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RIEMANN'S MAPPING THEOREM FOR VARIABLE METRICS*

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Introduction

A Riemannian metric in a plane domain can be expressed by $ds = \lambda\,|\,dz + \mu d\bar{z}\,|$ where $\lambda = \lambda(z) > 0$ and $\mu = \mu(z)$ is complex valued with $|\,\mu\,| < 1$. A mapping f is said to be conformal with respect to ds, or μ-conformal, if it satisfies the Beltrami equation

$$f_{\bar{z}} = \mu f_z \,, \tag{1}$$

where

$$f_z = \frac{1}{2}(f_x - if_y) \,, \qquad f_{\bar{z}} = \frac{1}{2}(f_x + if_y) \,. \tag{2}$$

If $|\,\mu\,| \leqq k$ with a constant $k < 1$, then f is quasi-conformal. There are two main problems: to find a μ-conformal mapping of the whole plane on itself, and of a disk on itself. In both cases a mapping exists, and with appropriate normalization, it is unique.

The primary aim of this paper is to investigate f in its dependence on μ. In particular, if μ depends analytically, differentiably, or continuously on real parameters, the same is true for f; in the case of the plane, the result holds also for complex parameters. In order to make the paper self-contained we include proofs of existence and uniqueness.

Bibliographical note. For a Hölder continuous μ the existence of local μ-conformal mappings is classical (Korn [8], Lichtenstein [10]; modernized proofs have been given by Bers [2] and Chern [7]). The global mapping theorems follow from the local theorem by use of the general uniformization theorem. Vekua [13] and Ahlfors [1] gave direct proofs of the global theorems.

For measurable μ the mapping theorem is due to Morrey [12]; earlier, the case of a continuous μ had been treated by Lavrentiev [9]. New proofs of Morrey's result were given by Bers and Nirenberg [3], and by Boyarskii [4]. One owes to Boyarskii the important observation that the generalized derivatives of f are of class L_p for some $p > 2$.

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1. Preliminaries

1.1. If $f(x, y)$ is of class C^1 the derivatives $f_z, f_{\bar z}$ are defined by (2). More generally, if f is locally integrable in a region Ω, then $f_z, f_{\bar z}$ are said to be generalized derivatives of f if they are locally integrable and satisfy

$$\begin{aligned}\iint_\Omega f_z \varphi\, dxdy &= -\iint_\Omega f\varphi_z\, dxdy \\ \iint_\Omega f_{\bar z} \varphi\, dxdy &= -\iint_\Omega f\varphi_{\bar z}\, dx\, dy\end{aligned} \tag{3}$$

for all $\varphi \in C^1$ with compact support in Ω.

LEMMA 1. *If $f_{\bar z} = 0$, then f is holomorphic.*

More precisely, there exists a holomorphic function which is almost everywhere equal to f. The lemma and the proof are well known.

LEMMA 2. *Suppose that p, q are continuous and $p_{\bar z} = q_z$ in a simply connected region Ω. Then there exists a function $f \in C^1(\Omega)$ which satisfies $f_z = p, f_{\bar z} = q$.*

While less familiar than Lemma 1, the proof follows the same lines (use of a smoothing operator).

1.2. From now on Ω will be the whole plane. We consider the operators P and T defined by

$$(Pg)(z) = \frac{1}{2\pi i}\iint_\Omega g(\zeta)\left(\frac{1}{\zeta - z} - \frac{1}{\zeta}\right) d\zeta\, d\bar\zeta$$

$$(Tg)(z) = \frac{1}{2\pi i}\iint_\Omega \frac{g(\zeta) - g(z)}{(\zeta - z)^2} d\zeta\, d\bar\zeta\ .$$

LEMMA 3. *Suppose that $g \in L_p(\Omega)$, $p > 2$. Then Pg exists everywhere as an absolutely convergent integral, and Tg exists almost everywhere as a Cauchy principal limit. The following relations hold:*

$$(Pg)_{\bar z} = g\ , \qquad (Pg)_z = Tg \tag{4}$$

$$|\, Pg(z_1) - Pg(z_2)\,| \leqq c_p \,\|\, g \,\|_p \,|\, z_1 - z_2\,|^{1-2/p} \tag{5}$$

$$\|\, Tg \,\|_p \leqq C_p \,\|\, g \,\|_p\ . \tag{6}$$

Actually, (6) holds for $p > 1$, and for $p = 2$ it can be replaced by

$$\|\, Tg \,\|_2 = \|\, g \,\|_2\ . \tag{7}$$

The relations (4) and (7) are easy to verify. Condition (5) is an immediate consequence of the Hölder inequality, while (6), a deep result, is due to Calderón and Zygmund [5].

LEMMA 4. $C_p \to 1$ *for* $p \to 2$.

This is a consequence of the Riesz convexity theorem together with (7).

2. The nonhomogeneous Beltrami equation

2.1. Throughout the paper all functions denoted by μ, with or without subscripts, are subject to the restriction $\|\mu\|_\infty \leqq k$ with a fixed $k < 1$. The exponent p will be any number which satisfies

$$(8) \qquad p > 2\,, \qquad kC_p < 1\,.$$

By Lemma 4 there are such p, whatever the value of k.

We introduce the Banach space B_p of functions ω, defined on the whole plane, which satisfy a global Hölder condition of order $1 - 2/p$, which vanish at the origin, and whose generalized derivatives ω_z and $\omega_{\bar z}$ exist and belong to $L_p(\Omega)$. The norm is defined by

$$\|\omega\|_{B_p} = \sup \frac{|\omega(z_1) - \omega(z_2)|}{|z_1 - z_2|^{1-2/p}} + \|\omega_z\|_p + \|\omega_{\bar z}\|_p\,.$$

THEOREM 1. *If* $\sigma \in L_p(\Omega)$ *the equation*

$$(9) \qquad \omega_{\bar z} = \mu\omega_z + \sigma$$

has a unique solution $\omega^{\mu,\sigma} \in B_p$. *This is the only solution with* $\omega(0) = 0$ *and* $\omega_z \in L_p(\Omega)$.

