RIEMANN-HILBERT PROBLEMS FOR MULTIPLE ORTHOGONAL POLYNOMIALS

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Abstract. In the early nineties, Fokas, Its and Kitaev observed that there is a natural Riemann-Hilbert problem (for 2×2 matrix functions) associated with a system of orthogonal polynomials. This Riemann-Hilbert problem was later used by Deift et al. and Bleher and Its to obtain interesting results on orthogonal polynomials, in particular strong asymptotics which hold uniformly in the complex plane. In this paper we will show that a similar Riemann-Hilbert problem (for $(r+1)\times(r+1)$ matrix functions) is associated with multiple orthogonal polynomials. We show how this helps in understanding the relation between two types of multiple orthogonal polynomials and the higher order recurrence relations for these polynomials. Finally we indicate how an extremal problem for vector potentials is important for the normalization of the Riemann-Hilbert problem. This ex-

tremal problem also describes the zero behavior of the multiple orthogonal polynomials.

1. Introduction

Recently it was observed that one can describe various aspects of the theory of orthogonal polynomials using a Riemann-Hilbert problem (Fokas, Its, Kitaev [12] [13] [16], Deift [6], Deift et al. [7] [8]). The Riemann-Hilbert problem is to find a complex 2×2 matrix valued function which is analytic in $\mathbb{C} \setminus \mathbb{R}$, having a prescribed growth as $z \to \infty$, which satisfies a jump condition when crossing the real line. The jump matrix contains the weight function w with respect to which the polynomials are orthogonal. In this paper we will show that an extension of this Riemann-Hilbert approach to $(r+1) \times (r+1)$ matrix valued functions describes certain polynomials which obey orthogonality conditions with respect to $r > 1$ weights on the real line. These polynomials are known as multiple orthogonal polynomials [2] [19] [21] [22] [26].

In this introduction we will explain the notion of multiple orthogonal polynomials. In Section 2 we describe the Riemann-Hilbert problem for type I multiple orthogonal polynomials and Section 3 gives the Riemann-Hilbert problem for type II multiple orthogonal polynomials. In Section 4 we will use these Riemann-Hilbert problems to give a relation between type I and type II multiple orthogonal polynomials. It is worth noting that one can also introduce various combinations of type I and type II multiple orthogonal polynomials, which can all be shown to be related. Section 5 shows that the type II multiple orthogonal polynomials satisfy some finite order recurrence relations, a (known) property which in this paper follows as a nice consequence of this Riemann-Hilbert approach. Finally, in Section 6 we show how to normalize the Riemann-Hilbert problem so that the growth condition for $z \to \infty$ is replaced by the condition that one obtains the identity matrix as $z \to \infty$. This normalization involves a vector of probability measures describing the asymptotic zero distribution of the multiple orthogonal polynomials, and this vector of measures solves an equilibrium problem in logarithmic potential theory. We will give a survey of how to obtain the asymptotic zero distribution for an Angelesco system (Section 6.1) and for a Nikishin system (Section 6.2) and indicate how this vector of equilibrium measures is used for the normalization of the Riemann-Hilbert problem. Strong asymptotics for multiple orthogonal polynomials has been obtained in a general setting for Angelesco systems [1] and Nikishin systems [3]. The Riemann-Hilbert approach gives an alternative way to study the asymptotics. This aspect, however, is outside the scope of this survey, but will be considered in forthcoming publications.

Multiple orthogonal polynomials are related to Hermite-Padé approximation to a system of Markov functions [21, Chapter 4]. Let $f_1, f_2, \ldots, f_r$ be r Markov functions, i.e.,

$$f_j(z) = \int_{\Delta_j} \frac{w_j(x)}{z-x}\, dx, \qquad z \notin \Delta_j,\ j = 1,2,\ldots,r,$$

where each Δ_j is a real interval. We will think of the weights w_j as weights on the real line such that $w_j(x) = 0$ for $x \notin \Delta_j$. We also assume that the limits

$$f_j^+(x) = \lim_{\epsilon \to 0+} f_j(x+i\epsilon), \quad f_j^-(x) = \lim_{\epsilon \to 0+} f_j(x-i\epsilon),$$

exist, so that the Sokhotsky-Plemelj formula holds

$$f_j^+(x) - f_j^-(x) = -2\pi i w_j(x), \qquad x \in \mathbb{R}. \tag{1.1}$$

1.1. TYPE I MULTIPLE ORTHOGONAL POLYNOMIALS

Multiple orthogonal polynomials are also known as poly-orthogonal polynomials or Hermite-Padé polynomials. Good references are Aptekarev [2], Mahler [19], the book by Nikishin and Sorokin [21], Nuttall [22], and [26]. Here we briefly explain what we mean by type I and type II multiple orthogonal polynomials, how they arise from a problem of simultaneous rational approximation, and state the orthogonality conditions.

Let $\vec{n} = (n_1, n_2, \ldots, n_r)$ be a multi-index in $\mathbb{N}^r$. For type I Hermite-Padé approximation we look for a vector of polynomials $\vec{A}_{\vec{n}} = (A_{\vec{n},1}, A_{\vec{n},2}, \ldots, A_{\vec{n},r})$, where $A_{\vec{n},j}$ has degree $n_j - 1$, and a polynomial $B_{\vec{n}}$ such that

$$\sum_{j=1}^r A_{\vec{n},j}(z) f_j(z) - B_{\vec{n}}(z) = \mathcal{O}(1/z^{n_1+n_2+\cdots+n_r}), \qquad z \to \infty. \tag{1.2}$$

The type I vector polynomial $(A_{\vec{n},1}, A_{\vec{n},2}, \ldots, A_{\vec{n},r})$ satisfies a number of orthogonality conditions, namely

$$\int x^k \sum_{j=1}^r A_{\vec{n},j}(x) w_j(x)\, dx = 0, \qquad k = 0,1,2,\ldots, n_1+n_2+\cdots+n_r-2. \tag{1.3}$$

This gives $n_1 + n_2 + \cdots + n_r - 1$ homogeneous equations for the $n_1 + n_2 + \cdots + n_r$ unknown coefficients, so that we can find the vector polynomial up to a common factor if the matrix of the linear system has full rank (the index $\vec{n}$ is called normal in this case). We will normalize the solution by imposing that

$$\int x^{n_1+n_2+\cdots+n_r-1} \sum_{j=1}^{r} A_{\vec{n},j}(x) w_j(x)\, dx = 1. \tag{1.4}$$

The solution will exist and be unique if the weights $w_1, w_2, \ldots, w_r$ or the functions $f_1, f_2, \ldots, f_r$ are sufficiently 'independent'. Some useful systems are

- Angelesco systems: in this case $\mathring{\Delta}_i \cap \mathring{\Delta}_j = \emptyset$ whenever $i \neq j$;
- AT systems: in this case $\Delta_j = \Delta$ for all j and

$$\begin{aligned} \{w_1, xw_1, \ldots, x^{n_1-1}w_1, w_2, xw_2, \ldots, x^{n_2-1}w_2, \\ \ldots, w_r, xw_r, \ldots, x^{n_r-1}w_r\} \end{aligned} \tag{1.5}$$

  is a Chebyshev system for every multi-index $(n_1, n_2, \ldots, n_r)$, i.e., every linear combination of the basis functions in (1.5) has at most $n_1 + n_2 + \cdots + n_r - 1$ zeros on Δ.

Both systems guarantee the existence and uniqueness of the type I multiple orthogonal polynomials. Mixtures of Angelesco and AT systems have been discussed recently [15]. The remaining polynomial $B_{\vec{n}}$ in (1.2) is given by

$$B_{\vec{n}}(z) = \int \sum_{j=1}^{r} \frac{A_{\vec{n},j}(z) - A_{\vec{n},j}(x)}{z-x} w_j(x)\, dx.$$

1.2. TYPE II MULTIPLE ORTHOGONAL POLYNOMIALS

Type II Hermite-Padé approximation consists of finding a polynomial $P_{\vec{n}}$ of degree $n_1 + n_2 + \cdots + n_r$ and polynomials $Q_{\vec{n},j}$ $(j = 1, 2, \ldots, r)$ such that

$$P_{\vec{n}}(z) f_j(z) - Q_{\vec{n},j}(z) = \mathcal{O}(1/z^{n_j+1}), \qquad z \to \infty,\ j = 1, 2, \ldots, r. \tag{1.6}$$

In this case we look for rational functions approximating $f_1, f_2, \ldots, f_r$ near infinity and with the same denominator $P_{\vec{n}}$. This denominator satisfies a number of orthogonality conditions and is known as a type II multiple orthogonal polynomial:

$$\int x^k P_{\vec{n}}(x) w_j(x)\, dx = 0, \qquad k = 0, 1, \ldots, n_j - 1,\ j = 1, 2, \ldots, r. \tag{1.7}$$

The orthogonality conditions are now distributed over the r weights. We have $n_1 + n_2 + \cdots + n_r$ homogeneous linear conditions for the $n_1 + n_2 + \cdots +$

n_r+1 unknown coefficients of $P_{\vec{n}}$. If the matrix of this linear system has full rank, then the index $\vec{n}$ is called normal and the multiple orthogonal polynomial is unique up to a constant factor. We will normalize the polynomial by taking it to be monic, i.e., with leading coefficient one. Again existence and uniqueness is guaranteed for Angelesco systems and AT systems. The numerator polynomials will be given by

$$Q_{\vec{n},j}(z) = \int \frac{P_{\vec{n}}(z) - P_{\vec{n}}(x)}{z-x} w_j(x)\, dx.$$

2. Riemann-Hilbert Problems for Type I Multiple Orthogonal Polynomials

Fokas, Its, and Kitaev [16] [12] [13] observed that there is a natural Riemann-Hilbert problem associated with a system of orthogonal polynomials with weight function $w(x)$ on the real line. Deift has described this Riemann-Hilbert problem and some of its applications in his lecture notes [6]. He and his collaborators [6] [7] and Bleher and Its [4] have used this idea to obtain many interesting results on orthogonal polynomials and random matrices. Here we want to extend this idea to the multiple orthogonal polynomials described in the previous section.

