Riddling Bifurcation in Chaotic Dynamical Systems

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When a chaotic attractor lies in an invariant subspace, as in systems with symmetry, riddling can occur. Riddling refers to the situation where the basin of a chaotic attractor is riddled with holes that belong to the basin of another attractor. We establish properties of the riddling bifurcation that occurs when an unstable periodic orbit embedded in the chaotic attractor, usually of low period, becomes transversely unstable. An immediate physical consequence of the riddling bifurcation is that an extraordinarily low fraction of the trajectories in the invariant subspace diverge when there is a symmetry breaking. [S0031-9007(96)00503-0]

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Recently, the phenomenon of riddled basins in chaotic dynamical systems has become an area of intensive study [1–5]. The description of riddled basins was introduced in Ref. [1] where the following was shown for certain classes of dynamical systems with an invariant subspace: (i) if there is a chaotic attractor in the invariant subspace; (ii) if there is another attractor in the phase space; and (iii) if the Lyapunov exponent transverse to the subspace is negative, then the basin of the chaotic attractor in the invariant subspace can be riddled with holes belonging to the basin of the other attractor. That is, for every initial condition that asymptotes to the chaotic attractor in the invariant subspace, there are initial conditions arbitrarily nearby that asymptote to the other attractor. Invariant subspaces are particularly common for systems with symmetry. Rigorous results on the dynamics of riddled basins for discrete maps were presented in Refs. [1] and [2]. The dynamics of riddled basins was subsequently investigated in [3] using a more realistic physical model. A more extreme type of basin structure referred to as "intermingled basins," in which the basins of more than one chaotic attractor are riddled, was also studied using both discrete maps [1] and a more realistic physical system [4]. Riddled basins have been verified in experiments conducted using coupled electrical oscillators [5,6]. The mechanism for riddling to occur, and the basin structure associated with the riddling, were investigated by Ashwin, Buescu, and Stewart [6].

In this Letter, we describe the riddling bifurcation in chaotic systems, and we investigate the behavior when a symmetry-breaking parameter is introduced. The onset of riddling is determined by a saddle-repeller bifurcation (eigenvalue $+1$) [7]. For simplicity, we emphasize the case of two-dimensional phase space and, hence, the invariant subspace is a line. Before the bifurcation, the chaotic attractor attracts all points in some of its neighborhood, and all the periodic orbits embedded in the chaotic attractor are saddles. At the bifurcation, one of the periodic orbits, usually of low period, becomes transversely unstable. Since this periodic orbit is already unstable in the attractor, it becomes a repeller in the two-dimensional phase space. Specifically, let $\mathbf{x}_p$ be an unstable periodic point embedded in the chaotic attractor in the invariant subspace. To simplify notation, we assume it is a fixed point. The unstable point is stable transversely to this subspace, as shown in Fig. 1(a). Riddling occurs when some $\mathbf{x}_p$ loses its transverse stability as a parameter p passes through the critical value p_c. For such systems, the loss of transverse stability is induced by the collision at $p = p_c$ of two repellers $\mathbf{r}_+$ and $\mathbf{r}_-$, located symmetrically with respect to the invariant subspace, with the saddle at $\mathbf{x}_p$ (a saddle-repeller pitchfork bifurcation). These two repellers exist only for $p \le p_c$, as shown in Fig. 1(a). For $p > p_c$, the saddle $\mathbf{x}_p$ becomes a repeller, and the two repellers $\mathbf{r}_+$ and $\mathbf{r}_-$ off the invariant subspace do not exist anymore.

As we will argue shortly, due to nonlinearity, a "tongue" opens at $\mathbf{x}_p$ allowing trajectories near the invariant subspace to escape for $p > p_c$, as shown in Fig. 1(b). Each preimage of $\mathbf{x}_p$ also develops a tongue simultaneously. Since preimages of $\mathbf{x}_p$ are dense in the invariant subspace, an infinite number of tongues open simultaneously at $p = p_c$, indicating that initial conditions arbitrarily close to the invariant subspace may asymptote to another attractor. Trajectories in the chaotic attractor, however, remain there even for $p > p_c$, since the subspace in which the chaotic attractor lies is invariant and each tongue has a zero width there. But trajectories near the chaotic attractor have a finite probability of being in the open and dense set of tongues. Trajectories having initial conditions in the tongues asymptote to the other attractor. So, for $p \gtrsim p_c$, most initial conditions, off but close to the chaotic attractor, asymptote to it, but there is an open and dense set of initial conditions that asymptote to the other attractor. Thus, the basin of attraction for the chaotic attractor is a Cantor set of leaves of positive Lebesgue measure, signifying riddling. Physically, since the onset of riddling induces the creation of these supernarrow tongues near the invariant subspace,