PROOF. To establish the uniqueness we must show that a solution of the homogeneous equation

$$(10) \qquad \omega_{\bar z} = \mu\omega_z$$

reduces to 0 if $\omega(0) = 0$ and $\omega_z \in L_p(\Omega)$. It follows by (4) and Lemma 1 that

$$\omega = p(\omega_{\bar z}) + F\,,$$

where F is holomorphic. We obtain

$$(11) \qquad \omega_z = T(\mu\omega_z) + F'\,,$$

and by (6)

$$\|F'\|_p \leqq (1 + kC_p)\|\omega_z\|_p\,.$$

But $\|F'\|_p < \infty$ only if F is a constant. Hence $F' = 0$, and (11) yields $\|\omega_z\|_p \leqq kC_p\|\omega_z\|_p$, a contradiction unless $\omega_z = 0$, $\omega = 0$.

To prove the existence we solve the equation

$$(12) \qquad q = T(\mu q) + T\sigma$$

in L_p. This is possible because $T\sigma \in L_p$ and the norm of the transformation $T\mu$ is $\leqq kC_p < 1$. We set

$$\omega = P(\mu q + \sigma) \tag{13}$$

and verify that $\omega_z = q$, $\omega_{\bar z} = \mu q + \sigma$. Hence ω is a solution of (9).

It follows from (12) that

$$\| q \|_p \leqq kC_p \| q \|_p + C_p \| \sigma \|_p ,$$

and thus

$$\| q \|_p \leqq c \| \sigma \|_p \tag{14}$$

where we use the notation c for unspecified constants that may depend on k and p. With the aid of (7) we obtain further

$$| \omega(z_1) - \omega(z_2) | \leqq c \| \sigma \|_p | z_1 - z_2 |^{1-2/p} . \tag{15}$$

We have shown that $\omega^{\mu,\sigma} \in B_p$, and we have also proved

LEMMA 5. *The mapping $\sigma \to \omega^{\mu,\sigma}$ is a bounded linear transformation from L_p to B_p with a bound that depends only on k and p.*

2.2. Because $\sigma \to \omega^{\mu,\sigma}$ is a bounded linear transformation $\omega^{\mu,\sigma}$ depends continuously on σ, and because the bound does not depend on μ the continuity is uniform with respect to μ.

We show now that $\omega^{\mu,\sigma}$ is also continuous in μ. The simultaneous continuity in μ and σ would follow, but it is just as easy to prove it directly.

LEMMA 6. *If $\mu_n \to \mu$ almost everywhere and $\| \sigma_n - \sigma \|_p \to 0$, then $\omega^{\mu_n \sigma_n} \to \omega^{\mu,\sigma}$ in B_p.*

PROOF. We set $\Omega = \omega^{\mu_n,\sigma_n} - \omega^{\mu,\sigma}$ and $q_n = \omega_z^{\mu_n,\sigma_n}$, $q = \omega_z^{\mu,\sigma}$. From

$$\Omega_z = \mu_n q_n - \mu q + \sigma_n - \sigma = \mu_n \Omega_{\bar z} + (\mu_n - \mu) q + \sigma_n - \sigma$$

we conclude that $\Omega = \omega^{\mu_n,\lambda}$ with $\lambda = (\mu_n - \mu) q + \sigma_n - \sigma$. But $\| \lambda \|_p \to 0$, for $\| (\mu_n - \mu) q \|_p \to 0$ by dominated convergence and $\| \sigma_n - \sigma \|_p \to 0$ by assumption. Hence, by Lemma 5, $\Omega \to 0$ in B_p.

For convenient reference we exhibit the estimate

$$\| \omega^{\mu_1,\sigma_1} - \omega^{\mu_2,\sigma_2} \|_{B_p} \leqq c(\| \mu_1 - \mu_2 \|_\infty \| \sigma_2 \|_p + \| \sigma_1 - \sigma_2 \|_p) . \tag{16}$$

2.3. We will now study the situation which arises when μ and σ are functions of real or complex parameters. Again, although the dependence on σ is quite trivial, it is a convenience to treat μ and σ simultaneously.

Let $t = (t_1, \cdots, t_n)$ be a real or complex vector variable, and suppose that $\mu(t)$ and $\sigma(t)$ are defined on an open set. We say that μ is differentiable at t if we can write

(17) $$\mu(t+s)-\mu(t)=\sum_1^n a_i(t)s_i+|s|\,\alpha(t,s)\ ,$$

where $a_i(t)\in L_\infty$, $\|\alpha(t,s)\|_\infty$ is bounded, and $\alpha(t,s)\to 0$ almost everywhere as $s\to 0$.

Similarly, σ is differentiable if

(18) $$\sigma(t+s)-\sigma(t)=\sum_1^n b_i(t)s_i+|s|\beta(t,s)$$

with $b_i(t)\in L_p$, $\|\beta(t,s)\|_p\to 0$ as $s\to 0$.

We say further that the partial derivatives $a_i(t)$ are continuous if $\|a_i(t+s)\|_\infty$ is bounded in a neighborhood of $s=0$, and if $a_i(t+s)\to a_i(t)$ almost everywhere as $s\to 0$. The continuity of $b_i(t)$ shall mean continuity in L_p.

