

RICHNESS OR SEMI-HAMILTONICITY OF QUASI-LINEAR SYSTEMS THAT ARE NOT IN EVOLUTION FORM

BY

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Abstract. The aim of this paper is to consider strictly hyperbolic quasi-linear systems of conservation laws which appear in the form $A(u)u_x + B(u)u_y = 0$. If one of the matrices $A(u), B(u)$ is invertible, then this system is in fact in the form of evolution equations. However, it may happen that traveling along characteristics one moves from the “chart” where $A(u)$ is invertible to another “chart” where $B(u)$ is invertible. We propose a new condition of richness or semi-Hamiltonicity for such a system that is “chart”-independent. This new condition enables one to perform the blow-up analysis along characteristic curves for all times, not passing from one “chart” to another. This opens a possibility to use this theory for geometric problems as well as for stationary solutions of 2D+1 systems. We apply the results to the problem of polynomial integral for geodesic flows on the 2-torus.

1. Motivation and the result. Consider a quasi-linear system for a vector function $u(x, y) = (u_1, \dots, u_n)$ that has the following form:

$$A(u)u_x + B(u)u_y = 0. \tag{1}$$

It may happen in practice that one of the matrices $A(u)$ or $B(u)$ can degenerate somewhere (and even both of them can degenerate somewhere). Such situations appear in geometric problems as well as for stationary solutions of quasi-linear systems with two space variables x, y .

Throughout this paper, our main assumption on these matrices is that the homogeneous polynomial P in α, β is not a zero polynomial at any point (x, y) :

$$P = \det(\alpha B - \beta A), \quad \deg(P) = n. \tag{P}$$

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This assumption is obviously satisfied if one of the matrices $A(u)$ or $B(u)$ is non-degenerate; however, we shall assume everywhere the weaker version: (P) . We shall see in the example of the last section that (P) is in fact the correct assumption.

Moreover, we shall assume in the following that the system is strictly hyperbolic; that is, the polynomial P has n distinct roots $[\beta_i : \alpha_i]$. We define unit characteristic vector fields on the plane $\mathbf{R}^2(x, y)$ by

$$v_i = \cos \phi_i \partial_x + \sin \phi_i \partial_y,$$

where the angles ϕ_i , which we shall call characteristic angles, are such that $[\sin \phi_i : \cos \phi_i] = [\beta_i : \alpha_i], \phi_i \neq \phi_j \pmod{\pi}$.

One can use a regular change of variables and multiplication from the left on an invertible matrix in order to transform system (1) to an equivalent one. Namely, if $u = \Phi(w)$ is a regular change of variables and $C(w)$ is an invertible matrix, then system (1) takes the form

$$C(w)A(\Phi(w))D\Phi(w)w_x + C(w)B(\Phi(w))D\Phi(w)w_y = 0,$$

where D is the differential. Obviously such a transformation preserves the roots of P .

Notice that if one of the matrices, say A , is non-degenerate, then the system is equivalent to one in the evolution form. Recall the notion of the evolution system to be rich or semi-Hamiltonian (see [13], [5], [11], [12] and also [9]), we shall call them rich for the sake of brevity. A strictly hyperbolic evolution system is called rich if it can be written in Riemann invariants (diagonal form):

$$(r_i)_x + \lambda_i(r_1, \dots, r_n)(r_i)_y = 0, i = 1, \dots, n,$$

and, moreover, the eigenvalues $\lambda_i = \beta_i/\alpha_i$ of $A^{-1}B$ satisfy the following identities:

$$\partial_{r_k} \left(\frac{\partial_{r_i} \lambda_j}{\lambda_i - \lambda_j} \right) = \partial_{r_i} \left(\frac{\partial_{r_k} \lambda_j}{\lambda_k - \lambda_j} \right). \tag{R}$$

This condition allows one to perform blow-up analysis along characteristics as it is shown in [11] and applied to a mechanical example in [1]. It was proved by B. Sevenec [12] (see also the differential-geometric interpretation in [7]) that a strictly hyperbolic system in evolution form that is written in Riemann invariants is rich if and only if there are local coordinates in which the system takes the form of conservation laws.