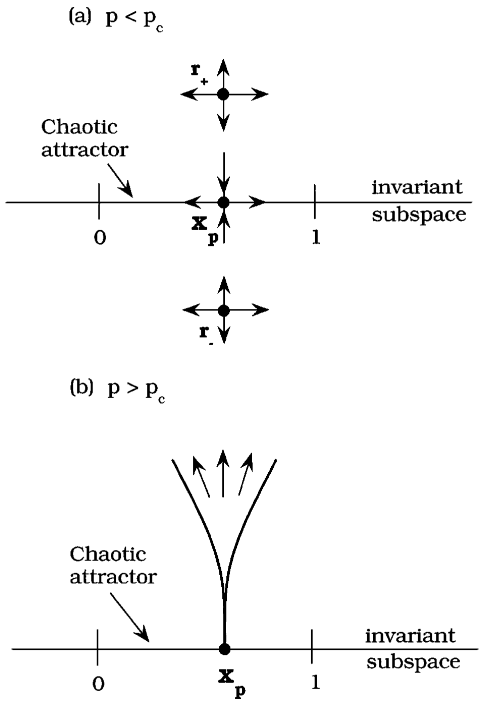

FIG. 1. (a) The unstable saddle fixed point in the invariant subspace and two repellers off the invariant subspace for $p < p_c$ (before the saddle-repeller pitchfork bifurcation). (b) The tongue structure formed for $p > p_c$, after the onset of riddling. Trajectories originated from initial conditions inside the tongues escape the invariant subspace.

it leads to superpersistent chaotic transient behavior [7] in the vicinity of the chaotic attractor. This should be contrasted to the typical average lifetime of transient chaos that scales algebraically [8]. For points chosen at random at a small distance d from the attractor, the probability of not being attracted depends on the distance d as

$$P(d) \sim \exp[-Kd^{-\gamma}], \quad (1)$$

where $\gamma > 0$ is a positive exponent, and $K > 0$ is a constant that can be expressed in terms of the Lyapunov exponent of the chaotic attractor.

We consider the following general class of dynamical systems:

$$\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n), \quad \mathbf{y}_{n+1} = \epsilon + pg(\mathbf{x}_n)\mathbf{y}_n + \text{high order odd terms of } \mathbf{y}_n, \quad (2)$$

where $\mathbf{x} \in R^N$ $(N \geq 1)$, $\mathbf{y} \in R^M$ $(M \geq 1)$, $\mathbf{f}(\mathbf{x}_n)$ is a map that has a chaotic attractor in the invariant subspace $\mathbf{y}_n = 0$, $g(\mathbf{x}_n) = 1$ at some unstable periodic orbit of $\mathbf{f}(\mathbf{x}_n)$, p is a system parameter, and $pg(\mathbf{x}_n)$ is assumed to be positive. We call $\epsilon \geq 0$ the symmetry-breaking parameter. Notice that, for initial $y_0 \geq 0$ and $\epsilon \geq 0$, trajectories have $y_n \geq 0$ for all times. Our main goal now is to understand how riddling occurs as p passes through p_c when $\epsilon \geq 0$. To illustrate our findings, we consider the following version of Eq. (2):

$$x_{n+1} = ax_n(1 - x_n), \quad y_{n+1} = \epsilon + pe^{-b(x-x_p)^2}y_n + y_n^3, \quad (3)$$

where, for $\epsilon = 0$, $y = 0$ defines the invariant subspace as a trajectory, $y = 0$ will remain so forever, and $a, b > 0$ are parameters. The symmetry-breaking parameter is for the symmetry $y \to -y$. Thus, the dynamics in the invariant subspace is described by the logistic map $x_{n+1} = ax_n(1 - x_n)$ for which chaotic attractors occur for parameter values in a positive Lebesgue measure set [9].