THEOREM 2. *If μ and σ are differentiable, then $\omega^{\mu,\sigma}$ is differentiable as an element of B_p. If μ and σ have continuous partial derivatives, so does $\omega^{\mu,\sigma}$.*

PROOF. We use subscripts 0 when the argument is t and no subscripts when it is $t+s$. As in the proof of Lemma 6, the difference $\Omega=\omega^{\mu,\sigma}-\omega^{\mu_0,\sigma_0}$ satisfies

$$\Omega_z=\mu\Omega_{\bar z}+(\mu-\mu_0)q_0+\sigma-\sigma_0\ ,$$

and thus $\Omega=\omega^{\mu,\lambda}$ with $\lambda=(\mu-\mu_0)q_0+\sigma-\sigma_0$. On using (17) and (18) we obtain, by linearity,

$$\Omega=\sum_1^n\omega^{\mu,a_iq_0+b_i}s_i+|s|\,\omega^{\mu,\rho}$$

with $\rho=\alpha q_0+\beta$. Because α is bounded and tends to 0 almost everywhere we may conclude that $\rho\to 0$ in L_p, and hence that $\omega^{\mu,\rho}\to 0$ in B_p. At the same time $\omega^{\mu,a_iq_0+b_i}\to\omega^{\mu_0,a_iq_0+b_i}$ in B_p, and therefore Ω has a development

$$\Omega=\sum_1^n c_i(t)s_i+|s|\gamma(t,s)$$

with $c_i=\omega^{\mu_0,a_iq_0+b_i}$ and $\|\gamma(t,s)\|_{B_p}\to 0$. The continuity of $c_i(t)$ follows by Lemma 5.

COROLLARY. *The generalized derivatives $\omega_z^{\mu,\sigma}$ and $\omega_{\bar z}^{\mu,\sigma}$ are differentiable as elements of L_p, and the derivations with respect to the parameters commute with the derivations with respect to z and $\bar z$.*

Indeed,

$$\Omega_z=\sum_1^n c_i(t)_z s_i+|s|\gamma_z(t,s)$$

with $\| \gamma_z(t, s) \|_p \to 0$, and a similar relation holds for $\Omega_{\bar{z}}$.

2.4. If t is complex, Theorem 2 permits us to conclude that $\omega^{\mu,\sigma}$ is a complex analytic function of each variable. It is true, even for functions with values in a Banach space, that analyticity in each variable implies simultaneous analyticity, at least if, as in the present case, the function is known to be continuous.

THEOREM 3. *Let μ and σ be analytic functions of the complex variables $\tau_1, \cdots, \tau_n$ when regarded as elements of L_∞ and L_p respectively. Then $\omega^{\mu,\sigma}$ depends analytically on the same variables when viewed as an element of B_p.*

For the case of a single variable we prefer to give a direct proof which has the advantage of leading to an explicit power series development.

THEOREM 3′. Suppose that μ and σ have power series developments

$$\begin{aligned}\mu &= m_0 + m_1\tau + \cdots + m_n\tau^n + \cdots \\ \sigma &= s_0 + s_1\tau + \cdots + s_n\tau^n + \cdots\end{aligned} \tag{19}$$

which converge in L_∞ and L_p respectively. Then

$$\omega^{\mu,\sigma} = \omega_0 + \omega_1\tau + \cdots + \omega_n\tau^n + \cdots \tag{20}$$

in B_p, where $\omega_n = \omega^{m_0,\lambda_n}$ and the λ_n are determined recursively by $\lambda_0 = s_0$,

$$\lambda_n = s_n + m_1\omega_{n-1,z} + m_2\omega_{n-2,z} + \cdots + m_n\omega_{0,z} . \tag{21}$$

PROOF. The relations (21) are obtained by formal substitution of (19) and (20) in (9). All that needs to be proved is the convergence of (20).

We have $\| \omega_{n,z} \|_p \leqq c \| \lambda_n \|_p$, by Lemma 5, and the convergence of (19) for some $\tau \neq 0$ implies $\| m_n \|_\infty \leqq cM^n$, $\| s_n \|_p \leqq cM^n$. It follows by (21) that

$$\| \lambda_n \|_p \leqq c(M^n + M \| \lambda_{n-1} \|_p + M^2 \| \lambda_{n-2} \|_p + \cdots + M^n \| \lambda_0 \|_p) . \tag{22}$$

By the method of majorants, or by induction, (22) implies

$$\| \lambda_n \|_p \leqq c(1 + c)^n M^n ,$$

and we conclude that (20) has a positive radius of convergence.

3. Local properties of μ-conformal mappings

3.1. In addition to the restriction $\| \mu \|_\infty \leqq k < 1$ we assume in this section that all functions denoted by μ vanish outside of a fixed compact set. We study the solutions of the homogeneous Beltrami equation

(23) $$f_{\bar z} = \mu f_z .$$

A continuous solution of this equation, not necessarily one to one, is said to be μ-conformal if f_z is locally of class L_2.

THEOREM 4. *There exists a unique μ-conformal function f^μ which vanishes at the origin and satisfies $f^\mu_z - 1 \in L_p(\Omega)$. It is given by*

(24) $$f^\mu(z) = z + \omega^{\mu,\mu}(z) .$$

It is evident that f^μ is a solution of (23). Since f^μ_z is locally of class L_p with $p > 2$ it is also locally of class L_2.

We remark that f^μ is independent of the choice of p. Indeed, because μ has compact support, f^μ is analytic in a neighborhood of ∞. Together with $f^\mu_z - 1 \in L_p(\Omega)$ this implies $f^\mu_z = 1 + O(1/|z|^2)$, and hence $f^\mu_z - 1 \in L_q(\Omega)$ as soon as $1 < q \leqq p$.

3.2. If μ is smooth, then f^μ can be expected to be smooth. We prove in this respect:

LEMMA 7. *If $\mu_z \in L_p$, then $f^\mu \in C^1$. Moreover, f^μ is a homeomorphism of the whole plane on itself, and the Jacobian is positive.*

PROOF. First of all, $\overline{\mu} = P\overline{\mu}_{\bar z} + F$ with an analytic F, which shows that μ is continuous.

Set $\lambda = \omega^{\mu,\mu_z}$ and $\rho = e^\lambda$. Then ρ satisfies

$$\rho_{\bar z} = \mu\rho_z + \mu_z\rho = (\mu\rho)_z .$$

By Lemma 2 there exists a function $f \in C^1$ which satisfies $f_z = \rho$, $f_{\bar z} = \mu\rho$. Its Jacobian is $|f_z|^2 - |f_{\bar z}|^2 = \rho^2(1 - |\mu|^2) > 0$. This implies that the mapping is locally one to one.