Theorem 2.1 *Consider the following Riemann-Hilbert problem: determine an $(r+1)\times(r+1)$ matrix function $Y(z)$ such that*

1. *$Y(z)$ is analytic in $\mathbb{C}\setminus\mathbb{R}$.*
2. *On the real line there is the jump condition*

$$Y^+(x) = Y^-(x) \begin{pmatrix} 1 & 0 & \cdots & 0 & -2\pi i w_1(x) \\ 0 & 1 & \cdots & 0 & -2\pi i w_2(x) \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & -2\pi i w_r(x) \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}, \tag{2.1}$$

   where $Y^\pm(x) = \lim_{\epsilon\to 0+} Y(x \pm i\epsilon)$.
3. *For $z\to\infty$ we have*

$$\lim_{z\to\infty} Y(z) \begin{pmatrix} z^{-n_1} & & & & 0 \\ & z^{-n_2} & & & \\ & & \ddots & & \\ & & & z^{-n_r} & \\ 0 & & & & z^{n_1+n_2+\cdots+n_r} \end{pmatrix} = I. \tag{2.2}$$

If the indices $\vec{n}$ *and* $\vec{n}+\vec{e}_k$ $(k = 1,2,\dots,r)$ *are normal, where* $\vec{e}_k = (0,\dots,0,1,0,\dots,0)$ *with* 1 *on the* k*th position, then the solution is unique and given by*

$$Y(z) = \begin{pmatrix} c_1^{-1}\vec{A}_{\vec{n}+\vec{e}_1}(z) & c_1^{-1}R_{\vec{n}+\vec{e}_1}(z) \\ c_2^{-1}\vec{A}_{\vec{n}+\vec{e}_2}(z) & c_2^{-1}R_{\vec{n}+\vec{e}_2}(z) \\ \vdots & \vdots \\ c_r^{-1}\vec{A}_{\vec{n}+\vec{e}_r}(z) & c_r^{-1}R_{\vec{n}+\vec{e}_r}(z) \\ \vec{A}_{\vec{n}}(z) & R_{\vec{n}}(z) \end{pmatrix} \tag{2.3}$$

where $\vec{A}_{\vec{n}}$ *is the vector containing the type I multiple orthogonal polynomials,*

$$R_{\vec{n}}(z) = \int \sum_{j=1}^{r} A_{\vec{n},j}(x) w_j(x)\, \frac{dx}{z-x},$$

and c_j *is the leading coefficient of* $A_{\vec{n}+\vec{e}_j,j}$*.*

Proof. Let us write the matrix Y as

$$Y = \begin{pmatrix} U & v \\ u^t & g \end{pmatrix}$$

where U is an $r \times r$ matrix function, v and u are column vector functions of size r (with u^t the transpose of u), and g is a complex function. The jump condition (2.1) then implies

$$U^+(x) = U^-(x),$$

so that U is analytic on the whole complex plane. The asymptotic condition (2.2) implies

$$\lim_{n\to\infty} U(z) \begin{pmatrix} z^{-n_1} & & & 0 \\ & z^{-n_2} & & \\ & & \ddots & \\ 0 & & & z^{-n_r} \end{pmatrix} = I,$$

so that each diagonal element $U_{k,k}(z)$ is a monic polynomial of degree n_k and each non-diagonal element $U_{k,j}(z)$ (with $k \neq j$) is a polynomial of degree at most $n_j - 1$. Here we used the fact that an entire function f for which $f(z)/z^n$ remains bounded as $z \to \infty$, is a polynomial of degree at most n (Liouville). For the vector u the jump condition (2.1) is

$$u^+(x) = u^-(x),$$

so that u is entire, and the asymptotic condition is

$$\lim_{n\to\infty} u_k(z)/z^{n_k} = 0, \qquad k = 1,\dots,r,$$

so that each $u_k(z)$ is a polynomial of degree at most $n_k - 1$. We will show that the polynomials in U and u satisfy the orthogonality conditions for type I multiple orthogonal polynomials. For this we use the jump condition (2.1) which for the vector v becomes

$$v_k^+(z) = v_k^-(z) - 2\pi i \sum_{j=1}^r U_{k,j}(x) w_j(x),$$

but then the Sokhotsky-Plemelj formula (1.1) implies that

$$v_k(z) = \int \sum_{j=1}^r \frac{U_{k,j}(x) w_j(x)}{z-x}\, dx, \qquad z \notin \mathbb{R}.$$

Use the expansion

$$\frac{1}{z-x} = \sum_{\ell=0}^{n-1} \frac{x^\ell}{z^{\ell+1}} + \frac{x^n}{z^n}\frac{1}{z-x}, \tag{2.4}$$

to find for any n

$$v_k(z) = \sum_{\ell=0}^{n-1} \frac{1}{z^{\ell+1}} \int x^\ell \sum_{j=1}^r U_{k,j}(x) w_j(x)\, dx + \frac{1}{z^n} \int x^n \sum_{j=1}^r \frac{U_{k,j}(x) w_j(x)}{z-x}\, dx. \tag{2.5}$$

The asymptotic condition (2.2) for v is

$$\lim_{z\to\infty} v_k(z) z^{n_1+n_2+\cdots+n_r} = 0, \qquad k = 1,2,\dots,r,$$

which combined with (2.5) implies

$$\int x^\ell \sum_{j=1}^r U_{k,j}(x) w_j(x)\, dx = 0, \qquad \ell = 0,1,\dots,n_1+n_2+\cdots+n_r-1.$$

Finally for the function g the jump condition (2.1) becomes

$$g^+(x) = g^-(x) - 2\pi i \sum_{j=1}^r u_j(x) w_j(x),$$

so that the Sokhotsky-Plemelj formula (1.1) implies that

$$g(z) = \int \sum_{j=1}^{r} \frac{u_j(x)w_j(x)}{z-x}\,dx, \qquad z \notin \mathbb{R}.$$

Again, this can be written as

$$g(z) = \sum_{\ell=0}^{n-1} \frac{1}{z^{\ell+1}} \int x^\ell \sum_{j=1}^{r} u_j(x)w_j(x)\,dx + \frac{1}{z^n} \int x^n \sum_{j=1}^{r} \frac{u_j(x)w_j(x)}{z-x}\,dx. \tag{2.6}$$

The asymptotic condition (2.2) for g becomes

$$\lim_{z\to\infty} g(z) z^{n_1+n_2+\cdots n_r} = 1,$$

which combined with (2.6) gives

$$\int x^\ell \sum_{j=1}^{r} u_j(x)w_j(x)\,dx = \begin{cases} 0, & \text{if } \ell = 0,1,2,\ldots, n_1+n_2+\cdots+n_r-2, \\ 1, & \text{if } \ell = n_1+n_2+\cdots+n_r-1. \end{cases}$$

If we compare all these orthogonality conditions with the orthogonality conditions (1.3) and (1.4) of type I multiple orthogonal polynomials, then we see that $u(z) = A_{\vec{n}}(z)$. Similarly, the kth row of $U(z)$ satisfies the orthogonality conditions (1.3) for $A_{\vec{n}+\vec{e}_k}(z)$, and the requirement that $U_{k,k}(z)$ is a monic polynomial of degree n_k shows that the appropriate normalizing factor is c_k^{-1}, where c_k is the leading coefficient of $A_{\vec{n}+\vec{e}_k,k}(z)$. □

3. Riemann-Hilbert Problems for Type II Multiple Orthogonal Polynomials

There is a similar Riemann-Hilbert problem for type II multiple orthogonal polynomials.

Theorem 3.1 *Consider the following Riemann-Hilbert problem: determine an $(r+1)\times(r+1)$ matrix function $Z(z)$ such that*

1. *$Z(z)$ is analytic in $\mathbb{C}\setminus\mathbb{R}$,*
2. *On the real line there is the jump condition*

$$Z^+(x) = Z^-(x) \begin{pmatrix} 1 & -2\pi i w_1(x) & -2\pi i w_2(x) & \cdots & -2\pi i w_r(x) \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \cdots & & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix} \tag{3.1}$$

3. For $z \to \infty$ we have

$$\lim_{z\to\infty} Z(z) \begin{pmatrix} z^{-n_1-n_2-\cdots-n_r} & & & & 0 \\ & z^{n_1} & & & \\ & & z^{n_2} & & \\ & & & \ddots & \\ 0 & & & & z^{n_r} \end{pmatrix} = I. \tag{3.2}$$

If the indices $\vec{n}$ and $\vec{n}-\vec{e}_k$ $(k = 1, 2 \dots, r)$ are normal, then the solution is unique and given by

$$Z(z) = \begin{pmatrix} P_{\vec{n}}(z) & \vec{R}_{\vec{n}}(z) \\ d_1 P_{\vec{n}-\vec{e}_1}(z) & d_1 \vec{R}_{\vec{n}-\vec{e}_1}(z) \\ d_2 P_{\vec{n}-\vec{e}_2}(z) & d_2 \vec{R}_{\vec{n}-\vec{e}_2}(z) \\ \vdots & \vdots \\ d_r P_{\vec{n}-\vec{e}_r}(z) & d_r \vec{R}_{\vec{n}-\vec{e}_r}(z) \end{pmatrix} \tag{3.3}$$

where $P_{\vec{n}}(z)$ is the type II multiple orthogonal polynomial and $\vec{R}_{\vec{n}} = (R_{\vec{n},1}, R_{\vec{n},2}, \dots, R_{\vec{n},r})$ is the vector containing

$$R_{\vec{n},j}(z) = \int P_{\vec{n}}(x) w_j(x) \, \frac{dx}{z-x},$$

and

$$\frac{1}{d_j} = \int x^{n_j-1} P_{\vec{n}-\vec{e}_j}(x) w_j(x) \, dx.$$

Proof. We write the matrix Z as

$$\begin{pmatrix} h & u^t \\ v & U \end{pmatrix},$$

where h is a complex function, u and v are column vectors of size r, and U is an $r \times r$ matrix function. The jump condition (3.1) implies that $h^+(x) = h^-(x)$ for $x \in \mathbb{R}$, hence h is an entire function. The asymptotic condition (3.2) shows that

$$\lim_{z\to\infty} h(z)/z^{n_1+\cdots+n_r} = 1,$$

hence $h(z)$ is a monic polynomial of degree $n_1+\cdots+n_r$. The jump condition (3.1) for u is

$$\begin{pmatrix} u_1^+(x) \\ u_2^+(x) \\ \vdots \\ u_r^+(x) \end{pmatrix} = \begin{pmatrix} -2\pi i h(x) w_1(x) + u_1^-(x) \\ -2\pi i h(x) w_2(x) + u_2^-(x) \\ \vdots \\ -2\pi i h(x) w_r(x) + u_r^-(x) \end{pmatrix},$$

hence for each $k \in \{1, 2, \dots, r\}$, the Sokhotsky-Plemelj formula gives

$$u_k(z) = \int \frac{h(x) w_k(x)}{z - x} \, dx.$$

If we use (2.4), then this gives

$$u_k(z) = \sum_{\ell=0}^{n_k - 1} \frac{1}{z^{\ell+1}} \int x^\ell h(x) w_k(x) \, dx + \frac{1}{z^{n_k}} \int x^{n_k} \frac{h(x) w_k(x)}{z - x} \, dx.$$

The asymptotic condition (3.2) gives

$$\lim_{z \to \infty} u_k(z) z^{n_k} = 0, \qquad k = 1, 2, \dots, r,$$

hence one needs to have

$$\int x^\ell h(x) w_k(x) \, dx = 0, \qquad \ell = 0, 1, \dots, n_k - 1,$$

so that the monic polynomial h of degree $n_1 + \ldots + n_r$ needs to satisfy the orthogonality conditions in (1.7). Hence h is the type II multiple orthogonal polynomial $P_{\vec{n}}$ and

$$u_k(x) = \int \frac{P_{\vec{n}}(x) w_k(x)}{z - x} \, dx.$$

This gives the first row of the matrix Z.

