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RICCION FROM HIGHER-DIMENSIONAL SPACE-TIME AND ONE-LOOP RENORMALIZATION

S.K. Srivastava
Department of Mathematics, North Eastern Hill University,
Shillong - 793022, India[1]
and
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

Using higher-dimensional gravity in (4+D)-dimensional space-time, lagrangian density of riccion is obtained with the quartic self-interacting potential. It is found that after compactification to 4-dimensional space-time the resulting theory is one-loop multiplicatively renormalizable. Renormalization group equations are solved and their solutions yield many interesting results such as (i) dependence of extra dimensions on the energy mass scale showing that these dimensions increase with the increasing energy mass scale such that $D = 6$ at extremely high energy, (ii) phase transition at 2.82×10^{16}GeV and (iii) dependence of gravitational as well as other coupling constants on energy scale. Results also suggest that space-time above 2.82×10^{16}GeV should be fractal.

MIRAMARE – TRIESTE

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[1]Permanent address.

1 Introduction

Higher-derivative gravity is an important candidate from the last many years. It obeys the principle of covariance as well as the principle of equivalence which are basic principles of the general relativity. While quantizing gravity, this theory has problems at the perturbation level where ghost terms appear in the Feynman propagator of graviton [1].

Recently a different feature of higher-derivative gravity has been noted. The present paper deals with the new feature of this theory. In refs. [2-5], it has been discussed that Ricci scalar, which is a geometrical field, also behaves like a matter field at high energy scales. Here dual roles of Ricci scalar R (like a matter field as well as a geometrical field) are exploited. The matter aspect of R is represented by a scalar field $\tilde{R} = \eta R$ (where η has length dimension and unit magnitude in natural units defined below). In quantum field theory, fields are treated as physical concepts describing particles. After the name of the great mathematician Ricci, a particle described by $\tilde{R}$ is called riccion.

In the earlier work [2-5], riccions are obtained from the 4-dim. space-time geometry which has been discussed in Appendix A. In what follows, riccions are obtained from the higher-dimensional geometry with topology $M^4 \otimes T^D$ (M^4 is the 4-dim. space-time with signature (+,-,-,-) and T^D is D-dimensional torus which is an extra-dimensional space). The distance function is defined as

$$dS^2 = g_{\mu\nu} dx^\mu dx^\nu - \rho_1^2 d\theta_1^2 - \rho_2^2 d\theta_2^2 - \cdots - \rho_D^2 d\theta_D^2, \tag{1.1}$$

where $g_{\mu\nu}(\mu, \nu = 0, 1, 2, 3)$ are components of the metric tensor, $\rho_i (i = 1, 2, \cdots, D)$ are radii of circle components of T^D and $0 \le \theta_1, \theta_2, \cdots, \theta_D \le 2\pi$.

The paper is organized as follows. In section 2, taking the action for higher-derivative gravity in (4+D)-dimensional space-time, action for the riccion is obtained. Section 3 contains one-loop quantum correction to riccion, calculation of counter-terms and renormalization. Renormalization group equations are obtained and solved in section 4. Section 5 is the concluding section where results are discussed.

Natural units are defined as $\kappa_B = \hbar = c = 1$ (where κ_B is Boltzman's constant, $\hbar$ is Planck's constant divided by 2π and c is the speed of light), which are used throughout the paper.

2 Riccions from (4+D)-dimensional geometry

The action for the higher-derivative gravity in (4+D)-dim. space-time is taken as

$$S_g^{(4+D)} = \int d^4x d^D y \sqrt{-g_{(4+D)}} \quad \Big[\frac{R_{(4+D)}}{16\pi G_{(4+D)}} + \alpha_{(4+D)} R^2_{(4+D)} + \gamma_{(4+D)} R^3_{(4+D)}\Big], \tag{2.1a}$$

where $G_{(4+D)} = G V_D, \quad \alpha_{(4+D)} = \alpha V_D^{-1}, \quad \gamma_{(4+D)} = \frac{\eta^2}{3!(D-2)} V_D^{-1}$,

$$V_D = (2\pi)^D \rho_1 \rho_2 \cdots \rho_D$$

and $g_{(4+D)}$ is the determinant of the metric tensor $g_{MN}(M, N = 0, 1, 2, \cdots, (4+D))$. Here $G_{(4+D)}$ is the (4+D)-dim. gravitational constant and $\alpha_{(4+D)}$ as well as $\gamma_{(4+D)}$ are coupling constants. α is a dimensionless coupling constant, $R_{(4+D)}$ is (4+D)-dim. Ricci scalar and G is the 4-dim. gravitational constant.

It is important to mention here that higher-derivative terms in the action given by eq.(2.1a) are significant at the energy mass scale given by

$$M^2 \geq \Big[\frac{2\eta^2}{3!(D-2)}\Big]^{-1}\Big[-\alpha \pm \sqrt{\alpha^2 - \frac{1}{24\pi G(D-2)}}\Big]. \qquad (2.1b)$$

M is obtained using the method described in Appendix A.