The unsatisfactory thing, however, with the condition (R) for the system (1) is the fact that characteristic curves can rotate in the plane and pass from the chart where A is non-degenerate to the chart where B is non-degenerate or even reach those points where both matrices degenerate. For understanding the local solutions, this does not play a role, and this does not allow one to analyze the long-time behavior of the solutions, either, since one cannot write the Riccati equations for all times.

We propose the following generalization of the richness condition whose naturality we shall justify below:

DEFINITION 1.1. We call the strictly hyperbolic system (1) rich if it can be written in the diagonal form

$$L_{v_i} r_i = \cos \phi_i (r_i)_x + \sin \phi_i (r_i)_y = 0, i = 1, \dots, n, \phi_i \neq \phi_j \pmod{\pi} \tag{2}$$

for a regular change of variables $(u_1, \dots, u_n) \rightarrow (r_1, \dots, r_n)$ and the following conditions on the characteristic angles $\phi_i(r_1, \dots, r_n)$ hold true:

$$\partial_{r_k} \left(\frac{\partial_{r_i} \phi_j}{\tan(\phi_i - \phi_j)} \right) = \partial_{r_i} \left(\frac{\partial_{r_k} \phi_j}{\tan(\phi_k - \phi_j)} \right). \tag{\Phi}$$

It is an important fact that this definition is invariant with respect to rotations of the plane. We shall continue to call r_i in (2) by Riemann invariants. Our first result is the following

THEOREM 1.2. If a strictly hyperbolic system (1) satisfying (P) is rich according to Definition 1.1 so that the conditions (2),(\Phi) hold true, then the derivatives of the i -th Riemann invariant $w_i = L_{v_i^\perp} r_i$ in the orthogonal direction to those characteristics satisfy the following Riccati equation:

$$L_{v_i}(\exp(-G_i)w_i) + \exp(G_i)\partial_{r_i}(\phi_i)(\exp(-G_i)w_i)^2 = 0,$$

for any $i = 1, \dots, n$, where G_j is a function of Riemann invariants satisfying

$$\partial_{r_i} G_j = \frac{\partial_{r_i} \phi_j}{\tan(\phi_i - \phi_j)}.$$

Here v^\perp stands for the vector field rotated from v by 90° counterclockwise.

We shall see in the lemma below that the conditions (R) and (\Phi) are almost equivalent. This lemma enables us to prove the following theorem, which is a generalization to our case of the result of [12].

THEOREM 1.3. Given any strictly hyperbolic diagonal system

$$\cos \phi_i(r_i)_x + \sin \phi_i(r_i)_y = 0, \quad i = 1, \dots, n,$$

the condition (\Phi) is satisfied if and only if the system can be written in the form of n conservation laws

$$(g_i)_x + (h_i)_y = 0, \quad i = 1, \dots, n.$$

We prove the main theorems in sections 2 and 3. The last section contains a geometric example originating from classical mechanics.

2. Derivation along characteristics. Proof of Theorem 1.2. Differentiate the j -th equation of (2) with respect to the field v_j^\perp . We have

$$0 = L_{v_j^\perp} L_{v_j} r_j = L_{v_j} L_{v_j^\perp} r_j - L_{[v_j, v_j^\perp]} r_j. \tag{3}$$

Compute now the derivative along the commutator:

$$\begin{aligned}
 L_{[v_j, v_j^\perp]} r_j &= L_{v_j} L_{v_j^\perp} r_j - L_{v_j^\perp} L_{v_j} r_j \\
 &= L_{v_j} (-\sin \phi_j (r_j)_x + \cos \phi_j (r_j)_y) - L_{v_j^\perp} (\cos \phi_j (r_j)_x + \sin \phi_j (r_j)_y) \\
 &= (r_j)_x (-\cos^2 \phi_j (\phi_j)_x - \cos \phi_j \sin \phi_j (\phi_j)_y) \\
 &\quad + (r_j)_y (-\sin \phi_j \cos \phi_j (\phi_j)_x - \sin^2 \phi_j (\phi_j)_y) \\
 &\quad + (r_j)_x (-\sin^2 \phi_j (\phi_j)_x + \sin \phi_j \cos \phi_j (\phi_j)_y) \\
 &\quad + (r_j)_y (\sin \phi_j \cos \phi_j (\phi_j)_x - \cos^2 \phi_j (\phi_j)_y) \\
 &= -(r_j)_x (\phi_j)_x - (r_j)_y (\phi_j)_y.
 \end{aligned} \tag{4}$$

Notice that the derivatives $(r_j)_x, (r_j)_y$ can be expressed by the following two identities:

$$\begin{aligned}
 \cos \phi_j (r_j)_x + \sin \phi_j (r_j)_y &= 0, \\
 -\sin \phi_j (r_j)_x + \cos \phi_j (r_j)_y &= L_{v_j^\perp} r_j.
 \end{aligned}$$

Therefore

$$\begin{aligned}
 (r_j)_x &= -\sin \phi_j L_{v_j^\perp} r_j, \\
 (r_j)_y &= \cos \phi_j L_{v_j^\perp} r_j.
 \end{aligned} \tag{5}$$

Substituting back to (4), we get

$$L_{[v_j, v_j^\perp]} r_j = (L_{v_j^\perp} r_j) (\sin \phi_j (\phi_j)_x - \cos \phi_j (\phi_j)_y) = -(L_{v_j^\perp} r_j) (L_{v_j^\perp} \phi_j). \tag{6}$$

By the chain rule for $L_{v_j^\perp} \phi_j$, the last equation can be rewritten as follows:

$$\begin{aligned}
 L_{[v_j, v_j^\perp]} r_j &= -L_{v_j^\perp} r_j \sum_{i=1}^n (\partial_{r_i} \phi_j) (L_{v_j^\perp} r_i) \\
 &= -(L_{v_j^\perp} r_j)^2 (\partial_{r_j} \phi_j) - (L_{v_j^\perp} r_j) \sum_{i \neq j} (\partial_{r_i} \phi_j) (L_{v_j^\perp} r_i).
 \end{aligned} \tag{7}$$

Let us express now the derivative

$$L_{v_j^\perp} r_i = -\sin \phi_j (r_i)_x + \cos \phi_j (r_i)_y \tag{8}$$

via $L_{v_j} r_i$ as follows. Write

$$\begin{aligned}
 \cos \phi_i (r_i)_x + \sin \phi_i (r_i)_y &= 0, \\
 \cos \phi_j (r_i)_x + \sin \phi_j (r_i)_y &= L_{v_j} r_i.
 \end{aligned} \tag{9}$$

From these two identities we have

$$(r_i)_x = \frac{\sin \phi_i}{\sin(\phi_i - \phi_j)} L_{v_j} r_i, \quad (r_i)_y = -\frac{\cos \phi_i}{\sin(\phi_i - \phi_j)} L_{v_j} r_i. \tag{10}$$

Substitute expressions (10) into (8) to get

$$L_{v_j^\perp} r_i = -\frac{L_{v_j} r_i}{\tan(\phi_i - \phi_j)}. \tag{11}$$

Denote by

$$w_i := L_{v_i^\perp} r_i.$$

Plug this together with (11) into equation (7) and then to (3):

$$L_{v_j}(w_j) + (\partial_{r_j}\phi_j)(w_j)^2 - w_j \sum_{i \neq j} (\partial_{r_i}\phi_j) \frac{1}{\tan(\phi_i - \phi_j)} L_{v_j}r_i = 0. \tag{12}$$

By richness (Φ) , we have that for all $j = 1, \dots, n$ there exist functions

$$G_j(r_1, \dots, r_n) : \quad \partial_{r_i}G_j = \frac{(\partial_{r_i}\phi_j)}{\tan(\phi_i - \phi_j)}, \quad i \neq j. \tag{13}$$

By (13) we can rewrite (12) as

$$L_{v_j}(w_j) + (\partial_{r_j}\phi_j)(w_j)^2 - w_j \sum_{i \neq j} (\partial_{r_i}G_j)L_{v_j}r_i = 0,$$

which is the same as

$$L_{v_j}(w_j) + (\partial_{r_j}\phi_j)(w_j)^2 - w_j L_{v_j}G_j = 0. \tag{14}$$

Multiplying (14) by $exp(-G_j)$, we get the Riccati equation of the first theorem for j instead of i . This completes the proof of Theorem 1.2.