The jump condition (3.1) for the vector v is $v^+(x) = v^-(x)$, so that v is a vector of entire functions. The asymptotic condition (3.2) gives

$$\lim_{z \to \infty} v_k(z) / z^{n_1 + n_2 + \cdots + n_r} = 0,$$

so that each v_k $(k = 1, 2, \dots, r)$ is a polynomial of degree $< n_1 + n_2 + \cdots + n_r$. The jump condition (3.1) for the kth row of U is

$$\begin{pmatrix} U_{k,1}^+(x) \\ U_{k,2}^+(x) \\ \vdots \\ U_{k,r}^+(x) \end{pmatrix}^t = \begin{pmatrix} -2\pi i v_k(x) w_1(x) + U_{k,1}^-(x) \\ -2\pi i v_k(x) w_2(x) + U_{k,2}^-(x) \\ \vdots \\ -2\pi i v_k(x) w_r(x) + U_{k,r}^-(x) \end{pmatrix}^t.$$

Hence, Sokhotsky-Plemelj gives

$$U_{k,j}(z) = \int \frac{v_k(x) w_j(x)}{z - x} \, dx.$$

We can expand this using (2.4) to find

$$U_{k,j}(z) = \sum_{\ell=0}^{n_k-2} \frac{1}{z^{\ell+1}} \int x^\ell v_k(x) w_j(x)\, dx + \frac{1}{z^{n_k-1}} \int x^{n_k-1} \frac{v_k(x)w_j(x)}{z-x}\, dx.$$

The asymptotic condition (3.2) implies

$$\lim_{z\to\infty} U_{k,j}(z) z^{n_j} = \begin{cases} 1 & \text{if } k=j, \\ 0 & \text{if } k \neq j. \end{cases}$$

Therefore we have

$$\begin{aligned} \int x^\ell v_k(x) w_j(x)\, dx &= 0, \qquad \ell = 0,1,\ldots,n_k-1,\ j\neq k \\ \int x^\ell v_k(x) w_k(x)\, dx &= 0, \qquad \ell = 0,1,\ldots,n_k-2, \\ \int x^{n_k-1} v_k(x) w_k(x)\, dx &= 1. \end{aligned}$$

This means that the polynomial v_k satisfies the orthogonality conditions of a type II multiple orthogonal polynomial with index $\vec{n}-\vec{e}_k$ and hence, since the index $\vec{n}-\vec{e}_k$ is normal, $v_k(x) = d_k P_{\vec{n}-\vec{e}_k}(x)$. Here the normalizing constant d_k is such that

$$\int x^{n_k-1} d_k P_{\vec{n}-\vec{e}_k}(x) w_k(x)\, dx = 1,$$

which gives the required result. □

4. Relation between Type I and Type II Multiple Orthogonal Polynomials

Theorem 4.1 *Denote by Y the matrix function solving the Riemann-Hilbert problem for type I multiple orthogonal polynomials (Theorem 2.1), and by Z the matrix function solving the Riemann-Hilbert problem for type II multiple orthogonal polynomials (Theorem 3.1). Then, assuming $\vec{n}$ and $\vec{n}\pm\vec{e}_k$ $(k=1,\ldots,r)$ are normal indices,*

$$Z = \begin{pmatrix} \vec{0}^{\,t} & 1 \\ -I_r & \vec{0} \end{pmatrix} (Y^{-1})^t \begin{pmatrix} \vec{0} & -I_r \\ 1 & \vec{0}^{\,t} \end{pmatrix}, \tag{4.1}$$

where I_r is the identity matrix of order r and $\vec{0}$ is the column vector containing r zeros.

Proof. For simplification, we write $\vec{w} = -2\pi i(w_1, w_2, \dots, w_r)^t$ for the column vector containing the weights. The jump condition for Y then becomes

$$Y^+ = Y^- \begin{pmatrix} I_r & \vec{w} \\ \vec{0}^{\,t} & 1 \end{pmatrix}.$$

The inverse of Y exists since $\det Y$ is analytic on $\mathbb{C} \setminus \mathbb{R}$, $\det Y^+ = \det Y^-$ on the real line, so there is no jump and $\det Y$ is an entire function, and the asymptotic condition tells us that $\det Y$ is bounded and equal to 1 as $z \to \infty$. Hence $\det Y = 1$ everywhere. Taking inverse and transpose gives

$$[(Y^+)^{-1}]^t = [(Y^-)^{-1}]^t \begin{pmatrix} I_r & \vec{0} \\ -\vec{w}^{\,t} & 1 \end{pmatrix}.$$

Multiplying to the left and right by the appropriate matrices and using

$$\begin{pmatrix} \vec{0} & -I_r \\ 1 & \vec{0}^{\,t} \end{pmatrix} \begin{pmatrix} \vec{0}^{\,t} & 1 \\ -I_r & \vec{0} \end{pmatrix} = I_{r+1}$$

gives

$$\begin{pmatrix} \vec{0}^{\,t} & 1 \\ -I_r & \vec{0} \end{pmatrix} [(Y^+)^{-1}]^t \begin{pmatrix} \vec{0} & -I_r \\ 1 & \vec{0}^{\,t} \end{pmatrix} = \begin{pmatrix} \vec{0}^{\,t} & 1 \\ -I_r & \vec{0} \end{pmatrix} [(Y^-)^{-1}]^t \begin{pmatrix} \vec{0} & -I_r \\ 1 & \vec{0}^{\,t} \end{pmatrix} \begin{pmatrix} 1 & \vec{w}^{\,t} \\ \vec{0} & I_r \end{pmatrix},$$

where the last jump matrix is the one for the Riemann-Hilbert problem for Z. The asymptotic condition for Y is

$$\lim_{z\to\infty} Y(z) \begin{pmatrix} z^{-n_1} & & & & 0 \\ & z^{-n_2} & & & \\ & & \ddots & & \\ & & & z^{-n_r} & \\ 0 & & & & z^{n_1+n_2+\cdots+n_r} \end{pmatrix} = I_{r+1},$$

so that

$$\lim_{z\to\infty} \begin{pmatrix} \vec{0}^{\,t} & 1 \\ -I_r & \vec{0} \end{pmatrix} [Y(z)^{-1}]^t \begin{pmatrix} \vec{0} & -I_r \\ 1 & \vec{0}^{\,t} \end{pmatrix}$$
$$\times \begin{pmatrix} \vec{0}^{\,t} & 1 \\ -I_r & \vec{0} \end{pmatrix} \begin{pmatrix} z^{n_1} & & & & 0 \\ & z^{n_2} & & & \\ & & \ddots & & \\ & & & z^{n_r} & \\ 0 & & & & z^{-(n_1+n_2+\cdots+n_r)} \end{pmatrix} \begin{pmatrix} \vec{0} & -I_r \\ 1 & \vec{0}^{\,t} \end{pmatrix} = I_{r+1}.$$

An easy calculation gives

$$\begin{pmatrix} \vec{0}^{\,t} & 1 \\ -I_r & \vec{0} \end{pmatrix} \begin{pmatrix} z^{n_1} & & & & 0 \\ & z^{n_2} & & & \\ & & \ddots & & \\ & & & z^{n_r} & \\ 0 & & & & z^{-(n_1+n_2+\cdots+n_r)} \end{pmatrix} \begin{pmatrix} \vec{0} & -I_r \\ 1 & \vec{0}^{\,t} \end{pmatrix}$$
$$= \begin{pmatrix} z^{-(n_1+n_2+\cdots+n_r)} & & & & 0 \\ & z^{n_1} & & & \\ & & z^{n_2} & & \\ & & & \ddots & \\ 0 & & & & z^{n_r} \end{pmatrix},$$

so that we have the asymptotic condition for the Riemann-Hilbert problem for Z. This means that

$$\begin{pmatrix} \vec{0}^{\,t} & 1 \\ -I_r & \vec{0} \end{pmatrix} [Y(z)^{-1}]^t \begin{pmatrix} \vec{0} & -I_r \\ 1 & \vec{0}^{\,t} \end{pmatrix} = Z,$$

hence proving the theorem. □

A consequence of this theorem is a relationship between the multiple orthogonal polynomials of type I and type II, which is well-known (see, e.g., Mahler [19]). If we take the $(1,1)$-entry of Z then

$$Z_{1,1} = P_{\vec{n}}(z).$$

On the other hand, from (4.1) we also have

$$Z_{1,1} = (Y^{-1})_{r+1,r+1},$$

so that

$$P_{\vec{n}}(z) = \frac{1}{c_1 c_2 \cdots c_r} \det \begin{pmatrix} \vec{A}_{\vec{n}+\vec{e}_1}(z) \\ \vec{A}_{\vec{n}+\vec{e}_2}(z) \\ \vdots \\ \vec{A}_{\vec{n}+\vec{e}_r}(z) \end{pmatrix}. \tag{4.2}$$

5. Recurrence Relations

It is very well known that a system of orthogonal polynomials on the real line always satisfies a three-term recurrence relation (a second order linear recurrence relation). Multiple orthogonal polynomials also satisfy a number of recurrence relations. This was already known to Padé [23] and one can also find various recurrence relations in the work of Mahler [19]. In this section we show that such recurrences follow rather easily from the Riemann-Hilbert problem for these multiple orthogonal polynomials. In view of Theorem 4.1 it is sufficient to study the matrix Z for type II multiple orthogonal polynomials. We will use the $Z_{\vec{n}}$ to denote the matrix function for the multi-index $\vec{n}$ and assume that all indices involved are normal. Consider the matrix function

$$X_{\vec{n},k}(z) = Z_{\vec{n}}(z) Z_{\vec{n}-\vec{e}_k}^{-1}(z), \qquad k = 1, 2, \dots, r.$$

The inverse Z^{-1} always exists since $\det Z = 1$ in the complex plane. Clearly $X_{\vec{n},k}$ is analytic in $\mathbb{C} \setminus \mathbb{R}$ and since $Z_{\vec{n}}$ and $Z_{\vec{n}-\vec{e}_k}$ have the same jump condition on the real line, we see that $X_{\vec{n},k}$ has no jump condition and hence $X_{\vec{n},k}$ is an entire matrix function. For $z \to \infty$ it has the asymptotic behavior

$$\begin{aligned} X_{\vec{n},k}(z) &= Z_{\vec{n}} \begin{pmatrix} z^{-n_1-n_2-\cdots-n_r} & & & & & \\ & z^{n_1} & & & & \\ & & \ddots & & & \\ & & & z^{n_k} & & \\ & & & & \ddots & \\ & & & & & z^{n_r} \end{pmatrix} \\ &\times \begin{pmatrix} z & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1/z & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \\ &\times \begin{pmatrix} z^{n_1+\cdots+n_r-1} & & & & & \\ & z^{-n_1} & & & & \\ & & \ddots & & & \\ & & & z^{-n_k+1} & & \\ & & & & \ddots & \\ & & & & & z^{-n_r} \end{pmatrix} Z_{\vec{n}-\vec{e}_k}^{-1} \end{aligned}$$

$$= \quad (I + \frac{A(\vec{n})}{z} + \mathcal{O}(\frac{1}{z^2})) \begin{pmatrix} z & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1/z & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

$$\times (I + \frac{B(\vec{n}-\vec{e}_k)}{z} + \mathcal{O}(\frac{1}{z^2})), \tag{5.1}$$

where $A_{i,j}(\vec{n})_{0\le i,j\le r}$ and $B_{i,j}(\vec{n}-\vec{e}_k)_{0\le i,j\le r}$ are matrices independent of z but depending on $\vec{n}$ and k. Hence $X_{\vec{n},k}$ is a matrix polynomial of degree 1. Comparing the coefficients of z and of z^0 in (5.1) shows that

$$X_{\vec{n},k}(z) = \begin{pmatrix} z + A_{0,0} + B_{0,0} & B_{0,1} & B_{0,2} & \cdots & B_{0,k} & \cdots & & B_{0,r} \\ A_{1,0} & 1 & & & & & & \\ A_{2,0} & & \ddots & & & & & \\ \vdots & & & 1 & & & & \\ A_{k,0} & & & & 0 & & & \\ \vdots & & & & & 1 & & \\ & & & & & & \ddots & \\ A_{r,0} & & & & & & & 1 \end{pmatrix}. \tag{5.2}$$