Invariance of $S_g^{(4+D)}$ under transformations $g_{MN} \to g_{MN} + \delta g_{MN}$ yields

$$(16\pi G_{(4+D)})^{-1}(R_{MN} - \frac{1}{2}g_{MN}R_{(4+D)}) + \alpha_{(4+D)}H^1_{MN} + \gamma_{(4+D)}H^2_{MN} = 0, \qquad (2.2a)$$

where

$$H^1_{MN} = 2R_{;MN} - 2g_{MN}\Box_{(4+D)}R_{(4+D)} - \frac{1}{2}g_{MN}R^2_{(4+D)} + 2R_{(4+D)}R_{MN}, \qquad (2.2b)$$

and

$$H^2_{MN} = 3R^2_{;MN} - 3g_{MN}\Box_{(4+D)}R^2_{(4+D)} - \frac{1}{2}g_{MN}R^3_{(4+D)} + 3R^2_{(4+D)}R_{MN} \qquad (2.2c)$$

with semi-colon (;) denoting curved space covariant derivative and

$$\Box_{(4+D)} = \frac{1}{\sqrt{-g_{(4+D)}}}\frac{\partial}{\partial x^M}\Big(\sqrt{-g_{(4+D)}} \quad g^{MN}\frac{\partial}{\partial x^N}\Big)$$

Trace of these field equations is obtained as

$$\frac{(D+2)}{(32\pi G_{(4+D)})}R_{(4+D)} + \alpha_{(4+D)}[2(D+3)\Box_{(4+D)}R_{(4+D)} + \frac{1}{2}DR^2_{(4+D)}]$$
$$+\gamma_{(4+D)}[3(D+3)\Box_{(4+D)}R^2_{(4+D)} + \frac{1}{2}(D-2)R^3_{(4+D)}] = 0. \qquad (2.3)$$

In the space-time described by the distance function defined in eq.(1.1),

$$R_{(4+D)} = R \quad \text{and} \quad \Box = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\Big(\sqrt{-g} \quad g^{\mu\nu}\frac{\partial}{\partial x^\nu}\Big), \qquad (2.4a, b)$$

where R is the 4-dim. Ricci scalar.

Connecting eqs.(2.3) and (2.4) as well as using $G_{(4+D)}, \alpha_{(4+D)}$ and $\gamma_{(4+D)}$ from eq.(2.1), one obtains

$$\frac{(D+2)}{(32\pi G)}R + \alpha[2(D+3)\Box R + \frac{1}{2}DR^2] + \frac{1}{3!(D-2)}\eta^2[3(D+3)\Box R^2 + \frac{1}{2}(D-2)R^3] = 0. \qquad (2.5)$$

Analogous to the Lorentz gauge, a gauge condition

$$\Box R^2 = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\Big(\sqrt{-g} \quad A^\mu\Big) = 0 \qquad (2.6)$$

with $A^\mu = g^{\mu\nu}\frac{\partial R^2}{\partial x^\nu}$ is used in eq.(2.5). As a result, eq.(2.5) is re-written as

$$\frac{(D+2)}{(32\pi G)}R + \alpha[2(D+3)\Box R + \frac{1}{2}DR^2] + \frac{1}{12}\eta^2 R^3 = 0. \tag{2.7}$$

Now multiplying eq.(2.7) by η and using $\tilde{R} = \eta R$, it is obtained that

$$\Box\tilde{R} + \xi R\tilde{R} + m^2\tilde{R} = -\frac{\lambda}{3!}\tilde{R}^3, \tag{2.8a}$$

where

$$\begin{aligned}\xi &= \frac{D}{4(D+3)} \\ m^2 &= \frac{(D+2)}{64\pi G\alpha(D+3)} \\ \lambda &= \frac{1}{4(D+3)\alpha}.\end{aligned} \tag{2.8b, c, d}$$

To exploit the matter aspect of the 4-dimensional Ricci scalar R obtained from the higher-dimensional geometry, $\tilde{R}$ is treated as a basic physical field, because it behaves like a matter field representing the matter aspect of R (see Appendix A for details) . As a result, the lagrangian density leading to eq.(2.8) is written as

$$L = \frac{1}{2}(g^{\mu\nu}\partial_\mu\tilde{R}\partial_\nu\tilde{R} - \xi R\tilde{R} - m^2\tilde{R}^2) - \frac{\lambda}{4!}\tilde{R}^4 \tag{2.9}$$

with the action $S_{\tilde{R}} = \int d^4xL$. For a further check, one finds that invariance of this action under transformation $\tilde{R} \to \tilde{R} + \delta\tilde{R}$ also yields eq.(2.8).

3 One-loop quantum correction and renormalization

The $S_{\tilde{R}}$ with the lagrangian density, given by eq.(2.9), can be expanded around the classical minimum $\tilde{R}_0$ in powers of quantum fluctuation $\tilde{R}_q = \tilde{R} - \tilde{R}_0$ as

$$S_{\tilde{R}} = S_{\tilde{R}}^{(0)} + S_{\tilde{R}}^{(1)} + S_{\tilde{R}}^{(2)} + \cdots,$$

where

$$\begin{aligned}S_{\tilde{R}}^{(0)} &= \int d^4x\sqrt{-g} \quad [\frac{1}{2}(g^{\mu\nu}\partial_\mu\tilde{R}_{(0)}\partial_\nu\tilde{R}_0 - \xi R\tilde{R}_0 - m^2\tilde{R}_0^2) - \frac{\lambda}{4!}\tilde{R}_0^4], \\ S_{\tilde{R}}^{(2)} &= -\frac{1}{2}\int d^4x\sqrt{-g} \quad \tilde{R}_q[\Box + \xi R + m^2 + (\lambda/2)\tilde{R}_0^2]\tilde{R}_q.\end{aligned}$$

and

$$S_{\tilde{R}}^{(1)} = 0$$

as usual, because this term contains the classical equation.