To understand how riddling occurs for $\epsilon = 0$, we note that the two eigenvalues of the unstable fixed point $\mathbf{x}_p$ $(x \equiv x_p = 1 - 1/a, y = 0)$ are $(\lambda_x, \lambda_y) = (2 - a, p)$. Thus, $\mathbf{x}_p$ is stable in the y direction for $p < 1$ and unstable for $p > 1$. This fixed point is a saddle for $a > 3$ and $p < 1$. For $p < 1$, there are two other unstable fixed points located at $\mathbf{r}_\pm = (x_p, \pm\sqrt{1 - p})$. These two fixed points have eigenvalues $(2 - a, 3 - 2p)$, both being pure repellers for $a > 3$ and $p < 1$, as shown in Fig. 1(a). These two repellers collide with each other and with the saddle at $p = p_c = 1$ in a saddle-repeller pitchfork bifurcation with eigenvalue $+1$ at $p = 1$; they do not exist for $p > 1$. Thus, for $p > 1$, two tongues, symmetrically located with respect to the invariant subspace, open at $x = x_p$ allowing trajectories near $y = 0$ to escape to $|y| = \infty$. To understand why these occur, observe that the cubic term in the y dynamics guarantees that if $|y_n| > 1$ then $|y_{n+1}| > |y_n| > 1$. Once a trajectory reaches $|y| = 1$, its y value asymptotes to infinity rapidly. So $|y| = \infty$ can be regarded as the second attractor of Eq. (3) besides the chaotic attractor in the $y = 0$ plane (invariant subspace). To understand why tongues are formed, take an open set $(|y| > 1)$ intersecting the transverse unstable manifold of $\mathbf{x}_p$. By taking inverse images, this open set approaches $\mathbf{x}_p$ asymptotically. There are two inverse images, but we choose only the one with $x = x_p$. By continuity, it remains an open set. Its inverse images are a subset of a tongue that opens up at $\mathbf{x}_p$, as shown in Fig. 1(b). The tongues are the intersection of all escaping open sets [10].

For $\epsilon > 0$, trajectories can leave the chaotic attractor at $y = 0$ ($y = 0$ is no longer an invariant subspace) and, hence, the chaotic attractor becomes a chaotic saddle. Computation of Eq. (3) shows one interesting phenomenon. Because of the tongue structure formed at $p_c = 1$, only a very small fraction of the points at $y = 0$ diverges toward the $|y| = \infty$ attractor. The transient time can easily be longer than, say, 10^5 iterates even when $\epsilon = 0.014$. As ϵ decreases towards zero, the transient time increases drastically. For instance, at $\epsilon = 0.01$, the typical transient time is over 3×10^6 iterations. This indicates a unique consequence of the onset of riddling: Trajectories in the vicinity of the $y = 0$ attractor belonging to the basin of the $|y| = \infty$ attractor spend an extremely long transient time near the $y = 0$ attractor before they asymptote to $|y| = \infty$ [7]. This is a physically observable phenomenon accompanying the onset of riddling.

To get the scaling on how the transient time increases as the symmetry-breaking parameter ϵ approaches zero, we decrease ϵ from 0.04 and compute the average transient time τ for a large number of trajectories at $p = p_c$.

Specifically, for each value of ϵ, we randomly choose 1000 initial conditions with x_0 uniformly distributed in (0,1) and $y_0 = 0$. A trajectory is regarded as having escaped once it satisfies $y > 1$. The average transient time increases faster than exponential growth as ϵ decreases towards zero. We find that, for Eq. (3), τ scales with ϵ as

$$\tau \sim \exp[K\epsilon^{-2/3}], \tag{4}$$

where K is a constant to be determined shortly. This behavior is shown in Fig. 2 as a plot of $\log_{10}\tau$ vs $\epsilon^{-2/3}$ for $0.01 < \epsilon < 0.04$. The plot can be fitted by a straight line, implying Eq. (4). The scaling relation Eq. (4) indicates that, as $\epsilon \to 0$, the average transient time approaches infinity rapidly, a superpersistent chaotic transient behavior.

We now derive Eq. (4) analytically. The first step is to estimate, for $\epsilon \geq 0$, the size δ of the opening at $y = 0$ for a trajectory of transient lifetime T. Note that T depends on ϵ. Since the $y = 0$ attractor is chaotic, its maximum Lyapunov exponent λ is positive. Let $L_u = e^{\lambda} > 1$, which is the expanding rate of an infinitesimal vector in the x direction. Since the transient time is T, we have $(L_u)^T \delta < 1$, which gives

$$\delta < 1/(L_u)^T . \tag{5}$$

We next examine the probability that a trajectory falls into the tongue of size δ at $y = 0$ for $\epsilon \geq 0$. This probability is proportional to δ. The average time for a trajectory to fall into the tongue is