Using this expression for X, the entry in the first row and first column in the equation

$$X_{\vec{n},k}(z) Z_{\vec{n}-\vec{e}_k} = Z_{\vec{n}} \tag{5.3}$$

gives

$$\begin{aligned} P_{\vec{n}}(z) &= [z + A_{0,0}(\vec{n}) + B_{0,0}(\vec{n}-\vec{e}_k)] P_{\vec{n}-\vec{e}_k}(z) \\ &\quad + \sum_{j=1}^{r} B_{0,j}(\vec{n}) d_j(\vec{n}-\vec{e}_k) P_{\vec{n}-\vec{e}_k-\vec{e}_j}(z). \end{aligned} \tag{5.4}$$

The entry in the first column and on row $j+1$ of (5.3) gives

$$\begin{aligned} d_j(\vec{n}) P_{\vec{n}-\vec{e}_j}(z) &= A_{j,0}(\vec{n}) P_{\vec{n}-\vec{e}_k}(z) + d_j(\vec{n}-\vec{e}_k) P_{\vec{n}-\vec{e}_k-\vec{e}_j}(z), \qquad j \neq k \\ d_k(\vec{n}) P_{\vec{n}-\vec{e}_k}(z) &= A_{k,0}(\vec{n}) P_{\vec{n}-\vec{e}_k}(z). \end{aligned} \tag{5.5}$$

The latter equation shows that $A_{k,0}(\vec{n}) = d_k(\vec{n})$. Summarizing we have

Theorem 5.1 *Type II multiple orthogonal polynomials satisfy the following recurrence relation*

$$\begin{aligned}(z + A_{0,0}(\vec{n}+\vec{e}_k) + B_{0,0}(\vec{n}))P_{\vec{n}}(z) &= P_{\vec{n}+\vec{e}_k}(z) \\ &- \sum_{j=1}^{r} B_{0,j}(\vec{n}+\vec{e}_k)d_j(\vec{n})P_{\vec{n}-\vec{e}_j}(z).\end{aligned} \tag{5.6}$$

Proof. This follows by replacing $\vec{n}$ by $\vec{n}+\vec{e}_k$ in (5.4). □

Corollary 1 *Type II multiple orthogonal polynomials also satisfy the recurrence relation*

$$(z - a_0(\vec{n}))P_{\vec{n}}(z) = P_{\vec{n}+\vec{e}_1}(z) + \sum_{j=1}^{r} a_j(\vec{n})P_{\vec{n}-\vec{e}_{r-j+1}-\cdots-\vec{e}_r}(z), \tag{5.7}$$

for certain coefficients $a_j(\vec{n})$ $(j = 0, 1, \ldots, r)$.

Proof. Take $k = 1$ in (5.6). Then

$$\begin{aligned}(z + A_{0,0}(\vec{n}+\vec{e}_1) + B_{0,0}(\vec{n}))P_{\vec{n}}(z) &= P_{\vec{n}+\vec{e}_1}(z) \\ &- \sum_{j=1}^{r} B_{0,j}(\vec{n}+\vec{e}_1)d_j(\vec{n})P_{\vec{n}-\vec{e}_j}(z).\end{aligned} \tag{5.8}$$

If we use (5.5) with $k = j + 1$, then

$$P_{\vec{n}-\vec{e}_j}(z) = P_{\vec{n}-\vec{e}_{j+1}}(z) + c_j(\vec{n})P_{\vec{n}-\vec{e}_j-\vec{e}_{j+1}}(z), \tag{5.9}$$

with $c_j(\vec{n})$ some constant. More general, for $0 \le k \le r - j - 1$ we have

$$P_{\vec{n}-\vec{e}_j-\cdots-\vec{e}_{j+k}}(z) = P_{\vec{n}-\vec{e}_{j+1}-\cdots-\vec{e}_{j+k+1}}(z) + c_{j,k}(\vec{n})P_{\vec{n}-\vec{e}_j-\cdots-\vec{e}_{j+k+1}}(z),$$

which follows from (5.5) with $\vec{n}$ replaced by $\vec{n} - \vec{e}_{j+1} - \cdots - \vec{e}_{j+k}$ and k replaced by $j + k + 1$. Repeated application of these identities then shows that $P_{\vec{n}-\vec{e}_j}(z)$ is a linear combination of the form

$$P_{\vec{n}-\vec{e}_j}(z) = \sum_{i=j}^{r} \hat{c}_{i,j}(\vec{n})P_{\vec{n}-\vec{e}_i-\cdots-\vec{e}_r}(z).$$

If we insert this into (5.8), then we get the required recurrence relation. □

6. Vector Potentials and g-Functions

The Riemann-Hilbert problems described in Theorems 2.1 and 3.1 are not normalized in the sense that conditions (2.2) and (3.2) impose some growth condition as $z \to \infty$. In order to normalize the Riemann-Hilbert problem, we would like to modify Y and Z in such a way that we get another Riemann-Hilbert problem with the same contours (but possibly different jump conditions) for which the solution tends to the identity matrix as $z \to \infty$. For the normalization we need to take into account the behavior of $Y(z)$ or $Z(z)$ for large z. This depends heavily on the distribution of the zeros of the multiple orthogonal polynomial. The zero distribution of orthogonal polynomials is usually given by an extremal problem in logarithmic potential theory [24] [25]. For multiple orthogonal polynomials one needs to study an extremal problem for vector potentials [14] [21]. Suppose μ and ν are two probability measures on the real line and define the *(logarithmic) energy* of μ by

$$I(\mu) = \iint \log \frac{1}{|x-y|} \, d\mu(x) \, d\mu(y), \tag{6.1}$$

and the *mutual energy* of μ and ν by

$$I(\mu,\nu) = \iint \log \frac{1}{|x-y|} \, d\mu(x) \, d\nu(y). \tag{6.2}$$

These quantities are bounded from below if the support of μ and ν is compact, but the energies can be $+\infty$. Observe that $I(\mu) = I(\mu,\mu)$. Let $C = (c_{i,j})_{i,j=1}^r$ be a positive definite matrix of order r. Then the extremal problem is to minimize

$$I(\vec{\mu}) = \sum_{i=1}^r \sum_{j=1}^r c_{i,j} I(\mu_i,\mu_j) \tag{6.3}$$

over all vectors $\vec{\mu} = (\mu_1,\ldots,\mu_r)$ of probability measures μ_i supported on given compact sets Δ_i $(i = 1,2,\ldots,r)$. Under fairly weak conditions on the sets Δ_i, this minimum is finite and attained at a (unique) vector of measures $\vec{\nu} = (\nu_1,\ldots,\nu_r)$, which is called the equilibrium. The support of ν_i will be denoted by Δ_i^* and is a subset of Δ_i. The solution of this minimization problem can also be described using *(logarithmic) potentials*

$$U(x;\mu_i) = \int_{\Delta_i} \log \frac{1}{|x-y|} \, d\mu_i(y). \tag{6.4}$$

Indeed, the variational conditions are the following: there exist constants $F_1,\ldots,F_r$ (Lagrange multipliers) such that

$$\sum_{i=1}^r c_{i,j} U(x;\nu_i) = F_j, \qquad x \in \Delta_j^*, \tag{6.5}$$

$$\sum_{i=1}^{r} c_{i,j} U(x;\nu_i) \quad \geq \quad F_j, \qquad x \in \Delta_j, \tag{6.6}$$

holds for $j = 1, 2, \ldots, r$. The precise form of the positive definite matrix C depends on the problem at hand.

6.1. ANGELESCO SYSTEMS

Suppose that each Δ_i is a finite interval and that the open intervals are disjoint: $\overset{\circ}{\Delta}_i \cap \overset{\circ}{\Delta}_j = \emptyset$ whenever $i \neq j$. In view of the connection between type I and type II multiple orthogonal polynomials, given by Theorem 4.1, we can limit the discussion to type II multiple orthogonal polynomials, which is the most convenient for Angelesco systems. Let $P_{\vec{n}}(x)$ be the type II multiple orthogonal polynomials for weights $(w_1, \ldots, w_r)$ defined on the intervals $(\Delta_1, \ldots, \Delta_r)$. The orthogonality relations on Δ_i imply that $P_{\vec{n}}$ has at least n_i zeros on $\overset{\circ}{\Delta}_i$. Indeed, suppose that $P_{\vec{n}}$ has $m < n_i$ sign changes on Δ_i at the points $x_1, \ldots, x_m$. Let $Q_m(x) = (x - x_1) \cdots (x - x_m)$. Then $P_{\vec{n}}(x) Q_m(x)$ does not change sign on $\overset{\circ}{\Delta}_i$, but since $m < n_i$, the orthogonality on Δ_i implies

$$\int_{\Delta_i} P_{\vec{n}}(x) Q_m(x) w_i(x)\, dx = 0,$$

which is not possible. This contradiction shows that $P_{\vec{n}}$ has at least n_i zeros on $\overset{\circ}{\Delta}_i$. But all these open intervals are disjoint and the degree of $P_{\vec{n}}$ is $n_1 + \cdots + n_r$. Hence $P_{\vec{n}}$ has precisely n_i simple zeros on each open interval $\overset{\circ}{\Delta}_i$. This means that

$$P_{\vec{n}}(x) = q_{n_1,1}(x) \cdots q_{n_r,r}(x),$$

where each $q_{n_i,i}$ has n_i simple zeros on $\overset{\circ}{\Delta}_i$. The orthogonality relation

$$\int_{\Delta_i} x^k q_{n_i,i}(x) \prod_{j \neq i} q_{n_j,j}(x) \; w_i(x)\, dx = 0, \qquad k = 0, \ldots, n_i - 1,$$

means that $q_{n_i,i}$ is the (monic) orthogonal polynomial of degree n_i for the weight

$$\prod_{j \neq i} |q_{n_j,j}(x)| \; w_i(x)$$

on Δ_i. This means that $q_{n_i,i}$ minimizes the integral

$$\int_{\Delta_i} |q(x)|^2 \prod_{j \neq i} |q_{n_j,j}(x)| \; w_i(x)\, dx \tag{6.7}$$

over all monic polynomials q of degree n_i. It is known that the distribution of the zeros of orthogonal polynomials with varying weights is given by the solution of an equilibrium problem with external field [24]. Suppose $\{x_{j,n_i},\ j = 1,\dots,n_i\}$ are the zeros of $q_{n_i,i}$, which are all on Δ_i, and let

$$\mu(q_{n_i,i}) = \frac{1}{n_i}\sum_{j=1}^{n_i}\delta(x_{j,n_i})$$

be the distribution of these n_i zeros, where $\delta(c)$ is the Dirac measure at c. Observe that we can write

$$|q_{n_i,i}(x)| = \prod_{j=1}^{n_i}|x - x_{j,n_i}| = \exp[-n_i U(x;\mu(q_{n_i,i}))].$$

If we set $\mu_i = \mu(q_{n_i,i})$, then the integrand in (6.7) becomes

$$\exp\left(-2n_i U(x;\mu_i) - \sum_{j\neq i} n_j U(x;\mu_j) + \log w_i(x)\right). \tag{6.8}$$

We want to minimize the integral of this on Δ_i over all probability measures μ_i supported on Δ_i. We will instead minimize the maximum on Δ_i over all probability measures μ_i supported on Δ_i. Assume that

$$\lim_{|\vec{n}|\to\infty}\frac{n_i}{|\vec{n}|} = p_i > 0, \qquad i = 1,\dots,r,$$

where $p_1 + \cdots + p_r = 1$, and that $w_i(x) > 0$ on Δ_i, then, as each $n_i \to \infty$, minimizing the maximum of (6.8) on Δ_i is equivalent with maximizing

$$\inf_{x\in\Delta_i}\left(2p_i U(x;\mu_i) + \sum_{j\neq i} p_j U(x;\mu_j)\right).$$