We have $\lambda_{\bar z} = 0$ near ∞. Because $\lambda = O(|z|^{1-2/p})$ it follows that λ has a limit $\lambda(\infty)$ for $z \to \infty$. f is analytic near ∞ and $f_z \to e^{\lambda(\infty)}$. This shows that $f \to \infty$ for $z \to \infty$, and for topological reasons f must be a homeomorphic mapping of the plane onto itself. It is clear that $f^\mu(z) = e^{-\lambda(\infty)}(f(z) - f(0))$.

3.3. It is possible to find a sequence $\{\mu_n\}$, $\mu_n \in C^1$, with fixed compact support such that $\mu_n \to \mu$ almost everywhere, and, as a consequence, $\|\mu_n - \mu\|_p \to 0$. It follows by Lemma 6 that $f^{\mu_n} - f^\mu \to 0$ in B_p. We use this approximation to prove:

LEMMA 8. *f^μ is always a homeomorphism of the whole plane on itself.*

PROOF. Let g_n be the inverse of $f_n = f^{\mu_n}$, known to exist by the preceding lemma. One finds that

$$g_{n,\bar z} = \nu_n g_{n,z}$$

with

$$\nu_n = -\left(\frac{f_{n,z}}{\overline{f}_{n,\bar{z}}}\cdot\mu_n\right)\circ g_n ,$$
$$|\nu_n| = |\mu_n|\circ g_n .$$

By use of Hölder's inequality we obtain

$$\iint_\Omega |\nu_n|^p dxdy = \iint_\Omega |\mu_n|^p(|f_{n,z}|^2 - |f_{n,\bar{z}}|^2)dxdy$$
$$\leqq \iint_\Omega |\mu_n|^{p-2}|f_{n,\bar{z}}|^2 dxdy \leqq \left(\iint_\Omega |\mu_n|^p dxdy\right)^{1-2/p}\left(\iint_\Omega |f_{n,\bar{z}}|^p dxdy\right)^{2/p}.$$

But $\| f_{n,\bar{z}} \|_p = \| \omega_{\bar{z}}^{\mu_n \cdot \mu_n} \|_p \leqq c \| \mu_n \|_p$, and we have shown that

(25) $$\| \nu_n \|_p \leqq c \| \mu_n \|_p .$$

It is evident that $g_n = f^{\nu_n}$, for $g_{n,z} \to 1$ as $z \to \infty$. We conclude that

$$| g_n(\zeta_1) - g_n(\zeta_2) | \leqq | \zeta_1 - \zeta_2 | + c \| \mu_n \|_p | \zeta_1 - \zeta_2 |^{1-2/p}$$

or, equivalently,

$$| z_1 - z_2 | \leqq | f_n(z_1) - f_n(z_2) | + c \| \mu_n \|_p | f_n(z_1) - f_n(z_2) |^{1-2/p} .$$

On letting n tend to ∞ it follows that $f^\mu(z_1) = f^\mu(z_2)$ implies $z_1 = z_2$, and we have proved that f^μ is one to one. The fact that $f_z^\mu \to 1$ for $z \to \infty$ is sufficient to show that the mapping is onto.

3.4. A homeomorphism f is said to be *measurable* if measurable sets are mapped on measurable sets. An equivalent condition is that null-sets are mapped on null-sets, or that the set-function mes $f(e)$, defined for Borel sets e, is absolutely continuous.

LEMMA 9. *The mapping f^μ and its inverse are measurable. Moreover, $f_z^\mu \neq 0$ almost everywhere, and*

(26) $$\text{mes } f^\mu(e) = \iint_e (|f_z^\mu|^2 - |f_{\bar{z}}^\mu|^2)dxdy$$

for any measurable e.

PROOF. Let e be an open set. Let χ denote the characteristic function of $f^\mu(e)$, and χ_n that of $f_n(e)$. Then $\chi \leqq \liminf \chi_n$, and consequently

$$\text{mes } f^\mu(e) \leqq \liminf \text{mes } f_n(e) .$$

We have

(27) $$\text{mes } f_n(e) = \iint_e (|f_{n,z}|^2 - |f_{n,\bar{z}}|^2)dxdy \leqq \iint_e |f_{n,z}|^2 dxdy ,$$

and by Hölder's inequality

$$\iint_e |f_{n,z}|^2 dxdy \leqq \left(\iint_e |f_{n,z}|^p dxdy\right)^{2/p} (\text{mes } e)^{1-2/p} .$$

Here

$$\left(\iint_e |f_{n,z}|^p dxdy\right)^{1/p} \leqq \|f_{n,z}-1\|_p + (\text{mes } e)^{1/p} \leqq c\|\mu_n\|_p + (\text{mes } e)^{1/p} ,$$

and we obtain

$$\text{mes } f_n(e) \leqq (c\|\mu_n\|_p + (\text{mes } e)^{1/p})^2(\text{mes } e)^{1-2/p} .$$

Because $\|\mu_n\|_p \to \|\mu\|_p$ these estimates show that mes $f^\mu(e)$ tends to 0 with mes e. Since every null set is contained in an open set of arbitrarily small measure we have proved that null sets are mapped on null sets, and thus f^μ is measurable.

Precisely the same reasoning can be applied to the inverse function, for $(f^\mu)^{-1} = \lim f^{\nu_n}$ and $\|\nu_n\|_p$ is bounded.

If e is open and bounded it follows by (27) and the preceding inequality that

$$\text{mes } f^\mu(e) \leqq \iint_e (|f_z^\mu|^2 - |f_{\bar z}^\mu|^2)\, dxdy ,$$

for $f_{n,z} \to f_z^\mu$ and $f_{n,\bar z} \to f_{\bar z}^\mu$ in $L_2(e)$. For compact sets e the reverse inequality is true, either by the same argument or by taking complements with respect to a large disk. On approximating an open set by compact subsets it is easily concluded that the equality (26) holds for all open sets. It will therefore hold for closed sets, for Borel sets, and finally for all measurable sets.