3. Conservation laws. Proof of Theorem 1.3. We shall need the following key observation.

LEMMA 3.1. Given two sets of functions $\lambda_i(r_1, \dots, r_n); \phi_i(r_1, \dots, r_n), i = 1, \dots, n$ such that

$$\lambda_i \neq \lambda_j, \quad \phi_i \neq \pi/2 \pmod{\pi}, \quad \lambda_i = \tan \phi_i,$$

the conditions (R) and (Φ) are equivalent.

The proof is computational. It would be interesting to find a more conceptual proof.

Proof. Let us prove first that (R) implies (Φ) .

Denote by

$$a_{ij} := \frac{\partial_{r_i}\lambda_j}{\lambda_i - \lambda_j}.$$

Then a_{ij} satisfy the following identities ([11]):

$$\partial_{r_i}a_{kj} = \partial_{r_k}a_{ij} = a_{ki}a_{ij} + a_{ik}a_{kj} - a_{kj}a_{ij}. \tag{15}$$

In order to prove them, differentiate the identity $\partial_{r_i}\lambda_j = a_{ij}(\lambda_i - \lambda_j)$ with respect to r_k , then interchange the order of i, k , subtract one from the other and divide by $\lambda_i - \lambda_k$.

Denote by

$$b_{ij} := \frac{\partial_{r_i}\phi_j}{\tan(\phi_i - \phi_j)} = \frac{\partial_{r_i}\lambda_j}{\lambda_i - \lambda_j} \frac{1 + \lambda_i\lambda_j}{1 + \lambda_j^2} = a_{ij} \frac{1 + \lambda_i\lambda_j}{1 + \lambda_j^2}. \tag{16}$$

To prove (Φ) , we have to verify that the difference

$$d = \partial_{r_k}b_{ij} - \partial_{r_i}b_{kj}$$

vanishes. Let us compute d explicitly:

$$d = (\partial_{r_k} a_{ij}) \frac{1 + \lambda_i \lambda_j}{1 + \lambda_j^2} - (\partial_{r_i} a_{kj}) \frac{1 + \lambda_k \lambda_j}{1 + \lambda_j^2} + a_{ij} \partial_{r_k} \left(\frac{1 + \lambda_i \lambda_j}{1 + \lambda_j^2} \right) - a_{kj} \partial_{r_i} \left(\frac{1 + \lambda_k \lambda_j}{1 + \lambda_j^2} \right).$$

By the identities (15) and the condition (R) we have

$$d = (a_{ki} a_{ij} + a_{ik} a_{kj} - a_{kj} a_{ij}) \frac{\lambda_j (\lambda_i - \lambda_k)}{1 + \lambda_j^2} + a_{ij} \frac{\partial_{r_k} (\lambda_i \lambda_j) (1 + \lambda_j^2) - (1 + \lambda_i \lambda_j) 2 \lambda_j \partial_{r_k} (\lambda_j)}{(1 + \lambda_j^2)^2} - a_{kj} \frac{\partial_{r_i} (\lambda_k \lambda_j) (1 + \lambda_j^2) - (1 + \lambda_k \lambda_j) 2 \lambda_j \partial_{r_i} (\lambda_j)}{(1 + \lambda_j^2)^2}. \tag{17}$$