The variational conditions for this extremal problem are

$$2p_i U(x;\nu_i) + \sum_{j\neq i} p_j U(x;\mu_j) = \ell_i, \qquad x\in\Delta_i^*,$$
$$2p_i U(x;\nu_i) + \sum_{j\neq i} p_j U(x;\mu_j) \geq \ell_i, \qquad x\in\Delta_i,$$

where ℓ_i is some constant and Δ_i^* is the support of the extremal measure ν_i. Now we have to do this for every $i \in \{1,2,\dots,r\}$, which finally gives us the variational conditions

$$2p_i U(x;\nu_i) + \sum_{j\neq i} p_j U(x;\nu_j) = \ell_i, \qquad x\in\Delta_i^*, \tag{6.9}$$

$$2p_iU(x;\nu_i) + \sum_{j\neq i} p_jU(x;\nu_j) \geq \ell_i, \qquad x \in \Delta_i, \tag{6.10}$$

for $i = 1, 2, \dots, r$. This equilibrium problem therefore corresponds to the vector equilibrium problem for the interaction matrix

$$C = \begin{pmatrix} 2p_1^2 & p_1p_2 & p_1p_3 & \cdots & p_1p_r \\ p_2p_1 & 2p_2^2 & p_2p_3 & \cdots & p_2p_r \\ \vdots & & \ddots & & \cdots \\ p_rp_1 & p_rp_2 & \cdots & p_rp_{r-1} & 2p_r^2 \end{pmatrix}, \tag{6.11}$$

and Lagrange multipliers $F_i = p_i\ell_i$. If we denote the intervals by $\Delta_i = [a_i, b_i]$, then one knows [21] that the supports of ν_i are again intervals which we denote by $\Delta_i^* = [a_i^*, b_i^*] \subset [a_i, b_i]$.

The normalization of the Riemann-Hilbert problem for an Angelesco system now is as follows. Let $(\nu_1, \dots, \nu_r)$ be the vector of equilibrium measures satisfying the variational conditions (6.9)–(6.10), and let $\vec{n}$ be a multi-index such that

$$n_k = np_k \in \mathbb{N}, \qquad k = 1, \dots, r.$$

This can be done if all $p_k = a_k/b_k$ are rational and n is a multiple of the least common multiple of the denominators $b_1, \dots, b_r$. Observe that $|\vec{n}| = n$. Define

$$g_k(z) = \int_{a_k^*}^{b_k^*} \log(z-y)\, d\nu_k(y), \qquad k = 1, \dots, r. \tag{6.12}$$

Then

$$\begin{aligned} g_k^+(x) &= -U(x;\nu_k), && x \geq b_k^*, \\ g_k^+(x) &= -U(x;\nu_k) + i\pi, && x \leq a_k^*, \\ g_k^+(x) &= -U(x;\nu_k) + \varphi_k(x), && x \in [a_k^*, b_k^*], \end{aligned}$$

where

$$\varphi_k(x) = i\pi \int_x^{b_k^*} d\nu_k(y).$$

Similarly

$$\begin{aligned} g_k^-(x) &= -U(x;\nu_k), && x \geq b_k^*, \\ g_k^-(x) &= -U(x;\nu_k) - i\pi, && x \leq a_k^*, \\ g_k^-(x) &= -U(x;\nu_k) - \varphi_k(x), && x \in [a_k^*, b_k^*]. \end{aligned}$$

It will be convenient to write the variational conditions (6.9)–(6.10) as

$$-p_k[g_k^+(x) + g_k^-(x)] - \sum_{j\neq k} p_jg_j^-(x) = \ell_k^*, \qquad x \in \Delta_k^*, \tag{6.13}$$

$$-p_k[g_k^+(x)+g_k^-(x)]-\sum_{j\neq k}p_jg_j^-(x)-\ell_k^*\geq 0, \qquad x\in\Delta_k, \tag{6.14}$$

where $\ell_k^*-\ell_k$ is an integer multiple of $i\pi/n$. If Z is the solution of the Riemann-Hilbert problem given by Theorem 3.1, then we define

$$\begin{aligned} M(z) &= \begin{pmatrix} 1 & & & & & \\ & e^{-n\ell_1^*} & & & & \\ & & \ddots & & & \\ & & & e^{-n\ell_k^*} & & \\ & & & & \ddots & \\ & & & & & e^{-n\ell_r^*} \end{pmatrix} Z(z) \\ &\times \begin{pmatrix} e^{-n_1g_1(z)-\cdots-n_rg_r(z)} & & & & & \\ & e^{n_1g_1(z)} & & & & \\ & & \ddots & & & \\ & & & e^{n_kg_k(z)} & & \\ & & & & \ddots & \\ & & & & & e^{n_rg_r(z)} \end{pmatrix} \\ &\times \begin{pmatrix} 1 & & & & & \\ & e^{n\ell_1^*} & & & & \\ & & \ddots & & & \\ & & & e^{n\ell_k^*} & & \\ & & & & \ddots & \\ & & & & & e^{n\ell_r^*} \end{pmatrix}. \end{aligned} \tag{6.15}$$

Then M is analytic on $\mathbb{C}\setminus\bigcup_{i=1}^r[a_i,b_i]$. Let us write the matrix product in (6.15) as

$$M(z)=LZ(z)G(z)L^{-1}.$$

Then

$$\lim_{z\to\infty}M(z)=L\left[\lim_{z\to\infty}Z(z)G(z)\right]\ L^{-1},$$

and since

$$\lim_{z\to\infty}(g_k(z)-\log z)=0, \qquad k=1,\dots,r,$$

the growth condition (3.2) implies that

$$\lim_{z\to\infty}M(z)=I, \tag{6.16}$$

which shows that the Riemann-Hilbert problem for M is normalized. In order to find the jump on the interval Δ_k, we compute

$$M^+(x)=LZ^+(x)G^+(x)L^{-1}.$$

Recall that the jump for the original Riemann-Hilbert problem on $\Delta_k = [a_k, b_k]$ is

$$Z^+(x) = Z^-(x) \begin{pmatrix} 1 & -2\pi i w_k(x) \vec{e}_k^{\,t} \\ 0 & I \end{pmatrix}, \qquad x \in [a_k, b_k],$$

where $\vec{e}_k^{\,t} = (0, 0, \dots, 0, 1, 0, \dots, 0)$ is the kth unit vector in $\mathbb{R}^r$. Hence we have

$$M^+(x) = LZ^-(x) \begin{pmatrix} 1 & -2\pi i w_k(x) \vec{e}_k^{\,t} \\ 0 & I \end{pmatrix} G^+(x) L^{-1}, \qquad x \in [a_k, b_k].$$

Use $M^-(x) = LZ^-(x) G^-(x) L^{-1}$, then

$$M^+(x) = M^-(x) L [G^-(x)]^{-1} \begin{pmatrix} 1 & -2\pi i w_k(x) \vec{e}_k^{\,t} \\ 0 & I \end{pmatrix} G^+(x) L^{-1}, \qquad x \in [a_k, b_k]. \tag{6.17}$$

First we compute the matrix product

$$[G^-(x)]^{-1} \begin{pmatrix} 1 & -2\pi i w_k(x) \vec{e}_k^{\,t} \\ 0 & I \end{pmatrix} G^+(x)$$

$$= \begin{pmatrix} e^{\sum_{j=1}^r n_j [g_j^-(x) - g_j^+(x)]} & 0 & \cdots & a(x) & \cdots & 0 \\ & e^{n_1 [g_1^+(x) - g_1^-(x)]} & & & & \\ & & \ddots & & & \\ & & & e^{n_k [g_k^+(x) - g_k^-(x)]} & & \\ & & & & \ddots & \\ & & & & & e^{n_r [g_r^+(x) - g_r^-(x)]} \end{pmatrix},$$

where

$$a(x) = -2\pi i w_k(x) e^{n_k [g_k^+(x) + g_k^-(x)] + \sum_{j \neq k} n_j g_j^-(x)}.$$

Observe that on Δ_k^* we have

$$e^{g_j^+(x) - g_j^-(x)} = \begin{cases} 1 & \text{if } j \neq k, \\ 2\varphi_k(x) & \text{if } j = k, \end{cases}$$

and that, taking into account (6.13) we also have

$$a(x) = -2\pi i w_k(x) e^{-n\ell_k^*}.$$

This means that the jump on Δ_k^* is given by

$$L [G^-(x)]^{-1} \begin{pmatrix} 1 & -2\pi i w_k(x) \vec{e}_k^{\,t} \\ 0 & I \end{pmatrix} G^+(x) L^{-1}$$

$$= \begin{pmatrix} e^{-2n_k\varphi_k(x)} & 0 & \cdots & 0 & -2\pi i w_k(x) & 0 & \cdots & 0 \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ & & & & e^{2n_k\varphi_k(x)} & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix}, \qquad x \in \Delta_k^*. \tag{6.18}$$

On $\Delta_k \setminus \Delta_k^*$ we have

$$e^{g_j^+(x)-g_j^-(x)} = 1, \qquad j = 1, \dots, r,$$

and (6.14) gives

$$a(x) = -2\pi i w_k(x) e^{-n\ell_k^* - nV_k(x)},$$

where $V_k(x) \geq 0$. Hence on $\Delta_k \setminus \Delta_k^*$ the jump is

$$L[G^-(x)]^{-1} \begin{pmatrix} 1 & -2\pi i w_k(x)\vec{e}_k^{\,t} \\ 0 & I \end{pmatrix} G^+(x) L^{-1}$$

$$= \begin{pmatrix} 1 & 0 & \cdots & 0 & -2\pi i w_k(x) e^{-nV_k(x)} & 0 & \cdots & 0 \\ & \ddots & & & & & & \\ & & & & 1 & & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix}, \qquad x \in \Delta_k \setminus \Delta_k^*. \tag{6.19}$$

The normalized Riemann-Hilbert matrix $M(z)$ therefore has rather simple jumps on the intervals Δ_k. On the supports of the measures ν_k the jump (6.18) has oscillatory terms $e^{\pm 2n_k\varphi_k(x)}$ on the diagonal (recall that $\varphi_k(x)$ is purely imaginary), and on $\Delta_k \setminus \Delta_k^*$ the jump (6.19) is the identity matrix, except for one entry on the first row which decreases exponentially fast as $n \to \infty$ provided $V_k(x) > 0$, which will be typically the case. Such a Riemann-Hilbert problem with oscillatory and exponentially decreasing jumps can be analysed asymptotically by using the steepest descent method introduced by Deift and Zhou [9] [10]. We will apply this deepest descent method to some relevant cases in future contributions, since this would be outside of the scope of this survey.