Because the inverse function maps null sets on null sets it is impossible that $|f_z^\mu|^2 - |f_{\bar z}^\mu|^2$ vanish on a set of positive measure.

3.5. It is important to formulate conditions under which a composite function $h \circ f^\mu$ has generalized derivatives.

For convenience we simplify the notation f^μ to f.

LEMMA 10. *Suppose that $h_z, h_{\bar z}$ are locally of class L_q, $q \geqq 2$. Then $h \circ f$ has generalized derivatives given by*

$$(h\circ f)_z = (h_z\circ f)f_z + (h_{\bar z}\circ f)\bar f_z$$
$$(h\circ f)_{\bar z} = (h_z\circ f)f_{\bar z} + (h_{\bar z}\circ f)\bar f_{\bar z} \tag{28}$$

and

$$\|(h\circ f)_z\|_r \leqq M(\|h_z\|_q + \|h_{\bar z}\|_q) , \quad r = \frac{pq}{p+q-2} , \tag{29}$$

where the norms are over corresponding bounded regions Ω_0, $f(\Omega_0)$, and M is independent of h.

PROOF. We show first that (28) implies (29). By Hölder's inequality

$$\iint_{\Omega_0} |(h_z\circ f)f_z|^r dxdy \leqq \left(\iint_{\Omega_0} |h_z\circ f|^q |f_z|^2 dxdy\right)^{r/q}\left(\iint_{\Omega_0}|f_z|^p dxdy\right)^{1-r/q},$$

and by use of Lemma 9,

$$\iint_{\Omega_0} |h_z\circ f|^q |f_z|^2 dxdy \leqq \frac{1}{1-k^2}\iint_{\Omega_0} |h_z\circ f|^q(|f_z|^2 - |f_{\bar z}|^2)dxdy$$

$$= \frac{1}{1-k^2}\iint_{f(\Omega_0)} |h_z|^q dxdy .$$

A similar estimate applies to $(h_{\bar z}\circ f)\bar f_z$, and (29) follows.

The formulas (28) hold if h and f are of class C^1. By well known properties of generalized derivatives it is possible to find $h_m, f_n \in C^1$ in such a way that $h_m \to h$, $f_n \to f$ and

$$\iint_{f(\Omega_0)} (|h_{m,z} - h_z|^2 + |h_{m,\bar z} - h_{\bar z}|^2)dxdy \to 0$$

$$\iint_{\Omega_0} |f_{n,z} - f_z|^2 + |f_{n,\bar z} - f_{\bar z}|^2 dxdy \to 0 .$$

Consider

$$I_1 = \iint_{\Omega_0} |(h_{m,z}\circ f) - (h_z\circ f)|\ |f_z|\, dxdy$$

$$I_2 = \iint_{\Omega_0} |(h_{m,z}\circ f) - (h_{m,z}\circ f_n)|\ |f_z|\, dxdy$$

$$I_3 = \iint_{\Omega_0} |h_{m,z}\circ f_n|\ |f_z - f_{n,z}|\, dxdy .$$

By the same estimate that was used to prove (29) it is seen that I_1 can be made arbitrarily small by choosing m sufficiently large. When m is fixed it is evident that I_2 and I_3 tend to 0 for $n \to \infty$. Consequently, it is possible to choose m and n so that

$$\iint_{\Omega_0} |(h_{m,z}\circ f_n)f_{n,z} - (h_z\circ f)f_z|\, dxdy$$

is arbitrarily small. The same is true of

$$\iint_{\Omega_0} |(h_{m,\bar z}\circ f_n)\bar f_{n,z} - (h_{\bar z}\circ f)\bar f_z|\, dxdy ,$$

and we conclude easily that the first relation (28) is valid. The second is proved in the same way.

3.6. The preceding lemma is used to prove:

LEMMA 11. $(f^\mu)^{-1} = f^\nu$ *with*

$$\nu = -\left(\frac{f_z^\mu}{\bar f_{\bar z}^\mu}\cdot \mu\right)\circ (f^\mu)^{-1} . \tag{30}$$

PROOF. We remark first that ν is measurable because f^μ is a measurable mapping. Therefore f^ν exists and f^ν_z is locally of class L_p.

We compute $(f^\nu \circ f^\mu)_{\bar z}$ by formula (28) and find that it is 0 almost everywhere. Hence $\Phi = f^\nu \circ f^\mu$ is analytic, and since it is a homeomorphism it must be a linear function. But $\Phi(0) = 0$ and $\Phi'(z) \to 1$ for $z \to \infty$. Hence $\Phi(z) = z$, and f^ν is the inverse of f^μ.

3.7. We collect in a single theorem a number of results which are easy consequences of Lemma 10. In this theorem we do not need to assume that μ has compact support.

THEOREM 5. *Let f be any μ-conformal homeomorphism, defined in a region Ω_0. The following is true:*

(i) *f_z is locally of class L_p.*
(ii) *$f_z \neq 0$ almost everywhere.*
(iii) *f^{-1} has generalized derivatives which are locally of class L_p.*
(iv) *$(f^{-1})_z$ and $(f^{-1})_{\bar z}$ are determined by the classical formulas.*
(v) *f and f^{-1} transform measurable sets into measurable sets.*
(vi) *Integrals are transformed according to the classical rule.*
(vii) *If φ is any μ-conformal function in Ω_0, then $\varphi \circ f^{-1}$ is analytic, and vice versa.*

PROOF. Since all assertions are of a local nature it is no restriction to assume that Ω_0 is a bounded region. We extend μ to the whole plane by setting $\mu = 0$ in $\Omega - \Omega_0$. The results (i)-(vi) follow from the fact that $f \circ (f^\mu)^{-1}$ is analytic, and (vii) follows because $\varphi \circ (f^\mu)^{-1}$ and $f \circ (f^\mu)^{-1}$ are both analytic.