The effective action of the theory is expanded in powers of $\hbar$ (with $\hbar = 1$) as

$$\Gamma(\tilde{R}) = S_{\tilde{R}} + \Gamma^{(1)} + \Gamma'$$

with one-loop correction given as [6]

$$\Gamma^{(1)} = \frac{i}{2} lnDet(D/\mu^2), \tag{3.1a}$$

where

$$D \equiv \frac{\delta^2 S_{\tilde{R}}}{\delta \tilde{R}^2}\Big|_{\tilde{R}=\tilde{R}_0} = \Box + \xi R + m^2 + (\lambda/2)\tilde{R}_0^2 \tag{3.1b}$$

and Γ' is a term for higher-loop quantum corrections. In eq.(3.1), μ is a mass parameter to keep $\Gamma^{(1)}$ dimensionless.

To evaluate $\Gamma^{(1)}$, the operator regularization method [7] is used . Upto adiabatic order 4 (potentially divergent terms are expected upto this order only in a 4-dim. theory), one-loop correction is obtained as

$$\begin{aligned} \Gamma^{(1)} &= (16\pi^2)^{-1} \frac{d}{ds} \Big[\int d^4x \sqrt{-g(x)} \Big(\frac{\tilde{M}^2}{\mu^2} \Big)^{-s} \Big\{ \frac{\tilde{M}^4}{(s-2)(s-1)} \\ &+ \frac{\tilde{M}^2}{(s-1)} \Big(\frac{1}{6} - \xi \Big) R + \Big[\frac{1}{6} \Big(\frac{1}{5} - \xi \Big) \Box R + \frac{1}{180} R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta} \\ &- \frac{1}{180} R^{\mu\nu} R_{\mu\nu} + \frac{1}{2} \Big(\xi - \frac{1}{6} \Big)^2 R^2 \Big] \Big\} \Big] \Big|_{s=0}, \end{aligned} \tag{3.2a}$$

where

$$\tilde{M}^2 = m^2 + (\lambda/2)\tilde{R}_0^2. \tag{3.2b}$$

After some manipulations, the lagrangian density in $\Gamma^{(1)}$ is obtained as

$$\begin{aligned} L^{(1)} &= (16\pi^2)^{-1} \Big[(m^2 + (\lambda/2)\tilde{R}_0^2)^2 \Big\{ \frac{3}{4} - \frac{1}{2} ln \Big(\frac{m^2 + (\lambda/2)\tilde{R}_0^2}{\mu^2} \Big) \\ &- \Big(\frac{1}{6} - \xi \Big) R (m^2 + (\lambda/2)\tilde{R}_0^2) \Big\{ 1 - ln \Big(\frac{m^2 + (\lambda/2)\tilde{R}_0^2}{\mu^2} \Big) \Big\} \\ &- ln \Big(\frac{m^2 + (\lambda/2)\tilde{R}_0^2}{\mu^2} \Big) \Big\{ \frac{1}{6} \Big(\frac{1}{5} - \xi \Big) \Box R + \frac{1}{180} R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta} \\ &- \frac{1}{180} R^{\mu\nu} R_{\mu\nu} + \frac{1}{2} \Big(\xi - \frac{1}{6} \Big)^2 R^2 \Big\} \Big]. \end{aligned} \tag{3.3}$$

Now the renormalized form of lagrangian density can be written as

$$\begin{aligned} L_{\rm ren} &= \frac{1}{2} \Big(g^{\mu\nu} \partial_\mu \tilde{R}_0 \partial_\nu \tilde{R}_0 - \xi R \tilde{R}_0^2 - m^2 \tilde{R}_0^2 \Big) - \frac{\lambda}{4!} \tilde{R}_0^4 + \Lambda \\ &+ \epsilon_0 R + \frac{1}{2} \epsilon_1 R^2 + \epsilon_2 R^{\mu\nu} R_{\mu\nu} + \epsilon_3 R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta} \\ &+ \epsilon_4 \Box R + L^{(1)} + L_{\rm ct} \end{aligned} \tag{3.4a}$$

with bare coupling constants $\lambda_i \equiv (m^2, \lambda, \Lambda, \xi, \epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$, $L^{(1)}$ given by eq.(3.3) and $L_{\rm ct}$ given as

$$\begin{aligned} L_{\rm ct} &= -\frac{1}{2}\delta\xi R\tilde{R}_0^2 - \frac{1}{2}\delta m^2\tilde{R}_0^2 - \frac{\delta\lambda}{4!}\tilde{R}_0^4 + \delta\Lambda + \delta\epsilon_0 R + \frac{1}{2}\delta\epsilon_1 R^2 \\ &+\delta\epsilon_2 R^{\mu\nu}R_{\mu\nu} + \delta\epsilon_3 R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} + \delta\epsilon_4\Box R. \end{aligned} \tag{3.4b}$$