6.2. NIKISHIN SYSTEMS

When all the weight functions $w_1, \dots, w_r$ are defined on the same interval Δ_r, then one needs appropriate conditions on these weights in order to guarantee that a multi-index $\vec{n}$ is normal. An interesting construction was suggested by Nikishin [20], which in the book [21] is called an MT-system, but which is nowadays known as a Nikishin system. The construction is by induction. A Nikishin system of order 1 on Δ_1 consists of a weight function $w_{1,1}$ on an interval Δ_1 of the real line. A Nikishin system of order 2 on Δ_2 consists of weight functions $(w_{1,2}, w_{2,2})$ on an interval Δ_2, with $w_{1,2}$ a positive weight on Δ_2 and

$$w_{2,2}(x) = w_{1,2}(x) \int_{\Delta_1} \frac{w_{1,1}(t)}{x-t}\, dt, \qquad x \in \Delta_2, \tag{6.20}$$

where $\mathring{\Delta}_1 \cap \mathring{\Delta}_2 = \emptyset$ and $w_{1,1}$ is a Nikishin system of order 1 on Δ_1. In general, a Nikishin system of order r on Δ_r consists of weights $(w_{1,r}, \dots, w_{r,r})$ on Δ_r, such that $w_{1,r}$ is a positive weight on Δ_r and for $k = 2, 3, \dots, r$

$$w_{k,r}(x) = w_{1,r}(x) \int_{\Delta_{r-1}} \frac{w_{k-1,r-1}(t)}{x-t}\, dt, \qquad x \in \Delta_r, \tag{6.21}$$

where $\mathring{\Delta}_{r-1} \cap \mathring{\Delta}_r = \emptyset$ and $(w_{1,r-1}, \dots, w_{r-1,r-1})$ is a Nikishin system of order $r-1$ on Δ_{r-1}. Observe that each $w_{k,r}$ has constant sign on Δ_r since the intervals $\mathring{\Delta}_r$ and $\mathring{\Delta}_{r-1}$ are disjoint. For a Nikishin system of order r one knows that the weights $(w_{1,r}, \dots, w_{r,r})$ form an AT-system on Δ_r for the multi-indices $\vec{n} = (n_1, \dots, n_r)$ with

$$n_1 \ge n_2 \ge \cdots \ge n_r, \tag{6.22}$$

so that these multi-indices are normal. It is still an open problem (except for $r = 2$ [11] [5]) whether or not every multi-index is normal, but one already knows that there are more normal indices than given by (6.22) [11]. For Nikishin systems it is more convenient to work with the type I multiple orthogonal polynomials. We will assume that (6.22) holds so that the multi-index $\vec{n}$ is normal.

We will assume that $w_{1,r}(x) > 0$ on Δ_r. Consider the function

$$L_{\vec{n}}(x) = A_{\vec{n},1}(x) + \sum_{j=2}^{r} A_{\vec{n},j}(x) \int_{\Delta_{r-1}} \frac{w_{j-1,r-1}(t)}{x-t}\, dt. \tag{6.23}$$

Then

$$w_{1,r}(x) L_{\vec{n}}(x) = \sum_{j=1}^{r} A_{\vec{n},j}(x) w_{j,r}(x),$$

and $L_{\vec{n}}$ has exactly $|\vec{n}|-1$ sign changes on $\mathring{\Delta}_r$. Indeed, since the weights form an AT-system on Δ_r, we already know that $L_{\vec{n}}$ has at most $|\vec{n}|-1$ sign changes on $\mathring{\Delta}_r$. If there are $m<|\vec{n}|-1$ sign changes on $\mathring{\Delta}_r$, then we can form the polynomial Q_m with zeros at these points and $L_{\vec{n}}Q_m$ will not change sign on Δ_r. But the orthogonality (1.3) gives

$$\int_{\Delta_r} w_{1,r}(x)L_{\vec{n}}(x)Q_m(x)\,dx=0,$$

and this contradiction shows that there are exactly $|\vec{n}|-1$ sign changes on $\mathring{\Delta}_r$. Denote by $H_{\vec{n}}$ the monic polynomial of degree $|\vec{n}|-1$ with zeros at the points where $L_{\vec{n}}$ changes sign on $\mathring{\Delta}_r$. Then the ratio $L_{\vec{n}}(z)/H_{\vec{n}}(z)$ is analytic on $\mathbb{C}\setminus\Delta_{r-1}$ and of constant sign on Δ_r. It turns out that $H_{\vec{n}}$ is the monic orthogonal of degree $|\vec{n}|-1$ on Δ_r for the weight function

$$\frac{|L_{\vec{n}}(x)|}{|H_{\vec{n}}(x)|}\,w_{1,r}(x), \tag{6.24}$$

because

$$\int_{\Delta_r} H_{\vec{n}}(x)x^k\frac{|L_{\vec{n}}(x)|}{|H_{\vec{n}}(x)|}\,w_{1,r}(x)\,dx=0,\qquad k=0,1,\dots,|\vec{n}|-2,$$

which follows from the orthogonality (1.3). This means that the zero distribution of $H_{\vec{n}}$ will be given by an equilibrium problem with an external field given by

$$Q_1(x)=-\lim_{|\vec{n}|\to\infty}\frac{1}{|\vec{n}|}\log\frac{|L_{\vec{n}}(x)|}{|H_{\vec{n}}(x)|},\qquad x\in\Delta_r. \tag{6.25}$$

Observe that $L_{\vec{n}}(x)/H_{\vec{n}}(x)$ is analytic in $\overline{\mathbb{C}}\setminus\Delta_{r-1}$ and that

$$\frac{L_{\vec{n}}(z)}{H_{\vec{n}}(z)}=\mathcal{O}(z^{-(n_2+\cdots+n_r)}),\qquad z\to\infty,$$

where we have taken into account (6.22). If we choose a contour Γ that goes around Δ_{r-1} counterclockwise but which does not contain any points of Δ_r, then the residue theorem applied to the domain outside Γ gives

$$\frac{1}{2\pi i}\int_\Gamma \frac{L_{\vec{n}}(z)}{H_{\vec{n}}(z)}\,z^k\,dz=0,\qquad k=0,1,\dots,n_2+\cdots+n_r-2.$$

We can evaluate this contour integral as

$$\frac{1}{2\pi i}\int_\Gamma \frac{L_{\vec{n}}(z)}{H_{\vec{n}}(z)}\,z^k\,dz$$

$$= \frac{1}{2\pi i}\int_\Gamma \frac{A_{\vec{n},1}(z)}{H_{\vec{n}}(z)}\, z^k\, dz + \sum_{j=2}^{r} \frac{1}{2\pi i}\int_\Gamma \frac{A_{\vec{n},j}(z)}{H_{\vec{n}}(z)}\, z^k \int_{\Delta_{r-1}} \frac{w_{j-1,r-1}(t)}{z-t}\, dt\, dz.$$

The first integral on the right hand side vanishes by applying the Cauchy theorem for the domain inside Γ (since all the zeros of $H_{\vec{n}}$ are on Δ_r). Changing the order of integration for the other terms and using Cauchy's theorem for the domain inside Γ gives

$$\frac{1}{2\pi i}\int_\Gamma \frac{A_{\vec{n},j}(z)}{H_{\vec{n}}(z)}\, z^k \int_{\Delta_{r-1}} \frac{w_{j-1,r-1}(t)}{z-t}\, dt\, dz = \int_{\Delta_{r-1}} \frac{A_{\vec{n},j}(t)}{H_{\vec{n}}(t)}\, t^k w_{j-1,r-1}(t)\, dt$$

so that we get

$$\int_{\Delta_{r-1}} t^k \textstyle\sum_{j=2}^r A_{\vec{n},j}(t) w_{j-1,r-1}(t)\, \dfrac{dt}{H_{\vec{n}}(t)} = 0, \qquad k = 0, 1, \ldots, n_2 + \cdots + n_r - 2. \tag{6.26}$$

This means that $(A_{\vec{n},2}, \ldots, A_{\vec{n},r})$ is the type I multiple orthogonal polynomial of multi-index $(n_2, \ldots, n_r)$ for the Nikishin system $(w_{1,r-1}, \ldots, w_{r-1,r-1})/H_{\vec{n}}$ of order $r-1$ on Δ_{r-1}. This way we have reduced the problem by going down one order, and we can repeat the reasoning. For the external field Q_1 given by (6.25), we can use Cauchy's formula with a contour Γ going counterclockwise around Δ_{r-1}, but not around x, so that for $x \notin \Delta_{r-1}$

$$\frac{L_{\vec{n}}(x)}{H_{\vec{n}}(x)} = -\frac{1}{2\pi i}\int_\Gamma \frac{L_{\vec{n}}(z)}{H_{\vec{n}}(z)} \frac{dz}{z-x},$$

which gives

$$\begin{aligned} \frac{L_{\vec{n}}(x)}{H_{\vec{n}}(x)} &= -\frac{1}{2\pi i}\int_\Gamma \frac{A_{\vec{n},1}(z)}{H_{\vec{n}}(z)} \frac{dz}{z-x} \\ &\quad - \sum_{j=2}^{r} \frac{1}{2\pi i}\int_\Gamma \frac{A_{\vec{n},j}(z)}{H_{\vec{n}}(z)} \int_{\Delta_{r-1}} \frac{w_{j-1,r-1}(t)}{z-t}\, dt\, \frac{dz}{z-x}. \end{aligned}$$

The first integral on the right vanishes because of Cauchy's theorem, and by interchanging the order of integration, the remaining integrals give

$$\frac{L_{\vec{n}}(x)}{H_{\vec{n}}(x)} = \int_{\Delta_{r-1}} \frac{1}{x-t} \sum_{j=2}^{r} A_{\vec{n},j}(t) w_{j-1,r-1}(t) \frac{dt}{H_{\vec{n}}(t)}.$$

If we set

$$w_{1,r-1}(t) L_{n_2,\ldots,n_r}(t) = \sum_{j=2}^{r} A_{\vec{n},j}(t) w_{j-1,r-1}(t),$$

and define $H_{n_2,\dots,n_r}$ to be the polynomial of degree $n_2 + \cdots + n_r - 1$ with the sign changes of $L_{n_2,\dots,n_r}$ on Δ_{r-1}, then

$$\begin{aligned} \frac{L_{\vec{n}}(x)}{H_{\vec{n}}(x)} &= \int_{\Delta_{r-1}} \frac{1}{x-t} L_{n_2,\dots,n_r}(t) \frac{w_{1,r-1}(t)}{H_{\vec{n}}(t)}\, dt \\ &= \frac{1}{H_{n_2,\dots,n_r}(x)} \int_{\Delta_{r-1}} \frac{H_{n_2,\dots,n_r}(x)}{x-t} L_{n_2,\dots,n_r}(t) \frac{w_{1,r-1}(t)}{H_{\vec{n}}(t)}\, dt. \end{aligned}$$

If we write

$$\frac{H_{n_2,\dots,n_r}(x)}{x-t} = \frac{H_{n_2,\dots,n_r}(x) - H_{n_2,\dots,n_r}(t)}{x-t} + \frac{H_{n_2,\dots,n_r}(t)}{x-t},$$

then the first term on the right hand side is a polynomial in t of degree at most $n_2 + \cdots + n_r - 2$, and hence the orthogonality gives

$$\frac{L_{\vec{n}}(x)}{H_{\vec{n}}(x)} = \frac{1}{H_{n_2,\dots,n_r}(x)} \int_{\Delta_{r-1}} \frac{H_{n_2,\dots,n_r}(t)}{x-t} L_{n_2,\dots,n_r}(t) \frac{w_{1,r-1}(t)}{H_{\vec{n}}(t)}\, dt. \tag{6.27}$$