Substitute now into the nominators of (17) the following expressions for the derivatives of λ_j from the definition of a_{ij} :

$$\partial_{r_i} \lambda_j = a_{ij} (\lambda_i - \lambda_j).$$

Then one has

$$d = (a_{ki} a_{ij} + a_{ik} a_{kj} - a_{kj} a_{ij}) \frac{\lambda_j (\lambda_i - \lambda_k)}{1 + \lambda_j^2} + a_{ij} \frac{a_{kj} (\lambda_k - \lambda_j) \lambda_i + \lambda_j a_{ki} (\lambda_k - \lambda_i)}{1 + \lambda_j^2} - a_{kj} \frac{a_{ij} (\lambda_i - \lambda_j) \lambda_k + \lambda_j a_{ik} (\lambda_i - \lambda_k)}{1 + \lambda_j^2} - 2 a_{ij} \frac{(1 + \lambda_i \lambda_j) \lambda_j a_{kj} (\lambda_k - \lambda_j)}{(1 + \lambda_j^2)^2} + 2 a_{kj} \frac{(1 + \lambda_k \lambda_j) \lambda_j a_{ij} (\lambda_i - \lambda_j)}{(1 + \lambda_j^2)^2}. \tag{18}$$

Notice that the identity (18) is a quadratic expression in a_{ij} 's. Collecting the coefficients of $a_{ij} a_{kj}$, $a_{ik} a_{kj}$, $a_{ki} a_{ij}$, one comes to $d = 0$. This proves the lemma in one direction.

Proof of the converse statement is very much analogous but with even harder computations. We shall give a sketch. Assume the identities (Φ) are satisfied. First, one can obtain the identity analogous to (15) for the derivatives $\partial_{r_k} b_{ij}$ in the following way. Write

$$\partial_{r_i} \phi_j = b_{ij} \tan(\phi_i - \phi_j) = b_{ij} \frac{\lambda_i - \lambda_j}{1 + \lambda_i \lambda_j}, \quad \partial_{r_i} \lambda_j = b_{ij} \frac{(1 + \lambda_j^2)(\lambda_i - \lambda_j)}{1 + \lambda_i \lambda_j}. \tag{19}$$

Differentiating the first equality of (19) with respect to r_k , using the identities (19) again and taking into account (16), one has

$$\partial_{r_k} \partial_{r_i} \phi_j = \partial_{r_k} (b_{ij}) \frac{(\lambda_i - \lambda_j)}{1 + \lambda_i \lambda_j} + b_{ij} \left(1 + \frac{(\lambda_i - \lambda_j)^2}{(1 + \lambda_i \lambda_j)^2} \right) \left(b_{ki} \frac{\lambda_k - \lambda_i}{1 + \lambda_k \lambda_i} - b_{kj} \frac{\lambda_k - \lambda_j}{1 + \lambda_k \lambda_j} \right).$$

Interchanging the order of indexes i and k in this identity and using $\partial_{r_k} b_{ij} = \partial_{r_i} b_{kj}$, one has the identity

$$\begin{aligned} & \partial_{r_k} (b_{ij}) \frac{(\lambda_i - \lambda_k)(1 + \lambda_j^2)}{(1 + \lambda_i \lambda_j)(1 + \lambda_k \lambda_j)} \\ &= b_{kj} \left(1 + \frac{(\lambda_k - \lambda_j)^2}{(1 + \lambda_k \lambda_j)^2} \right) \left(b_{ik} \frac{\lambda_i - \lambda_k}{1 + \lambda_i \lambda_k} - b_{ij} \frac{\lambda_i - \lambda_j}{1 + \lambda_i \lambda_j} \right) \\ & - b_{ij} \left(1 + \frac{(\lambda_i - \lambda_j)^2}{(1 + \lambda_i \lambda_j)^2} \right) \left(b_{ki} \frac{\lambda_k - \lambda_i}{1 + \lambda_i \lambda_k} - b_{kj} \frac{\lambda_k - \lambda_j}{1 + \lambda_k \lambda_j} \right). \end{aligned} \tag{20}$$