In eq.(3.4b), $\delta\lambda_i \equiv (\delta m^2, \delta\lambda, \delta\Lambda, \delta\xi, \delta\epsilon_0, \delta\epsilon_1, \delta\epsilon_2, \delta\epsilon_3, \delta\epsilon_4)$ are counter-terms, which are calculated using the following renormalization conditions [8]

$$\begin{aligned} \Lambda &= L_{\rm ren}|_{\tilde{R}_0=\tilde{R}_{(0)0}, R=0} \\ \lambda &= -\frac{\partial^4}{\partial\tilde{R}_0^4}L_{\rm ren}\Big|_{\tilde{R}_0=\tilde{R}_{(0)1}, R=0} \\ m^2 &= -\frac{\partial^2}{\partial\tilde{R}_0^2}L_{\rm ren}\Big|_{\tilde{R}_0=0, R=0} \\ \xi &= -\frac{\partial^3}{\partial R\partial\tilde{R}_0^2}L_{\rm ren}\Big|_{\tilde{R}_0=\tilde{R}_{(0)2}, R=0} \\ \epsilon_0 &= \frac{\partial}{\partial R}L_{\rm ren}\Big|_{\tilde{R}_0=0, R=0} \\ \epsilon_1 &= \frac{\partial^2}{\partial R^2}L_{\rm ren}\Big|_{\tilde{R}_0=0, R=R_5} \\ \epsilon_2 &= \frac{\partial}{\partial(R^{\mu\nu}R_{\mu\nu})}L_{\rm ren}\Big|_{\tilde{R}_0=0, R=R_6} \\ \epsilon_3 &= \frac{\partial}{\partial(R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta})}L_{\rm ren}\Big|_{\tilde{R}_0=0, R=R_7} \\ \epsilon_4 &= \frac{\partial}{\partial(\Box R)}L_{\rm ren}\Big|_{\tilde{R}_0=0, R=R_8}. \end{aligned} \tag{3.5a, b, c, d, e, f, g, h, i}$$

As $\tilde{R} = \eta R$, so when $R = 0, \tilde{R}_{(0)0} = \tilde{R}_{(0)1} = \tilde{R}_{(0)2} = 0$ and $R_5 = R_6 = R_7 = R_8 = 0$ when $\tilde{R}_0 = 0$.

Eqs.(3.4) and (3.5) yield counter-terms as

$$\begin{aligned} 16\pi^2\delta\Lambda &= \frac{m^4}{2}ln(m^2/\mu^2) \\ 16\pi^2\delta\lambda &= -3\lambda^2 ln(m^2/\mu^2) \\ 16\pi^2\delta m^2 &= -\lambda m^2 ln(m^2/\mu^2) \\ 16\pi^2\delta\xi &= -\lambda\Big(\xi - \frac{1}{6}\Big)ln(m^2/\mu^2) \\ 16\pi^2\delta\epsilon_0 &= m^2\Big(\xi - \frac{1}{6}\Big)ln(m^2/\mu^2) \\ 16\pi^2\delta\epsilon_1 &= \Big(\xi - \frac{1}{6}\Big)^2 ln(m^2/\mu^2) \\ 16\pi^2\delta\epsilon_2 &= -\frac{1}{180}ln(m^2/\mu^2) \end{aligned}$$

$$
\begin{aligned}
16\pi^2\delta\epsilon_3 &= \frac{1}{180}ln(m^2/\mu^2) \\
16\pi^2\delta\epsilon_4 &= \frac{1}{6}\Big(\frac{1}{5}-\xi\Big)ln(m^2/\mu^2)
\end{aligned}
$$

$$(3.6a, b, c, d, e, f, g, h, i)$$

4 Renormalization group equations and their solutions

The effective renormalized lagrangian can be improved further by solving renormalization group equations for coupling constants λ_i. For this purpose one-loop β-functions, defined by the equation [1, 8-9]

$$\beta_{\lambda_i} = \mu\frac{d}{d\mu}(\lambda_{i0} + \delta\lambda_i)\Big|_{\delta\lambda_i=0} \tag{4.1}$$

with counter-terms $\delta\lambda_i$ from eqs.(3.6), are obtained as

$$
\begin{aligned}
\beta_\Lambda &= -\frac{m^4}{16\pi^2} \\
\beta_\lambda &= \frac{3\lambda^2}{16\pi^2} \\
\beta^2_m &= \frac{\lambda m^2}{16\pi^2} \\
\beta_\xi &= \frac{\lambda(\xi-\frac{1}{6})}{16\pi^2} \\
\beta_{\epsilon_0} &= -\frac{m^2(\xi-\frac{1}{6})}{16\pi^2} \\
\beta_{\epsilon_1} &= -\frac{(\xi-\frac{1}{6})^2}{16\pi^2} \\
\beta_{\epsilon_2} &= \frac{1}{2880\pi^2} \\
\beta_{\epsilon_3} &= -\frac{1}{2880\pi^2} \\
\beta_{\epsilon_4} &= -\frac{1}{96\pi^2}\Big(\frac{1}{5}-\xi\Big)
\end{aligned}
$$

$$(4.2a, b, c, d, e, f, g, h, i)$$

The renormalization group equations are given as

$$\frac{d\lambda_i}{dt} = \beta_{\lambda_i}, \tag{4.3}$$

where $t = \frac{1}{2}ln(m_c^2/\mu^2)$ with μ being a mass parameter defined above and m_c being a reference mass scale such that $\mu \geq m_c$. Using β-functions for different coupling constants given by eqs.(4.2), solutions of differential equations (4.3) are derived as