4. Homeomorphisms of the plane and the disk

4.1. In this section w^μ denotes a μ-conformal homeomorphism of the whole plane onto itself which is normalized by $w^\mu(0) = 0$, $w^\mu(1) = 1$, $w^\mu(\infty) = \infty$. Similarly, W^μ is a μ-conformal homeomorphism of the closed unit disk onto itself which satisfies $W^\mu(0) = 0$, $W^\mu(1) = 1$. If they exist, it follows from Theorem 5 that w^μ and W^μ are uniquely determined.

The existence is proved in several steps.

LEMMA 12. *If $\mu = 0$ in a neighborhood of ∞, then*

$$w^\mu(z) = f^\mu(z)/f^\mu(1) . \tag{31}$$

If $\mu = 0$ in a neighborhood of the origin, then

$$w^\mu(z) = 1/w^\lambda(1/z) \tag{32}$$

with $\lambda(z) = \mu(1/z)z^2/\bar z^2$.

LEMMA 13. *Set* $\mu = \mu_1 + \mu_2$ *where* μ_1, μ_2 *vanish near* 0 *and* ∞ *respectively. Then* $w^\mu = w^\lambda \circ w^{\mu_2}$ *with*

$$\lambda = \left(\frac{\mu_1}{1 - \bar{\mu}_2 \mu} \cdot \frac{f_z^{\mu_2}}{\bar{f}_{\bar{z}}^{\mu_2}}\right) \circ (w^{\mu_2})^{-1} . \tag{33}$$

LEMMA 14. *If* $\overline{\mu(z)} = \mu(1/\bar{z})\bar{z}^2/z^2$, *then* W^μ *is the restriction of* w^μ *to the unit disk.*

Lemmas 12 and 13 are proved by direct verification. Together they show that w^μ exists, for an arbitrary μ can be decomposed in the supposed manner. As for Lemma 14, one proves by means of the uniqueness that $w^\mu(z) = 1/\bar{w}^\mu(1/\bar{z})$. Hence $|w^\mu(z)| = 1$ for $|z| = 1$, and it follows that the restriction of w^μ maps the unit disk onto itself. It is evident that Lemma 14 proves the existence of W^μ.

THEOREM 6. *There exist unique* μ*-conformal homeomorphisms of the whole plane and the unit disk onto themselves with fixpoints at* 0, 1, ∞ *and* 0,1 *respectively.*

4.2. The following preliminary result will be needed:

LEMMA 15. *If* $\mu = 0$ *for* $|z| \geqq 1$, *then* $c^{-1} \leqq |w^\mu(e^{i\theta})| \leqq c$.

PROOF. We have $\|\mu\|_p < c$ and therefore, by (24) and Lemma 5,

$$|f^\mu(z)| \leqq |z| + c|z|^{1-2/p} .$$

By virtue of Lemma 11 and the inequality (25) the same reasoning can be applied to the inverse function and yields

$$|z| \leqq |f^\mu(z)| + c|f^\mu(z)|^{1-(2/p)} .$$

These estimates imply

$$(1 + c)^{-(2p-2)/(p-2)} \leqq |w^\mu(e^{i\theta})| \leqq (1 + c)^{(2p-2)/(p-2)} .$$

4.3. Let $[z_1, z_2]$ denote spherical distance. The following spherical Hölder condition holds for all μ.

LEMMA 16.

$$[w^\mu(z_1), w^\mu(z_2)] \leqq c[z_1, z_2]^\alpha , \quad \alpha > 0 . \tag{34}$$

PROOF. We represent w^μ as in Lemma 13, choosing $\mu_1(z) = 0$ for $|z| > 1$, $\mu_1(z) = \mu(z)$ for $|z| \leqq 1$. Then $\|\lambda\|_\infty \leqq k_1(k) < 1$, and by Lemma 15, together with (33), λ vanishes in the disk $|z| \leqq 1/c$. It is sufficient to prove (34) for w^{μ_2} and w^λ. Also, because the spherical distance is invariant under inversion, we way replace w^λ by $w^\nu(z) = 1/w^\lambda(1/z)$ where $\nu(z) = \lambda(1/z)z^2/\bar{z}^2$ (see Lemma 12). Thus we need to prove (34) only for the case that $\mu = 0$ for $|z| > R$; c and α may depend on R.

We know that $|f^{\mu}(1)| \geqq c^{-1}$(see the proof of Lemma 15). It follows by (31), (24) and Lemma 5 that

$$| w^{\mu}(z_1) - w^{\mu}(z_2) | \leqq c \, | z_1 - z_2 |^{1-2/p} .$$

This implies (34) if, for instance, $|z_1|, |z_2| \leqq 3R$.

The function $g(z)=1/w^{\mu}(1/z)$ is holomorphic for $|z|<1/R$, and if $R>1$, as we may suppose, Lemma 15 yields $|g(z)| \leqq c$ for $|z| \leqq 1/R$. Hence $|g'(z)| \leqq 4cR$ for $|z| \leqq 1/2R$. This implies (34) if $|z_1|, |z_2| \geqq 2R$.

Finally, (34) is trivially fulfilled if $|z_1| \leqq 2R$, $|z_2| \geqq 3R$.

COROLLARY. $| W^{\mu}(z_1) - W^{\mu}(z_2) | \leqq c | z_1 - z_2 |^{\alpha}$.

In this estimate the best possible value of α is $(1-k)/(1+k)$, and the best value of c which does not depend on k is 16 (Mori [11]).

4.4. For a fixed R, $0 < R < \infty$, we consider w^{μ} as an element of the Banach space $B_{R,p}$ with norm

$$\| w \|_{B_{R,p}} = \sup_{|z_1|,|z_2|\leq R} \frac{| w(z_1) - w(z_2) |}{| z_1 - z_2 |^{1-2/p}} + \left(\iint_{|z|\leq R} | w_z |^p dxdy \right)^{1/p} .$$

The L_{∞}-norm of the restriction to $|z| < R$ will be denoted by $\| w \|_{R,\infty}$, and constants that depend only on R, k and p will be designated by $c(R)$.

