_1}e_2, e_1)g(e_1, [e_1, e_2])\\
&\qquad + g(\nabla_{e_2}e_2, e_1)g(e_2.[e_1, e_2]) + \sum_{k\geq 3} g(\nabla_{e_k}e_2, e_1)g(e_k, [e_1, e_2])\\
&= g(\nabla_\xi e_2, e_1)g(\xi, \nabla_{e_1}e_2)\\
&\qquad + \sum_{k\geq 3}(g(\nabla_{e_k}e_2, e_1)g(e_k, \nabla_{e_1}e_2) - g(\nabla_{e_k}e_2, e_1)g(e_k, \nabla_{e_2}e_1)).
\end{aligned}
$$

On the other hand, by (8), when $j \geq 3$,

$$
\begin{aligned}
a_1 g(\nabla_{e_j}e_2, e_1) - a_j g(\nabla_{e_1}e_2, e_j) = 0,\\
(a_1 - a_j)g(\nabla_{e_2}e_1, e_j) + a_j g(\nabla_{e_1}e_2, e_j) = 0.
\end{aligned}
$$

Thus, if $a_1 = a_j$, then we see that $a_j \neq 0$ and hence $g(\nabla_{e_1}e_2, e_j) = 0$ since $a_1 \neq 0$. Next, when $a_1 \neq a_j$ we have

$$
g(\nabla_{e_2}e_1, e_j) = -\frac{a_j}{(a_1 - a_j)} g(\nabla_{e_1}e_2, e_j).
$$

On the other hand,

$$
g(\nabla_{e_j}e_2, e_1) = \frac{a_j}{a_1} g(\nabla_{e_1}e_2, e_j) = -\frac{(a_1 - a_j)}{a_1} g(\nabla_{e_2}e_1, e_j).
$$

So we have

$$
\begin{aligned}
\sum_{k\geq 3}&(g(\nabla_{e_k}e_2, e_1)g(e_k, \nabla_{e_1}e_2) - g(\nabla_{e_k}e_2, e_1)g(e_k, \nabla_{e_2}e_1)\\
&= -\sum_{k\geq 3} g(\nabla_{e_2}e_1, e_k)g(e_k, \nabla_{e_1}e_2) = -\sum_{k\geq 3} g(\phi\nabla_{e_2}e_1, e_k)g(\phi e_k, \nabla_{e_1}e_2)\\
&= \sum_{l\geq 3} g(\nabla_{e_1}e_1, e_l)g(e_l, \nabla_{e_2}e_2) = 0.
\end{aligned}
$$

Thus we obtain

$$\begin{aligned} g(\nabla_{[e_1,e_2]}e_2, e_1) &= g(\nabla_\xi e_2, e_1)g(\xi, \nabla_{e_1}e_2) \\ &= -g(\nabla_\xi e_2, e_1)g(\phi A e_1, e_2) = -a_1 g(\nabla_\xi e_2, e_1), \end{aligned}$$

and so

$$g(R(e_1, e_2)e_2, e_1) = a_1 g(\nabla_\xi e_2, e_1).$$

On the other hand, by (9),

$$-(2c + \alpha a_1) + h_1 g(\nabla_{e_1} e_2, e_1) + (e_2 h_1) = 0.$$

Using Lemma 4.7 and $a_1\alpha - h_1^2 = 3c$, we have

$$(e_2 h_1) = 2c + \alpha a_1 = 5c + h_1^2.$$

By (13),

$$-(c + h_1^2) + a_1 g(\nabla_\xi e_2, e_1) + e_2 h_1 = 0,$$

from which we obtain

$$a_1 g(\nabla_\xi e_2, e_1) = -4c,$$

and so

$$g(R(e_1, e_2)e_2, e_1) = -4c.$$

On the other hand, the equation of Gauss implies

$$g(R(e_1, e_2)e_2, e_1) = 4c,$$

and hence $c = 0$. This is a contradiction.

Consequently, M is a ruled real hypersurface.

From (2), any ruled real hypersurface satisfies $g((\nabla_X S)Y, \xi) = 0$ for any X and Y orthogonal to ξ, and $S\xi = \beta\xi$ for some function β. □

From Theorems 3.1 and 4.1, we have Theorem 1.1.

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