$$\begin{aligned}
\Lambda &= \Lambda_0 + \frac{m_0^4}{\lambda_0}\Big[1 - \Big(1 - \frac{3\lambda_0 t}{16\pi^2}\Big)^{-1/3}\Big] \\
\lambda &= \lambda_0\Big[1 - \frac{3\lambda_0 t}{16\pi^2}\Big]^{-1} \\
m^2 &= m_0^2\Big[1 - \frac{3\lambda_0 t}{16\pi^2}\Big]^{-2/3} \\
\xi &= \frac{1}{6} + \Big(\xi_0 - \frac{1}{6}\Big)\Big[1 - \frac{3\lambda_0 t}{16\pi^2}\Big]^{-1/3} \\
\epsilon_0 &= \epsilon_{00} + \frac{m_0^2\big(\xi_0 - \frac{1}{6}\big)}{\lambda_0}\Big[\Big(1 - \frac{3\lambda_0 t}{16\pi^2}\Big)^{1/3} - 1\Big] \\
\epsilon_1 &= \epsilon_{10} + \frac{\big(\xi_0 - \frac{1}{6}\big)^2}{\lambda_0}\Big[\Big(1 - \frac{3\lambda_0 t}{16\pi^2}\Big)^{1/3} - 1\Big] \\
\epsilon_2 &= \epsilon_{20} + \frac{t}{2880\pi^2} \\
\epsilon_3 &= \epsilon_{30} - \frac{t}{2880\pi^2} \\
\epsilon_4 &= \epsilon_{40} - \frac{t}{2880\pi^2} - \frac{\big(\xi_0 - \frac{1}{6}\big)}{12\lambda_0}\Big[\Big(1 - \frac{3\lambda_0 t}{16\pi^2}\Big)^{2/3} - 1\Big],
\end{aligned} \quad (4.4a, b, c, d, e, f, g, h, i)$$

where $\lambda_{i0} = \lambda_i(t = 0)$ and $t = 0$ at $\mu = m_c$ according to the definition of t given above.

These results show that as $\mu \to \infty (t \to -\infty)$ $\lambda \to 0, m^2 \to 0$ and $\xi \to \frac{1}{6}$. Thus it follows from these expressions that in the limit $\mu \to \infty$, theory is asymptotically free and matter sector coupling constants tend to approach their conformally invariant values.

Further it is assumed that $D = 0$ at energy mass scale $\mu = m_c$. Now recalling the definition of ξ from eq.(2.8b), one obtains $\xi_0 = 0$ and

$$D = \frac{6\Big[1 - \Big(1 - \frac{3\lambda_0 t}{16\pi^2}\Big)^{-1/3}\Big]}{\Big[1 + 2\Big(1 - \frac{3\lambda_0 t}{16\pi^2}\Big)^{-1/3}\Big]}. \quad (4.5)$$

It is interesting to see from this result that D increases with increasing energy mass scale μ and it is equal to 6 when $\mu \to \infty$. Moreover, it is not necessarily an integer but a real number. Thus this result suggests that dimension of the space-time will be equal to 4 at $\mu = m_c$ and will increase continuously with increasing energy mass scale upto 10 at extremely large energy scale. Non - integer values of the dimension also indicate that the space-time above the energy mass scale $\mu = m_c$ should be fractal [10-11], as fractal dimensions need not be integers like topological dimensions.

Eqs.(2.8d) and (4.4b) yield

$$\alpha = \frac{1}{36}\Big[1 - \frac{3t}{16\pi^2}\Big]\Big[1 + 2\Big(1 - \frac{3t}{16\pi^2}\Big)^{-1/3}\Big] \quad (4.6)$$

and $\alpha_0 = \frac{1}{12}$, if $\lambda_0 = 1$.

Recalling the definition of m^2 from eq.(2.8c) and using eqs.(4.5)-(4.6) with $\lambda_0 = 1$, one obtains

$$G = G_N \frac{\Big[4 - \Big(1 - \frac{3t}{16\pi^2}\Big)^{-1/3}\Big]}{\Big[1 + 2\Big(1 - \frac{3t}{16\pi^2}\Big)^{-1/3}\Big]\Big[1 - \frac{3t}{16\pi^2}\Big]^{1/3}}, \tag{4.7}$$

where $G_0 = G_N$ (G_N is the Newtonian gravitational constant). Now from eq.(2.8c)

$$m_0^2 = \frac{1}{8\pi G_N} = \frac{M_P^2}{8\pi} \tag{4.8}$$

as $D = 0$ at $t = 0$ or $\mu = m_c$ and $G_N = M_P^2$ (M_P is Planck's constant). Thus eqs.(4.4a) and (4.8) imply

$$\Lambda - \Lambda_0 = \frac{M_P^4}{(8\pi)^2}\Big[1 - \Big(1 - \frac{3t}{16\pi^2}\Big)^{-1/3}\Big]. \tag{4.9}$$