$$\tau \sim \delta^{-1} > (L_u)^T = \exp(\lambda T) . \tag{6}$$

The final step is to evaluate T, the time it takes for the trajectory to exit once it has fallen into the tongue. Near x_p, we have $\exp[-b(x - x_p)^2] \approx 1$. For initial conditions chosen at $y_0 = 0$, we have $y_n \geq \epsilon$ for $n \geq 1$. For small ϵ it takes many iterations for a trajectory to reach $y = 1$. Thus, the y dynamics within the tongue can be approximated by the differential equation $dy/dt = \epsilon + y^3$. This gives

$$\begin{aligned} T &= \int_0^1 \frac{dy}{\epsilon + y^3} \\ &= \epsilon^{-2/3}\left\{ \frac{\sqrt{3}}{3}\left[\tan^{-1}\frac{2 - \epsilon^{-1/3}}{\sqrt{3}\epsilon^{1/3}} - \frac{\pi}{6} \right] \right. \\ &\quad \left. + \frac{1}{6}\ln\frac{(\epsilon^{1/3} + 1)^2}{1 - \epsilon^{1/3} + \epsilon^{2/3}} \right\} \\ &= C\epsilon^{-2/3}, \end{aligned} \tag{7}$$

where $C = O(1)$ and $C \to \pi/3^{3/2}$ as $\epsilon \to 0$. Substituting this expression into Eq. (6), we obtain the lower bound of the average transient time,

$$\tau \sim \exp[C\lambda\epsilon^{-2/3}], \tag{8}$$

which is Eq. (4), where $K \equiv C\lambda$. Alternatively, instead of looking at the width of the tongue at $y = 0$ and $\epsilon > 0$, we could have estimated the width at height $y = d$ and $\epsilon = 0$. In this case, we get Eq. (1) with $\gamma = 2/3$.

Note that Eq. (4) is the lower bound for the average transient time because of the inequality in Eq. (6). The actual transient time could be longer than that predicted by Eq. (4). Thus, the exponent could be larger than 2/3. Note that this exponent is a consequence of the y^3 term in the y dynamics. If we replace the y^3 term by, say, a y^2 term, then the exponent would be 1/2. Thus, the exponent 2/3 in Eq. (8) is specific to our model system Eq. (3). However, the scaling relation Eq. (1) is general, with the exponent γ being positive.

The escaping behavior of trajectories, once they have fallen into the tongue, can be seen by monitoring their traces in the phase space before they reach $y = 1$. Since the tongues are supernarrow at $p = p_c$, it is numerically convenient to examine the case where $p > p_c$. Figure 3 shows the last 50 points for 600 trajectories before they reach $y = 1$, where $p = 1.18$ and $\epsilon = 0.005$. We see that there is a "mushroom-shape" (tongue) crowd of trajectory points in the phase space located above the fixed point $x_p \approx 0.7368$. The red curves in Fig. 3 indicate the envelope of the tongue. These curves can be derived analytically by considering the escaping dynamics in the vicinity of x_p [10]. After a trajectory falls into the tongue at x_p, they move inside the tongue to escape the $y = 0$ attractor. There are also many other narrower tongues in Fig. 3. These correspond to the preimages of the tongue at x_p— an infinite number of them, though of course the number is limited since we examine only 50 iterates before the exit, and the tongues become narrower very fast. Thus, immediately after the fixed point x_p loses its transverse stability, an infinite number of tongues open immediately, allowing trajectories in the vicinity of the $y = 0$ attractor to escape.

In summary, we have studied the fundamental bifurcation for riddling to occur in chaotic dynamical systems; namely, the riddling bifurcation is induced by the loss of transverse stability of an unstable periodic orbit embedded in the chaotic attractor in the invariant

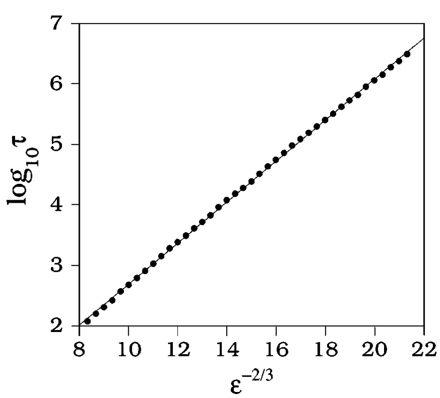

FIG. 2. Average transient time τ vs the symmetry-breaking parameter ϵ for $0.01 \leq \epsilon \leq 0.04$ at $p = p_c = 1$. We used 1000 random initial conditions with $x \in [0, 1]$ and $y = 0$ to compute τ. The parameter setting is $a = 3.8$ and $b = 5.0$ in Eq. (3). The plot is $\log_{10}\tau$ vs $\epsilon^{-2/3}$.