In order to verify (R), one computes

$$\begin{aligned} \partial_{r_k} a_{ij} - \partial_{r_i} a_{kj} &= \partial_{r_k} b_{ij} \frac{\lambda_j(\lambda_k - \lambda_i)(1 + \lambda_j^2)}{(1 + \lambda_i \lambda_j)(1 + \lambda_k \lambda_j)} \\ & + b_{ij} \partial_{r_k} \left(\frac{1 + \lambda_j^2}{1 + \lambda_i \lambda_j} \right) - b_{kj} \partial_{r_i} \left(\frac{1 + \lambda_j^2}{1 + \lambda_k \lambda_j} \right). \end{aligned} \tag{21}$$

The last step is to plug the expression (20) into (21) and also to differentiate the last two brackets of (21) using the expression for the derivatives (19). Then one finally gets a quadratic expression in $b_{ij}s$. Collecting similar terms, one verifies that the right-hand side of (21) vanishes. Therefore (R) holds true. This proves the lemma. \square

It is easy now to prove Theorem 1.3.

Proof. Notice first of all that the statement of the second theorem is local. Given a system that is strictly hyperbolic and written in the diagonal form

$$\cos \phi_i (r_i)_x + \sin \phi_i (r_i)_y = 0, \quad i = 1, \dots, n,$$

let us give a proof first in one direction; namely, assume that the system can be written in the form of conservation laws

$$(g_i)_x + (h_i)_y = 0, \quad i = 1, \dots, n.$$

Let me explain that then it must satisfy condition (Φ) . If among ϕ_i there is one with $\cos \phi_i = 0$, then one can apply a small rotation of the plane $\mathbf{R}^2(x, y)$ to get a new system that has all angles different from $\pm\pi/2$, $\phi_i \neq \phi_j \pmod{\pi}$. Notice that the rotated system remains in the form of conservation laws and, in addition, the differential Dg becomes a non-singular matrix, since otherwise $\alpha = 1, \beta = 0$ would be the root of (P), but this is impossible by $\phi_i \neq \pm\pi/2$. Denote

$$\lambda_i := \tan \phi_i.$$

Use now Sevenec's theorem saying that the diagonal system

$$(r_i)_x + \lambda_i (r_i)_y = 0,$$

which can be written in the form of conservation laws

$$(g_i)_x + (h_i)_y = 0, \quad i = 1, \dots, n$$

with the non-singular Jacobi matrix $(\partial_{r_j} g_i)$, must satisfy (R). But by the lemma in this case, (R) and (Φ) are equivalent. So we get condition (Φ) for the rotated system. But this condition is obviously rotationally invariant. Thus it holds also for the original system.

The proof in the opposite direction is analogous. First rotate the plane exactly as above. Condition (Φ) remains valid since it is rotationally invariant. Then by the lemma, (R) is valid as well, and then by Sevenec's theorem the rotated system can be written in the form of conservation laws. Obviously, the original one can be as well. This completes the proof. \square

4. Geometric example. In this section we give a geometric example originating from classical mechanics where the results of the previous sections apply.

Let ρ be a Riemannian metric on the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\Gamma$; ρ^t denotes the geodesic flow. Assume that ρ is written in a conformal way:

$$ds^2 = \Lambda(x, y)(dx^2 + dy^2).$$

Let $F : T^*\mathbb{T}^2$ be a function on the cotangent bundle that is homogeneous polynomial of degree n with respect to the fibre:

$$F = \sum_{k=0}^n a_k(x, y)p_1^{n-k}p_2^k.$$

We are looking for such an F which is an integral of motion for the geodesic flow ρ^t , i.e. $F \circ \rho^t = F$. We shall also assume that this F is irreducible, i.e., of minimal possible degree. Let us mention that this problem is classical; there are very well-studied examples of the geodesic flows on the 2-torus that have integrals F of degree one and two. We refer to the books [4] and [10] for the history and discussion of this classical question with references therein. In our recent papers with A. E. Mironov, we used the so-called semi-geodesic coordinates. In these coordinates one arrives at a remarkably rich quasi-linear system of equations in evolution form on the coefficients of the integral F ([2], [3]). However, it is very natural to be able to work in conformal coordinates as well. In this case, the quasi-linear system on the coefficients no longer has evolution form but looks like:

$$A(U)U_x + B(U)U_y = 0.$$

Let me write down explicitly the matrices for the case $n = 3$ (this case is already very interesting and not trivial; see, for example, [6]).