5 Discussion and concluding remarks

The equation (4.9) implies that vacuum energy density Λ increases with increasing energy mass scale μ (or decreasing t). Taking $\lambda_0 = 1$, it is obtained that $(\Lambda - \Lambda_0) = 1.58 \times 10^{73}\text{Gev}^4$ when $\mu \to \infty (t \to -\infty)$. In Table 1 $(\Lambda - \Lambda_0)$ is exhibited for different values of μ. It is interesting to note that when μ comes down from infinity to $(1 + 4 \times 10^{-6})m_c$, $(\Lambda - \Lambda_0)$ decreases slowly. But when it comes down from $(1+4\times10^{-6})m_c$ to $(1+3\times10^{-6})m_c$, there is a sudden drop in the value of $(\Lambda - \Lambda_0)$ from $6.29 \times 10^{65}\text{GeV}^4$ to zero. It means that all of a sudden, a huge amount of energy (with density equal to $6.29 \times 10^{65}\text{GeV}^4$) is released at $\mu = (1 + 3 \times 10^{-6})m_c$, which is sufficient to heat the universe upto a temperature 2.82×10^{16}GeV. It corresponds to the energy scale 2.82×10^{16}GeV in natural units (as Boltzman constant $\kappa_B = 1$ in these units). Sudden release of energy indicates a phase transition at 2.82×10^{16}GeV. According to the standard model of grand unified theories, strong and electro-weak interactions unite at 10^{15}GeV. But gravity maintains its identity different from these interactions upto this energy scale also. So unification of these interactions are expected above 10^{15}GeV [12]. Thus symmetry breaking, due to phase transition at 2.82×10^{16}GeV (discussed above) is expected to be the energy mass scale where gravity decouples from strong-electroweak interaction. In other words, these results suggest that unification of gravity with strong-electroweak interaction should take place at 2.82×10^{16}GeV. It implies that

$$m_c = 2.82 \times 10^{16}\text{GeV}. \tag{5.1}$$

As discussed in the preceding section, another interesting result is given by eq.(4.5) which suggests that above the energy mass scale $\mu = (1 + 3 \times 10^{-6})m_c \simeq 2.28 \times 10^{16}$GeV, space-time should be fractal [10-11]. Moreover, according to this result, dimension of space-time increases from 4 to 10 with increasing energy mass scale.

Equation (4.7) implies that the gravitational constant G decreases with increasing μ with $G = G_N$ at $\mu = 2.82 \times 10^{16}$GeV.

At the energy scales $\mu \le 2.82 \times 10^{16}\text{GeV}, D = 0, \alpha = \alpha_0 = \frac{1}{12}$ and $G = G_N = M_P^{-2}$, so M given by eq.(2.1b) is evaluated to be

$$M \ge (\frac{3}{4\pi})^{1/4} \times \sqrt{\frac{M_P}{\eta^2}} = 2.2 \times 10^9\text{GeV}. \tag{5.2}$$

Eqs.(4.4) also show that $\alpha, \epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3$ and ϵ_4 increase with increasing μ showing that higher-derivative terms grow stronger with increasing energy scale.

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Appendix A

The 4-dim. higher-derivative gravitational action is taken as[2-5]

$$S_g = \int d^4x\sqrt{-g} \quad \Big[\frac{R}{16\pi G} + \tilde{\alpha} R^{\mu\nu}R_{\mu\nu} + \tilde{\beta}R^2 - (1/3!)\lambda\eta^2 R^3\Big], \tag{1A}$$

where $\tilde{\alpha}, \tilde{\beta}$ and λ are dimensionless coupling constants.

Imposing invariance of S_g under transformations $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$, one obtains field equations

$$\begin{aligned}
&\frac{1}{16\pi G}\Big(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\Big) + \tilde{\alpha}\Big(2R^{\rho}_{\mu;\nu\rho} - \Box R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\Box R \\
&+2R^{\rho}_{\mu}R_{\rho\nu} - \frac{1}{2}g_{\mu\nu}R^{\gamma\delta}R_{\gamma\delta}\Big) + \tilde{\beta}\Big(2R_{:\mu\nu} - 2g_{\mu\nu}\Box R - \frac{1}{2}g_{\mu\nu}R^2 \\
&2RR_{\mu\nu}\Big) - \frac{\lambda}{3!}\eta^2\Big(3R^2_{;\mu\nu} - 3g_{\mu\nu}\Box R^2 - \frac{1}{2}g_{\mu\nu}R^3 + 3R^2R_{\mu\nu}\Big) \\
&= 0,
\end{aligned} \tag{2A}$$

where semi-colon (;) denotes covariant derivative with respect to the geometry of space-time.

Trace of these field equations yields

$$-\frac{R}{16\pi G} - 2(\tilde{\alpha} + 3\tilde{\beta})\Box R - \frac{\lambda}{3!}\eta^2 R^3 + \frac{3}{2}\lambda\eta^2\Box R^2 = 0. \tag{3A}$$

$R^2_{;\mu} = R^2_{,\mu}$ are vector components as R^2 is a scalar. So, analogous to Lorentz gauge used in the case of gauge fields, one can use a gauge condition

$$\Box R^2 = (R^2_{,\mu})^{;\mu} = 0. \tag{4A}$$

Connecting eqs. (3 A) and (4 A), one obtains

$$\Box R + m^2 R = -\frac{\lambda}{2(3!)(\tilde{\alpha} + 3\tilde{\beta})}\eta^2 R^3, \tag{5A}$$

where $m \equiv [32\pi G(\tilde{\alpha} + 3\tilde{\beta})]^{-1/2}$. Here $\tilde{\alpha}$ and $\tilde{\beta}$ are chosen such that $(\tilde{\alpha} + 3\tilde{\beta}) > 0$ to avoid the ghost problem.