Observe that $w_{1,r-1}(t)L_{n_2,\dots,n_r}(t)H_{n_2,\dots,n_r}(t)/H_{\vec{n}}(t)$ does not change sign on Δ_{r-1} so that

$$\int_{\Delta_{r-1}} \frac{H_{n_2,\dots,n_r}(t)}{x-t} L_{n_2,\dots,n_r}(t) \frac{w_{1,r-1}(t)}{H_{\vec{n}}(t)}\, dt$$

is the Stieltjes transform (Markov function, Cauchy transform) of a positive weight, which we can turn into a probability weight after an appropriate normalization. The nth root of this Stieltjes function hence converges to 1 uniformly on compact subsets of $\mathbb{C} \setminus \Delta_{r-1}$. This means that the external field Q_1 in (6.25) depends only on the distribution of the zeros of $H_{n_2,\dots,n_r}$. Let

$$\nu_k = \lim_{|\vec{n}|\to\infty} \frac{1}{n_k + \cdots + n_r - 1} \sum_{j=1}^{n_k+\cdots+n_r-1} \delta(x_{j,n_k,\dots,n_r}), \qquad k = 1, \dots, r, \tag{6.28}$$

where $x_{j,n_k,\dots,n_r}$ are the zeros of $H_{n_k,\dots,n_r}$ (which are all on Δ_{r+1-k}), and assume that

$$\lim_{|\vec{n}|\to\infty} \frac{n_k + \cdots + n_r}{n_1 + \cdots + n_r} = q_{k-1}, \qquad k = 1, \dots, r, \tag{6.29}$$

with $q_0 = 1$. Then

$$Q_1(x) = -q_1 U(x; \nu_2), \qquad x \in \Delta_r. \tag{6.30}$$

Hence the distribution of the zeros of $H_{\vec{n}}$ is governed by the variational conditions

$$2U(x;\nu_1) - q_1 U(x;\nu_2) = \ell_1, \qquad x \in \Delta_r. \tag{6.31}$$

Now repeat the reasoning for the type I multiple orthogonal polynomials $(A_{\vec{n},2}, \ldots, A_{\vec{n},r})$ for the Nikishin system $(w_{1,r-1}, \ldots, w_{r-1,r-1})/H_{\vec{n}}$ on Δ_{r-1}. First of all, the polynomial $H_{n_2,\ldots,n_r}$ will be an orthogonal polynomial of degree $n_2 + \cdots + n_r - 1$ on Δ_{r-1} with weight function

$$\frac{|L_{n_2,\ldots,n_r}(x)|}{|H_{n_2,\ldots,n_r}(x)|} \frac{w_{1,r-1}(x)}{|H_{\vec{n}}(x)|}, \qquad x \in \Delta_{r-1}. \tag{6.32}$$

Hence the zero distribution ν_2 of $H_{n_2,\ldots,n_r}$ is governed by an equilibrium problem with external field

$$\begin{aligned} Q_2(x) &= -\lim_{|\vec{n}|\to\infty} \frac{1}{|\vec{n}|} \log \frac{|L_{n_2,\ldots,n_r}(x)|}{|H_{n_2,\ldots,n_r}(x)|} \frac{w_{1,r-1}(x)}{|H_{\vec{n}}(x)|} \\ &= -q_2 U(x;\nu_3) - U(x;\nu_1), \qquad x \in \Delta_{r-1}, \end{aligned} \tag{6.33}$$

where the last equality follows because

$$\begin{aligned} &\frac{L_{n_2,\ldots,n_r}(x)}{H_{n_2,\ldots,n_r}(x)} \\ &= \frac{1}{H_{n_3,\ldots,n_r}(x)} \int_{\Delta_{r-2}} \frac{H_{n_3,\ldots,n_r}(t)}{x-t} L_{n_3,\ldots,n_r}(t) \frac{w_{1,r-2}(t)}{H_{n_2,\ldots,n_r}(t) H_{\vec{n}}(t)}\, dt. \end{aligned}$$

The variational condition hence becomes

$$2q_1 U(x;\nu_2) - q_2 U(x;\nu_3) - U(x;\nu_1) = \ell_2, \qquad x \in \Delta_{r-1}. \tag{6.34}$$

The remaining vector $(A_{\vec{n},3}, \ldots, A_{\vec{n},r})$ consists of the type I multiple orthogonal polynomial for the Nikishin system $(w_{1,r-2}, \ldots, w_{r-2,r-2})/H_{n_2,\ldots,n_r}$ of order $r-2$ on Δ_{r-2}. In general the polynomial $H_{n_k,\ldots,n_r}$ with the sign changes of

$$L_{n_k,\ldots,n_r}(x) = A_{\vec{n},k}(x) + \sum_{j=k+1}^{r} A_{\vec{n},j}(x) \int_{\Delta_{r-k}} \frac{w_{j-k,r-k}(t)}{x-t}\, dt$$

on Δ_{r-k+1} has a zero distribution ν_k which is given by the variational condition

$$2q_{k-1} U(x;\nu_k) - q_k U(x;\nu_{k+1}) - q_{k-2} U(x;\nu_{k-1}) = \ell_k, \qquad x \in \Delta_{r-k+1}, \tag{6.35}$$

and this for $k=2,\dots,r-1$. For $k=1$ one has (6.31), which corresponds to $q_{-1}=0$, and for $k=r$ the polynomial H_{n_r} is precisely equal to the (monic version of) the polynomial $A_{\vec{n},r}$, which is orthogonal on Δ_1 with respect to the weight $w_{1,1}/H_{n_{r-1},n_r}$. Hence the zero distribution for ν_r is determined by

$$2q_{r-1}U(x;\nu_r)-q_{r-2}U(x;\nu_{r-1})=\ell_r, \qquad x\in\Delta_1, \tag{6.36}$$

which corresponds to (6.35) with $k=r$ and $q_r=0$. The numbers ℓ_k ($k=1,\dots,r$) in (6.35) are constants. All this means that the relevant vector potential problem for Nikishin systems has an interaction matrix of the form

$$C=\begin{pmatrix} 2q_0^2 & -q_0q_1 & & & & \\ -q_0q_1 & 2q_1^2 & -q_1q_2 & & & \\ & -q_1q_2 & 2q_2^2 & -q_2q_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -q_{r-3}q_{r-2} & 2q_{r-2}^2 & -q_{r-2}q_{r-1} \\ & & & & -q_{r-2}q_{r-1} & 2q_{r-1}^2 \end{pmatrix}, \tag{6.37}$$

and the Lagrange multipliers are $F_k=q_{k-1}\ell_k$. It turns out that for each $k=1,\dots,r$ the support of ν_k is the full interval Δ_k (see, e.g., [15]).

We will consider the Riemann-Hilbert problem of Theorem 2.1 for the matrix function Y which is analytic in $\mathbb{C}\setminus\Delta_r$, with the jump condition

$$Y^+(x)=Y^-(x)\begin{pmatrix}1 & 0 & \cdots & 0 & -2\pi i w_{r,r}(x) \\ 0 & 1 & \cdots & 0 & -2\pi i w_{r-1,r}(x) \\ \vdots & & \dots & & \vdots \\ 0 & \cdots & 0 & 1 & -2\pi i w_{1,r}(x) \\ 0 & \cdots & 0 & 0 & 1\end{pmatrix}, \qquad x\in\Delta_r, \tag{6.38}$$

and growth condition

$$\lim_{z\to\infty} Y(z)\begin{pmatrix} z^{-n_r} & & & & \\ & \ddots & & & \\ & & z^{-n_2} & & \\ & & & z^{-n_1} & \\ & & & & z^{n_1+\cdots+n_r}\end{pmatrix}=I. \tag{6.39}$$

Observe that, compared with Theorem 2.1, we have reversed the order of the weights, which means that for the solution Y we must also reverse the order of the indices in the multi-index $\vec{n}$, i.e.,

$$Y(z) = \begin{pmatrix} c_r^{-1}A_{\vec{n}+\vec{e}_r,r}(z) & \cdots & c_r^{-1}A_{\vec{n}+\vec{e}_r,1}(z) & c_r^{-1}R_{\vec{n}+\vec{e}_r}(z) \\ \vdots & \cdots & \vdots & \vdots \\ c_1^{-1}A_{\vec{n}+\vec{e}_1,r}(z) & \cdots & c_1^{-1}A_{\vec{n}+\vec{e}_1,1}(z) & c_1^{-1}R_{\vec{n}+\vec{e}_1}(z) \\ A_{\vec{n},r}(z) & \cdots & A_{\vec{n},1}(z) & R_{\vec{n}}(z) \end{pmatrix}. \tag{6.40}$$

The normalization of this Riemann-Hilbert matrix Y for type I multiple orthogonal polynomials for a Nikishin system now goes as follows. Suppose

$$n_k = n(q_{k-1} - q_k) \in \mathbb{N},$$

where $q_0 = 1 \geq q_1 \geq \cdots \geq q_{r-1} \geq q_r = 0$ and $n = |\vec{n}|$. First we introduce the matrix

$$U_1(z) = Y(z) \begin{pmatrix} 1 & 0 & \cdots & 0 & \int_{\Delta_{r-1}} \frac{w_{r-1,r-1}(t)}{z-t}\,dt & 0 \\ & 1 & & 0 & \int_{\Delta_{r-1}} \frac{w_{r-2,r-1}(t)}{z-t}\,dt & 0 \\ & & \ddots & & \vdots & \vdots \\ & & & 1 & \int_{\Delta_{r-1}} \frac{w_{1,r-1}(t)}{z-t}\,dt & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix}. \tag{6.41}$$

Clearly U_1 is analytic in $\mathbb{C}\backslash(\Delta_r \cup \Delta_{r-1})$ and it has the same growth condition as Y as $z \to \infty$ because if we write (6.41) as $U_1(z) = Y(z)W_{r-1}(z)$, then

$$\begin{pmatrix} z^{n_r} & & & & \\ & \ddots & & & \\ & & z^{n_2} & & \\ & & & z^{n_1} & \\ & & & & z^{-n_1-\cdots-n_r} \end{pmatrix} W_{r-1}(z) \begin{pmatrix} z^{-n_r} & & & & \\ & \ddots & & & \\ & & z^{-n_2} & & \\ & & & z^{-n_1} & \\ & & & & z^{n_1+\cdots+n_r} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & \mathcal{O}(z^{-n_1+n_r-1}) & 0 \\ & \ddots & & \vdots & \\ & & 1 & \mathcal{O}(z^{-n_1+n_2-1}) & 0 \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix},$$

and since we assume (6.22) we see that this is $I + \mathcal{O}(1/z)$ as $z \to \infty$. The jump condition on Δ_r is

$$U_1^+(x) = Y^+(x) W_{r-1}(x) = Y^-(x) \begin{pmatrix} I_r & \vec{w}_r \\ \vec{0}^{\,t} & 1 \end{pmatrix} W_{r-1}(x),$$

where $\vec{w}_r^{\,t} = -2\pi i (w_{r,r}, \dots, w_{1,r})$. Since $U_1^-(x) = Y^-(x) W_{r-1}(x)$, we get

$$U_1^+(x) = U_1^-(x) [W_{r-1}(x)]^{-1} \begin{pmatrix} I_r & \vec{w}_r \\ \vec{0}^{\,t} & 1 \end{pmatrix} W_{r-1}(x).$$