THEOREM 7.

$$\| w^{\mu} \|_{B_{R,p}} \leqq c(R) . \tag{35}$$

PROOF. If Lemma 16 is applied to $(w^{\mu})^{-1}$ we obtain

$$[z, \infty] \leqq c[w^{\mu}(z), \infty] ,$$

and this implies

$$\| w^{\mu} \|_{R,\infty} \leqq c(R) . \tag{36}$$

Let $\lambda(z)$ be a fixed function of class C^1 with $0 \leqq \lambda \leqq 1$, $\lambda(z) = 1$ for $|z| \leqq R$, $\lambda(z) = 0$ for $|z| \geqq 2R$. We have

$$(\lambda w^{\mu})_{\bar z} = \mu(\lambda w^{\mu})_z + \nu$$

where

$$\nu = (\lambda_{\bar z} - \mu \lambda_z) w^{\mu} . \tag{37}$$

By use of Lemma 5 we conclude that

$$\| w^{\mu} \|_{B_{R,p}} \leqq \| \lambda w^{\mu} \|_{B_p} = \| \omega^{\mu,\nu} \|_{B_p} \leqq c \| \nu \|_p ,$$

while (36) and (37) show that $\| \nu \|_p \leqq c(R)$. This proves (35).

COROLLARY. $\| w^{\mu} \|_{B_{1,p}} \leqq c$.

5. Continuity, differentiability, and analyticity

5.1. We proceed to investigate the dependence of w^μ on μ. In this section μ and ν will both be subject to the restriction $\|\mu\|_\infty \leqq k$, $\|\nu\|_\infty \leqq k$.

LEMMA 17.

$$[w^\mu(z), w^\nu(z)] \leqq c\,\|\mu - \nu\|_\infty . \tag{38}$$

PROOF. We can write $w^\mu = w^\rho \circ w^\nu$ with

$$\rho = \left(\frac{\mu - \nu}{1 - \mu\bar{\nu}}\,\frac{w_z^\nu}{\bar{w}_{\bar{z}}^\nu}\right)\circ(w^\nu)^{-1} .$$

Since $\|\rho\|_\infty \leqq c\|\mu - \nu\|_\infty$ it follows that it is sufficient to prove (38) for $\nu = 0$. In other words, we need to show that

$$[w^\mu(z), z] \leqq c\,\|\mu\|_\infty . \tag{39}$$

We use again the representation $\mu = \mu_1 + \mu_2$ with $\mu_2 = 0$ for $|z| \geqq 1$, $\mu_1 = 0$ for $|z| < 1$. On setting $w^\mu = w^\lambda \circ w^{\mu_2}$ we have $\|\lambda\|_\infty \leqq c\|\mu\|_\infty$ and

$$[w^\mu(z), z] \leqq [w^\lambda \circ w^{\mu_2}, w^{\mu_2}] + [w^{\mu_2}, z] .$$

It will therefore suffice to prove (39) for λ and μ_2, and because spherical distance is invariant under inversion we have reduced the problem to the case where $\mu = 0$ for $|z| \geqq R$, say.

Under this condition we have $\|\mu\|_p \leqq c(R)\|\mu\|_\infty$, and Lemma 5 yields

$$|f^\mu(z) - z| = |\omega^{\mu,\mu}(z)| \leqq c(R)\,\|\mu\|_\infty \tag{40}$$

for $|z| \leqq R$. Because $f^\mu(z) - z$ is holomorphic for $|z| > R$, including $z = \infty$, this inequality is valid in the whole plane.

There is nothing to prove unless $\|\mu\|_\infty$ is small, and we may therefore assume that $c(R)\|\mu\|_\infty < 1/2$. Then (40) gives the preliminary estimate $|f^\mu(1)| > 1/2$, and by use of

$$|w^\mu(z) - z| \leqq \frac{|f^\mu(z) - z|}{|f^\mu(1)|} + |z|\frac{|f^\mu(1) - 1|}{|f^\mu(1)|}$$

we are able to conclude that (39) holds for $|z| \leqq R$.

On assuming that $R \geqq 1$ it is seen by Lemma 15 that (39) implies

$$\left|\frac{1}{w^\mu(z)} - \frac{1}{z}\right| \leqq c(R)\,\|\mu\|_\infty$$

for $|z| = R$, and by the maximum principle the same is true for $|z| \geqq R$. It follows that (39) holds for all z.

5.2. We can now prove

THEOREM 8. $\| w^\mu - w^\nu \|_{B_{R,p}} \leqq c(R) \| \mu - \nu \|_\infty$.

PROOF. Let λ be as in the proof of Theorem 7. One verifies that

$$[\lambda(w^\mu - w^\nu)]_{\bar z} = \mu[\lambda(w^\mu - w^\nu)]_z + \sigma$$

with

$$\sigma = (\lambda_{\bar z} - \mu\lambda_z)(w^\mu - w^\nu) + \lambda(\mu - \nu)w_z^\nu .$$

By Theorem 7 and Lemma 17 we see that $\| \sigma \|_p \leqq c(R) \| \mu - \nu \|_\infty$, while Lemma 5 yields the inequality

$$\| w^\mu - w^\nu \|_{B_{R,p}} \leqq \| \omega^{\mu,\sigma} \|_{B_p} \leqq c \| \sigma \|_p .$$

COROLLARY. $\| W^\mu - W^\nu \|_{B_{1,p}} \leqq c \| \mu - \nu \|_{1,\infty}$.