From eqs. (1 A) - (5 A), it is clear that eq.(5 A) can be obtained from S_g, given by eq.(1 A), only when higher-derivative terms $\Big[\tilde{\alpha} R^{\mu\nu} R_{\mu\nu} + \tilde{\beta} R^2 - (1/3!)\lambda \eta^2 R^3\Big]$ are not insignificant compared to $\frac{R}{16\pi G.}$ At this stage, it is important to know the energy mass scale M where this possibility exists. To decide the relative dominance of $\frac{R}{16\pi G}$ and higher-derivative terms, mass scale representation of these terms can be useful. In natural units, $\frac{R}{16\pi G}$ corresponds to $\frac{M^2 M_P^2}{16\pi (M_P}$ is the Planck mass and $G = M_P^{-2}$, $\Big[\tilde{\alpha} R^{\mu\nu} R_{\mu\nu} + \tilde{\beta} R^2\Big]$ corresponds to $\Big[\tilde{\alpha} + \tilde{\beta}\Big] M^4$ as well as $(1/3!)\lambda \eta^2 R^3$ corresponds to $(1/3!)\lambda \eta^2 M^6$, because R and $R_{\mu\nu}$ are linear combinations of second derivatives and squares of first derivatives of components of the metric tensor $g_{\mu\nu}$ (being defined through $dS^2 = g_{\mu\nu} dx^\mu dx^\nu$) w.r.t.space-time coordinates. $g_{\mu\nu}$ are dimensionless. Thus, it is found that higher-derivative terms are significant only when

$$M^2 \geq \frac{3[\tilde{\alpha} + \tilde{\beta}] + \sqrt{9(\tilde{\alpha} + \tilde{\beta})^2 + (3/8\pi)\lambda \eta^2 M_P^2}}{\lambda \eta^2}. \tag{6A}$$

According to eq.(6 A), M can be obtained exactly only when $\tilde{\alpha}, \tilde{\beta}$ and λ are known. But roughly it indicates that M should be above 10^9GeV.

Now the question arises how to interpret the physical meaning of the equation (5 A). An equation for the scalar matter field ϕ given as

$$\Box\phi + m_\phi^2 \phi = -\frac{\lambda}{(3!)}\phi^3, \tag{7A}$$

is well known in quantum field theory. This equation is derived from the action

$$S_\phi = \int d^4x \sqrt{-g}\Big[\frac{1}{2}\{g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m_\phi^2\phi^2\} - \frac{\lambda}{4!}\phi^4\Big] \tag{8A}$$

using its invariance under transformation $\phi \to \phi + \delta\phi$.

Comparing eqs.(5 A) and (7 A), it is found that

(i) mass dimension of ϕ is 1 in natural units whereas mass dimension of R is 2

(ii) $\Box$ and R both depend on $g_{\mu\nu}$ whereas ϕ does not depend on $g_{\mu\nu}$.

Now equation (5 A) is multiplied by η having length dimension and unit magnitude in natural units and ηR is recognized as $\tilde{R}$. As a result, eq.(5 A) looks like

$$\Box\tilde{R} + m^2\tilde{R} = -\frac{\lambda}{2(3!)(\tilde{\alpha} + 3\tilde{\beta})}\tilde{R}^3, \tag{9A}$$

Thus $\tilde{R}$ has mass dimension 1 like ϕ.

It has been discussed above that equations of type (5 A) are possible at high energy level. High energy modes exite the physical system at small length scales. So it is appropriate to use asymptotic expansion of $g_{\mu\nu}$ given as

$$\begin{aligned} g_{\mu\nu}(x) &= g_{\mu\nu}(x_0) + \frac{1}{3}R_{\mu\alpha\nu\beta}(x_0)y^\alpha y^\beta - \frac{1}{6}\partial_\gamma R_{\mu\alpha\nu\beta}(x_0)y^\alpha y^\beta y^\gamma \\ &+ \Big[\frac{1}{20}R_{\mu\alpha\nu\beta;\gamma\delta} + \frac{2}{45}R_{\mu\alpha\beta\lambda}R^\lambda_{\gamma\nu\delta}\Big](x_0)y^\alpha y^\beta y^\gamma y^\delta + \cdots \end{aligned} \tag{10A}$$

in the small vicinity of a space-time point with coordinates $\{x_0^\mu; \mu = 0,1,2,3\}$. Here $y^\alpha = x^\alpha - x_0^\alpha (\alpha = 0.1,2,3)$ and $g_{\mu\nu}(x_0) = \eta_{\mu\nu}$. Using these expressions, one obtains the operator

$$\Box = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\Big(\sqrt{-g} \quad g^{\mu\nu}\frac{\partial}{\partial x^\nu}\Big)$$

as

$$\Box = g^{\mu\nu}(x_0)\frac{\partial^2}{\partial x^\mu \partial x^\nu} + B^\nu(x;x_0)\frac{\partial}{\partial x^\nu}, \tag{11A}$$