Working out the matrix product gives

$$U_1^+(x) = U_1^-(x) \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ & \ddots & & & \vdots \\ & & 1 & 0 & 0 \\ & & & 1 & -2\pi i w_{1,r}(x) \\ & & & & 1 \end{pmatrix}, \qquad x \in \Delta_r, \tag{6.42}$$

because the entry on row $r-k+1$ $(k = 2, \dots, r)$ and the last column is

$$-2\pi i w_{k,r}(x) + 2\pi i w_{1,r}(x) \int_{\Delta_{r-1}} \frac{w_{k-1,r-1}(t)}{x-t} \, dt,$$

which vanishes by (6.21). There will also be a jump on Δ_{r-1}, which is given by

$$U_1^+(x) = U_1^-(x) [W_{r-1}^-(x)]^{-1} W_{r-1}^+(x),$$

since Y itself has no jump on Δ_{r-1}. This matrix product is easily evaluated and if we use the Sokhotsky-Plemelj formula (1.1) then we find

$$U_1^+(x) = U_1^-(x) \begin{pmatrix} I_{r-1} & \vec{w}_{r-1} & \vec{0}_{r-1} \\ \vec{0}_{r-1}^{\,t} & 1 & 0 \\ \vec{0}_{r-1}^{\,t} & 0 & 1 \end{pmatrix}, \qquad x \in \Delta_{r-1}, \tag{6.43}$$

where $\vec{w}_{r-1}^{\,t} = -2\pi i (w_{r-1,r-1}(x), \dots, w_{1,r-1}(x))$. The next step is to consider

$$U_2(z) = U_1(z) \begin{pmatrix} 1 & & & \int_{\Delta_{r-2}} \frac{w_{r-2,r-2}(t)}{z-t} \, dt & 0 & 0 \\ & 1 & & \int_{\Delta_{r-2}} \frac{w_{r-3,r-2}(t)}{z-t} \, dt & 0 & 0 \\ & & \ddots & \vdots & & \\ & & 1 & \int_{\Delta_{r-2}} \frac{w_{1,r-2}(t)}{z-t} \, dt & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix}, \tag{6.44}$$

and in general

$$U(z) = Y(z) \prod_{j=1}^{r-1} \begin{pmatrix} I_{r-j} & \int_{\Delta_{r-j}} \frac{\vec{w}_{r-j}(t)}{z-t}\, dt & 0_{r-j,j} \\ 0_{1,r-j} & 1 & 0_{1,j} \\ 0_{j,r-j} & 0_{j,1} & I_j \end{pmatrix}, \tag{6.45}$$

where the matrix product is taken from left ($j = 1$) to right ($j = r - 1$) and $0_{m,n}$ is a zero matrix with m rows and n columns. The matrix function U is then analytic on $\mathbb{C} \setminus \bigcup_{j=1}^{r} \Delta_j$. The growth condition is again

$$\lim_{z\to\infty} U(z) \begin{pmatrix} z^{-n_r} & & & 0 \\ & \ddots & & \\ & & z^{-n_1} & \\ 0 & & & z^{n_1+n_2+\cdots+n_r} \end{pmatrix} = I,$$

since

$$\begin{pmatrix} z^{n_r} & & & 0 \\ & \ddots & & \\ & & z^{n_1} & \\ 0 & & & z^{-n_1-\cdots-n_r} \end{pmatrix} \begin{pmatrix} I_{r-j} & \int_{\Delta_{r-j}} \frac{\vec{w}_{r-j}(t)}{z-t}\, dt & 0_{r-j,j} \\ 0_{1,r-j} & 1 & 0_{1,j} \\ 0_{j,r-j} & 0_{j,1} & I_j \end{pmatrix}$$

$$\begin{pmatrix} z^{-n_r} & & & 0 \\ & \ddots & & \\ & & z^{-n_1} & \\ 0 & & & z^{n_1+\cdots+n_r} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & \mathcal{O}(z^{-n_j+n_r-1}) & & \\ & \ddots & & & \\ & & 1 & \mathcal{O}(z^{-n_j+n_{j+1}-1}) & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} = I + \mathcal{O}(1/z),$$

since we assume (6.22). The jump condition requires some computation, but the use of the Sokhotsky-Plemelj formula (1.1) and the relation (6.21) for the weight functions in a Nikishin system, gives for $x \in \mathbb{R}$

$$U^+(x) = U^-(x) \begin{pmatrix} 1 & -2\pi i w_{1,1}(x) & 0 & \cdots & 0 \\ & 1 & -2\pi i w_{1,2}(x) & \cdots & 0 \\ & & 1 & \ddots & \vdots \\ & & & 1 & -2\pi i w_{1,r}(x) \\ & & & & 1 \end{pmatrix}. \tag{6.46}$$

Observe that there will only be jumps on the intervals Δ_k and that this jump matrix only contains the first weight function $w_{1,k}$ of each Nikishin system of order k. In case all the intervals Δ_k are disjoint, the jump will only contain one entry outside the diagonal on each interval Δ_k.

Let $(\nu_1, \dots, \nu_r)$ be the vector of equilibrium measures for the extremal problem with interaction matrix C given by (6.37) and variational conditions (6.35). The normalization will be of the form

$$\begin{aligned} M(z) &= \begin{pmatrix} e^{n(\ell_1^*+\cdots+\ell_r^*)} & & & & \\ & \ddots & & & \\ & & e^{n(\ell_1^*+\ell_2^*)} & & \\ & & & e^{n\ell_1^*} & \\ & & & & 1 \end{pmatrix} U(z) \\ &\times \begin{pmatrix} e^{-nq_{r-1}g_r(z)} & & & & & \\ & \ddots & & & & \\ & & e^{-n[q_{k-1}g_k(z)-q_kg_{k+1}(z)]} & & & \\ & & & \ddots & & \\ & & & & e^{-n[q_0g_1(z)-q_1g_2(z)]} & \\ & & & & & e^{nq_0g_1(z)} \end{pmatrix} \\ &\times \begin{pmatrix} e^{-n(\ell_1^*+\cdots+\ell_r^*)} & & & & \\ & \ddots & & & \\ & & e^{-n(\ell_1^*+\ell_2^*)} & & \\ & & & e^{-n\ell_1^*} & \\ & & & & 1 \end{pmatrix} \end{aligned} \tag{6.47}$$

where

$$g_k(z) = \int_{\Delta_{r-k+1}} \log(z-x)\, d\nu_k(x).$$

The numbers ℓ_k^* are obtained by rewriting the variational conditions (6.35) in the form

$$-q_{k-1}[g_k^+(x)+g_k^-(x)] + q_k g_{k+1}^-(x) + q_{k-2}g_{k-1}^+(x) = \ell_k^*, \qquad x \in \Delta_{r-k+1}, \tag{6.48}$$

where $\ell_k^* - \ell_k$ is an integer multiple of $i\pi/n$. Observe that

$$n[q_{k-1}g_k(z) - q_k g_{k+1}(z)] = n(q_{k-1}-q_k)\log z + \mathcal{O}(1/z), \qquad z \to \infty,$$

so that

$$e^{-n[q_{k-1}g_k(z)-q_kg_{k+1}(z)]} = z^{-n_k}[1+\mathcal{O}(1/z)], \qquad k = 1, \dots, r,$$

and

$$e^{nq_0 g_1(z)} = z^{n_1+\cdots+n_r}[1+\mathcal{O}(1/z)],$$

which means that

$$\lim_{z\to\infty} M(z) = I.$$

This shows that the matrix function M is normalized as $z\to\infty$. If we write (6.47) as $M(z) = L^{-1}U(z)G(z)L$, then the jump condition for this function on the real line is

$$M^+(x) = M^-(x)L^{-1}[G^-(x)]^{-1}W(x)G^+(x)L,$$

where W is the jump matrix for U given by (6.46). First we compute the matrix product

$$[G^-(x)]^{-1}W(x)G^+(x) = \begin{pmatrix} a_r(x) & b_r(x) & 0 & \cdots & 0 \\ & a_{r-1}(x) & b_{r-1}(x) & 0 & \vdots \\ & & \ddots & \ddots & 0 \\ & & & a_1(x) & b_1(x) \\ & & & & a_0(x) \end{pmatrix}, \tag{6.49}$$

where for $k=1,\ldots,r$

$$\begin{aligned} a_k(x) &= e^{n[q_k(g_{k+1}^+(x)-g_{k+1}^-(x))-q_{k-1}(g_k^+(x)-g_k^-(x))]}, \\ b_k(x) &= -2\pi i w_{1,r-k+1}(x)e^{n[-q_k g_{k+1}^-(x)+q_{k-1}(g_k^+(x)+g_k^-(x))-q_{k-2}g_{k-1}^+(x)]}, \end{aligned}$$

and

$$a_0(x) = e^{n(g_1^+(x)-g_1^-(x))}.$$

We have

$$g_k^\pm(x) = \begin{cases} -U(x;\nu_k), & x>b_k, \\ -U(x;\nu_k)\pm i\pi, & x<a_k, \\ -U(x;\nu_k)\pm\varphi_k(x), & x\in[a_k,b_k], \end{cases},$$

where we define $\Delta_{r-k+1}=[a_k,b_k]$, and

$$\varphi_k(x) = i\pi\int_x^{b_k} d\nu_k(t).$$

With this information and (6.48) we find

$$\begin{aligned} a_k(x) &= e^{2n[q_k\varphi_{k+1}(x)-q_{k-1}\varphi_k(x)]}, \\ b_k(x) &= -2\pi i w_{1,r-k+1}(x)e^{-n\ell_k^*}, \end{aligned}$$

for $k = 1, \dots, r$, and

$$a_0(x) = e^{2n\varphi_1(x)}.$$

The jump for M thus becomes

$$L^{-1}[G^-(x)]^{-1}W(x)G^+(x)L$$
$$= \begin{pmatrix} a_r(x) & -2\pi i w_{1,1}(x) & 0 & \cdots & 0 \\ & a_2(x) & -2\pi i w_{1,2}(x) & 0 & 0 \\ & & \ddots & \ddots & \\ & & & a_1(x) & -2\pi i w_{1,r}(x) \\ & & & & a_0(x) \end{pmatrix}, \quad (6.50)$$

where each a_k is an oscillatory function. This means that this normalized Riemann-Hilbert problem now has oscillatory jumps on each of the intervals Δ_k, so that the steepest descent method of Deift and Zhou [9] [10] can be used for the asymptotic analysis of $M(z)$ as $|\vec{n}|$ tends to infinity. We will return to this in future work.

7. Conclusion

We have shown that one can find a Riemann-Hilbert problem for $(r+1)\times(r+1)$ matrix functions for studying multiple orthogonal polynomials of type I and type II (Hermite-Padé polynomials). This Riemann-Hilbert problem easily gives an important relationship between type I and type II multiple orthogonal polynomials (Section 4) and a recurrence relation for contiguous type II multiple orthogonal polynomials. For the asymptotic analysis of these Riemann-Hilbert problems as the multi-index $\vec{n}$ is large, it is more convenient to normalize the Riemann-Hilbert problem. We have worked out the proper normalization for two important systems of weights, namely for Angelesco systems (Section 6.1) and for Nikishin systems (Section 6.2). This involves the asymptotic zero distribution for multiple orthogonal polynomials, which is related to an extremal problem for vector potentials. This is explained in Section 6. Details of this equilibrium problem are given again for Angelesco systems and for Nikishin systems. We hope that the material in Section 6 is a useful survey of these interesting systems for multiple orthogonal polynomials. The normalized Riemann-Hilbert problems has oscillatory jumps on the real line, which allows the use of a steepest descent method introduced by Deift and Zhou, but a detailed account of this would be out of the scope of the present survey.

Acknowledgements. The authors like to thank Percy Deift and Alexander I. Aptekarev for their interest in this work and for many fruitful discussions. This work was supported by the project INTAS-2000-272.

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