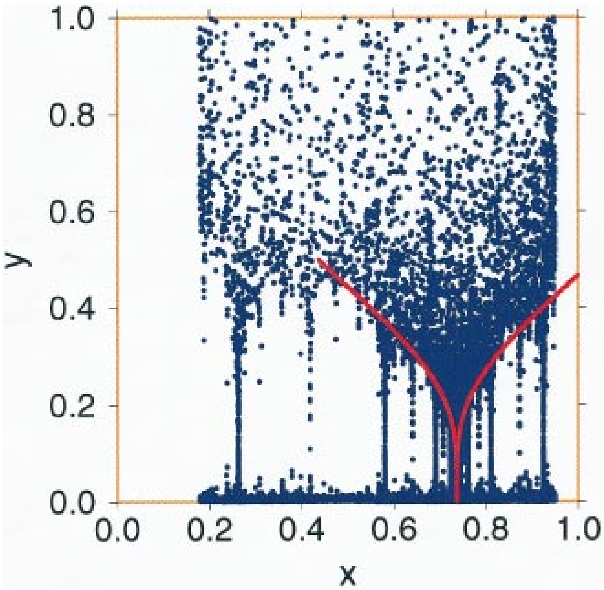

FIG. 3(color). Mushroom-shaped phase-space regions (tongues) through which trajectories escape the $y = 0$ chaotic attractor at $p = 1.18$ and $\epsilon = 0.005$.

subspace. The most interesting consequence accompanying riddling is the occurrence of a superpersistent chaotic transient behavior [11]. The basin structure of the attractor not in the invariant subspace is made up of an open and dense set of tongues. We stress that the model system Eq. (3), in which we rigged the unstable fixed point to lose its transverse stability first, is only for the purpose of illustrating the fundamental mechanism for riddling to occur and showing how symmetry breaking yields superpersistent transients. For more complicated systems, it is difficult to determine which unstable periodic orbits would lose transverse stability first. In all examples we have studied, it is a low-period periodic orbit, but we have no proof that this is the generic case.

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[10] After a trajectory falls into the escaping channel located at x_p, it dynamics can be approximated by (1) $(x_{n+1} - x_p) \approx (2 - a)(x_n - x_p)$ and (2) $y_{n+1} \approx \epsilon + p y_n + y^3$. Let $z_n \equiv |x_n - x_p|$; the x dynamics becomes $z_{n+1} = |2 - a| z_n = (a - 2) z_n$ (we only consider the case where a yields chaotic dynamics, $a > 3.6$). For p close to p_c and ϵ small, it takes a huge number of iterations for a typical trajectory to escape due to the long chaotic transient. It is thus an excellent approximation to describe the discrete dynamics inside the tongue via differential equations. The z and y dynamics become $dz/dt = \rho z$ (where $\rho = a - 3$) and $dy/dt = \epsilon + (p - 1)y + y^3$. We then obtain a formula for the edge of the tongue when $\epsilon = 0$ and $p > 1$,
$$z = \left(\frac{y}{\sqrt{(p - 1) + y^2}} \right)^{\rho/(p-1)} \quad \text{for } p > p_c = 1 .$$
The solid curves in Fig. 3 are $x_p \pm z$, respectively. Clearly, this is a good representation for the envelope of the tongue. In principle, one must consider additional terms such as cxy (c is constant) in the dy/dt equation, but analysis shows that this has a negligible effect on the results. In more general cases, when the system does not have a skew-product structure, one should also consider terms in the x equation such as ϵy, ϵy^2, or even higher-order terms in y. But for y small (near the invariant subspace) we have $\epsilon y \ll y$. Thus, these terms have a negligible effect on our conclusions.
[11] Superpersistent chaotic transients have also been identified in spatiotemporal chaotic systems [for example, J. P. Crutchfield and K. Kaneko, Phys. Rev. Lett. **60**, 2715 (1988); A. Hastings and K. Higgins, Science **263**, 1133 (1994); Y. C. Lai and R. L. Winslow, Phys. Rev. Lett. **74**, 5208 (1995)]. While we have investigated superpersistent chaotic transients associated with riddling bifurcation in this paper, the mechanism for spatiotemporal transients remains unknown.