$$A(U) = \begin{pmatrix} 1 & 0 & 3a \\ 0 & 1 & 3b \\ \Lambda & 0 & u \end{pmatrix}, \quad B(U) = \begin{pmatrix} 0 & -1 & 3b \\ 1 & 0 & -3a \\ 0 & \Lambda & v \end{pmatrix}, \quad U = \begin{pmatrix} u \\ v \\ \Lambda \end{pmatrix}. \quad (22)$$

Here a, b, u, v are related to the coefficients of the integral a_i by the following:

$$a_0 = a + \frac{u}{\Lambda}, \quad a_1 = 3b + \frac{v}{\Lambda}, \quad a_2 = -3a + \frac{u}{\Lambda}, \quad a_3 = -b + \frac{v}{\Lambda}.$$

It was noticed in [8] that a, b are in fact constants. Computing polynomial $P = \det(\alpha B - \beta A)$, one has:

$$P = \alpha^3(v + 3b\Lambda) + \alpha^2\beta(-u - 9a\Lambda) + \alpha\beta^2(v - 9b\Lambda) + \beta^3(-u + 3a\Lambda).$$

Let us remark that it may happen at some points that both matrices A, B are degenerate; however, polynomial P for any point cannot vanish identically. This is because otherwise both constants a, b vanish, but then one checks that in such a case the integral F is a

product of the Hamiltonian with an integral of degree one in momenta, and is therefore reducible.

Notice that quasi-linear system (22) is written in the form of conservation laws:

$$(g_i)_x + (h_i)_y = 0,$$

$$g_1 = u + 3a\Lambda, \quad g_2 = v + 3b\Lambda, \quad g_3 = u\Lambda,$$

$$h_1 = -v + 3b\Lambda, \quad h_2 = u - 3a\lambda, \quad h_3 = v\Lambda.$$

Moreover, by a very general argument, in the hyperbolic region, this system can be written in the diagonal form of (2). Indeed, introduce an angular coordinate ϕ on the fibres of the energy level

$$\left\{ \frac{1}{\Lambda}(p_1^2 + p_2^2) = 1 \right\} : p_1 = \sqrt{\Lambda} \cos \phi, p_2 = \sqrt{\Lambda} \sin \phi.$$

Then one can verify that the condition on a function F to be an integral of the flow reads

$$F_x \cos \phi + F_y \sin \phi + F_\phi \left(\frac{\Lambda_y}{2\Lambda} \cos \phi - \frac{\Lambda_x}{2\Lambda} \sin \phi \right) = 0.$$

At the points where F_ϕ vanishes, this equation takes a particularly nice form:

$$F_x \cos \phi + F_y \sin \phi = 0.$$

Therefore, critical values of F on the fibre are Riemann invariants. One can check also that the polynomial P is proportional in fact to the derivative of F in the direction of the fibre. Moreover, one can check, as we did in [2], that in the hyperbolic region Riemann invariants form a regular change of variables. As a consequence of Theorem 1.3, one concludes that in the hyperbolic region the system of this example is rich in our generalized sense. And therefore Theorem 1.2 tells us that the Riccati equation along characteristics applies. This result is in fact general and is not restricted to the case $n = 3$. For any n , the quasi-linear system (1) on the coefficients of the polynomial integral of motion is rich in the generalized sense. The details will appear elsewhere.

5. Questions. Several questions are very natural:

1. It would be interesting to find a more conceptual proof of the lemma in the framework of the differential-geometric approach by Dubrovin-Novikov [5].
2. How does the generalized hodograph method by Tsarev [13] work in our case?
3. How does one analyze the behavior of the Riccati equation for the example of the previous section? It seems that a genuine non-linearity condition cannot be expected for all eigenvalues.

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