with

$$\begin{aligned} g^{\mu\nu}(x) &= g^{\mu\nu}(x_0) - \frac{1}{3}R^{\mu\nu}_{\alpha\beta}(x_0)y^\alpha y^\beta - \frac{1}{6}\partial_\gamma R^{\mu\nu}_{\alpha\beta}(x_0)y^\alpha y^\beta y^\gamma \\ &+ \Big[\frac{1}{20}R_{\mu\alpha\nu\beta;\gamma\delta} + \frac{2}{45}R_{\mu\alpha\beta\lambda}R^\lambda_{\gamma\nu\delta}\Big](x_0)y^\alpha y^\beta y^\gamma y^\delta + \cdots \end{aligned} \tag{12A}$$

and

$$\begin{aligned} B^\nu(x;x_0) &= \Big[\frac{1}{6}\partial_\gamma R^{\gamma\nu}_{\alpha\beta} - \frac{1}{12}\partial^\nu R_{\alpha\beta}\Big](x_0)y^\alpha y^\beta - \Big[\frac{1}{20}R^\nu_{\beta;\gamma\delta} + \\ &\frac{2}{45}R_{\beta\lambda}R^{\lambda\nu}_{\gamma\delta}\Big](x_0)y^\beta y^\gamma y^\delta + \Big[\frac{1}{20}R^\nu_{\alpha;\gamma\delta} + \\ &\frac{2}{45}R_{\beta\lambda}R^{\lambda\nu}_{\gamma\delta}\Big](x_0)y^\alpha y^\gamma y^\delta - \Big[\frac{1}{20}R^{\mu\nu}_{\alpha;\gamma\delta} + \\ &\frac{2}{45}R^\mu_{\alpha\beta\gamma}R^{\gamma\nu}_{\mu\delta}\Big](x_0)y^\alpha y^\beta y^\delta - \Big[\frac{1}{20}R^{\mu\nu}_{\alpha\beta;\gamma\mu} + \\ &\frac{2}{45}R^\mu_{\alpha\beta\lambda}R^{\lambda\nu}_{\gamma\mu}\Big](x_0)y^\alpha y^\beta y^\gamma - \frac{1}{6}R_{\gamma\delta}(x_0)\Big[\frac{1}{6}\partial_\gamma R^{\gamma\nu}_{\alpha\beta} - \\ &\frac{1}{12}\partial^\nu R_{\alpha\beta}\Big](x_0)y^\alpha y^\beta y^\gamma y^\delta + \cdots . \end{aligned} \tag{13A}$$

Thus, at high energy level, one can work in the small neighbourhood of a point $\{x_0\}$, where $\Box$ depends on curvature terms evaluated at the point $\{x_0\}$ and $\tilde{R}$ is defined at arbitrary points of the neighbourhood. So, at high energy level, it is possible to have $\Box$ and $\tilde{R}$ independent.

It means that ,at high energy level, $\tilde{R}$ behaves like spinless matter field ϕ and treating $\tilde{R}$ as a basic physical field , eq.(9 A) can be derived from the action

$$S_{\tilde{R}} = \int d^4x\sqrt{-g}\Big[\frac{1}{2}\{g^{\mu\nu}\partial_\mu\tilde{R}\partial_\nu\tilde{R} - m^2\tilde{R}^2\} - \frac{\lambda}{2(4!)(\tilde{\alpha}+3\tilde{\beta})}\tilde{R}^4\Big] \tag{14A}$$

using its invariance under transformation $\tilde{R} \to \tilde{R} + \delta\tilde{R}$.

Thus the Ricci scalar has a dual role at high energy (i) as a spinless matter field represented by $\tilde{R} = \eta R$ and (ii) as a geometrical field.

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Table 1

$(\Lambda - \Lambda_0)$, G/G_N and dimension of space-time $(4 + D)$ are tabulated below against μ/m_c with $m_c = 2.82 \times 10^{16}$GeV taking $\lambda_0 = 1$.

μ/m_c	$(\Lambda - \Lambda_0)$ *in* GeV^4	G/G_N	$(4 + D)$
1	0	1	4
$1 + 3 \times 10^{-6}$	0	1	4
$1 + 4 \times 10^{-6}$	6.29×10^{65}	1	$4 + 7.9 \times 10^{-8}$
$1 + 9 \times 10^{-6}$	6.29×10^{65}	1	$4 + 7.9 \times 10^{-8}$
$1 + 10^{-5}$	1.26×10^{66}	1	$4 + 1.59 \times 10^{-7}$
$1 + 10^{-4}$	1.0×10^{67}	1	$4 + 1.27 \times 10^{-6}$
$1 + 10^{-3}$	1.0×10^{68}	1	$4 + 1.26 \times 10^{-5}$
$1 + 10^{-2}$	9.99×10^{68}	1	$4 + 1.26 \times 10^{-4}$
1.1	9.56×10^{69}	1	$4 + 1.2 \times 10^{-3}$
2	6.9×10^{70}	1	$4 + 8.74 \times 10^{-3}$
10	2.24×10^{71}	1	4.03
10^5	1.19×10^{72}	1	4.16
10^{10}	1.94×10^{72}	0.99	4.27
10^{20}	3.1×10^{72}	0.985	4.449
10^{30}	3.94×10^{72}	0.975	4.596
∞	1.58×10^{73}	0	10