5.3. LEMMA 18. *If $\mu_n \to \mu$ almost everywhere, then*

(40) $$\| w^{\mu_n} - w^\mu \|_{R,\infty} \to 0 .$$

PROOF. It follows by Theorem 7 that there exists a subsequence of the w^{μ_n} which converges to a limit function w, uniformly on every disk $|z| \leqq R$. The same theorem, applied to the inverse functions, shows that w is a homeomorphism. In view of the local weak compactness of L_2 we may also assume that $w_z^{\mu_n} \to u$, $w_{\bar z}^{\mu_n} \to v$ in the sense of weak L_2-convergence on every disk. It follows easily that $u = w_z$, $v = w_{\bar z}$, and $w_{\bar z} = \mu w_z$. Since w is normalized we may conclude that $w = w^\mu$, and that $\| w^{\mu_n} - w^\mu \|_{R,\infty} \to 0$ for the original sequence.

Lemma 18 can be strengthened to

THEOREM 9. *If $\mu_n \to \mu$ almost everywhere, then*

$$\| w^{\mu_n} - w^\mu \|_{B_{R,p}} \to 0 .$$

PROOF. Let λ be the same function as in the proof of Theorem 7. One verifies that

$$[\lambda(w^{\mu_n} - w^\mu)]_{\bar z} = \mu[\lambda(w^{\mu_n} - w^\mu)]_z + \sigma_n$$

with

$$\sigma_n = (\lambda_{\bar z} - \mu\lambda_z)(w^{\mu_n} - w^\mu) + (\mu - \mu_n)(\lambda_z w + \lambda w_z) .$$

The hypothesis, together with Theorem 7 and Lemma 18, shows that $\| \sigma_n \|_p \to 0$. Therefore, by Lemma 5, $\| \lambda(w^{\mu_n} - w^\mu) \|_{B_p} \to 0$, and the assertion follows.

COROLLARY. *If $\mu_n \to \mu$ almost everywhere, then*

$$\| W^{\mu_n} - W^\mu \|_{B_{1,p}} \to 0 .$$

5.4. If μ depends differentiably on one or more parameters we wish to show that the same is true of w^μ. The following preliminary lemma is an immediate consequence of the corresponding result for $\omega^{\mu,\nu}$ (Theorem 2).

LEMMA 19. *Let a and $\alpha(s)$ be bounded measurable functions of z, the latter depending on a parameter s, $0 \leqq s < s_0$. Suppose that $\|\alpha(s)\|_\infty \leqq c$, and that $\alpha(s) \to 0$ almost everywhere as $s \to 0$. In addition, we assume that a and α are identically 0 in a fixed neighborhood of ∞.*

Then

$$\theta^a = \lim_{s\to 0} \frac{w^{sa+s\alpha(s)} - z}{s}$$

exists as a limit in $B_{R,p}$ for all $R > 0$ and is given explicitly by

$$\theta^a(z) = Pa(z) - zPa(1) . \tag{41}$$

PROOF. Because $w^\mu(z) = f^\mu(z)/f^\mu(1)$ and $f^\mu(z) = z + \omega^{\mu,\mu}(z)$ the existence of θ^a follows directly from Theorem 2. We know by the corollary to the same theorem that derivations commute. Therefore

$$\theta^a_{\bar z} = \lim_{s\to 0} \frac{1}{s} w^{sa+s\alpha(s)}_{\bar z} = \lim (a + \alpha(s)) w_z^{sa+s\alpha(s)} = a$$

where the limits are in L_p over $|z| \leqq R$ and the last equality is a consequence of Theorem 9.

Furthermore, near infinity θ^a is expressed as a limit of holomorphic functions with at most a simple pole at $z = \infty$. Therefore θ^a has itself at most a simple pole at ∞, and because it vanishes at $z = 0$ and $z = 1$ it must be of the form (41).

LEMMA 20. *The conclusion of the preceding lemma remains valid if a and α vanish in a fixed neighborhood of 0, except that θ^a will be given by*

$$\theta^a(z) = z^2 P\hat a(z) - zP\hat a(1) \tag{42}$$

where $\hat a(z) = a(z)z^{-2}$.

PROOF. Set $\tilde a(z) = a(1/z)z^2/\bar z^2$ and $\tilde \alpha(z) = \alpha(1/z)z^2/\bar z^2$. By (32) we have $w^{sa+a\alpha}(z) = 1/w^{s\tilde a + s\tilde\alpha}(1/z)$. It follows by use of Lemma 19 that θ^a exists, at least as a pointwise limit, and that

$$\theta^a(z) = -z^2\theta^{\tilde a}(1/z) = -z^2(P\tilde a)(1/z) + zP\tilde a(1) .$$

One verifies, however, that

$$(P\hat a)(z) + (P\tilde a)(1/z) \tag{43}$$

is holomorphic in the whole plane, excepting $z = 0$. For $z \to \infty$ we know that $(P\hat a)(z) = O(|z|^{1-2/p})$ while $(P\tilde a)(1/z) \to 0$. For $z \to 0$ both terms tend to 0. Hence (43) is identically 0, and we see that θ^a can be written in the form (42).

The contention that θ^a is actually a limit in $B_{R,p}$ is easily proved by use of the fact that $w^{sa+s\alpha(s)}$ is holomorphic in a fixed neighborhood of the origin.

5.5. By a change of variable the preceding result can be generalized to the case where $s = 0$ corresponds to an arbitrary μ.

LEMMA 21. *Under the same assumptions as before*

$$\theta^{\mu,a} = \lim_{s\to 0}\frac{w^{\mu+sa+s\alpha(s)} - w^{\mu}}{s} \tag{44}$$

exists in $B_{R,p}$ and satisfies

$$\theta^{\mu,a} = \theta^b \circ w^{\mu} \tag{45}$$

with

$$b = \left(\frac{a}{1-|\mu|^2}\,\frac{w_z^{\mu}}{\bar{w}_{\bar z}^{\mu}}\right)\circ(w^{\mu})^{-1}\ . \tag{46}$$

PROOF. One verifies that $w^{\mu+sa+s\alpha(s)} = w^{sb+s\beta(s)}\circ w^{\mu}$ where b is given by (46) and $\beta(s)\to 0$ almost everywhere. It is true, moreover, that $\|\beta(s)\|_{\infty}$ is bounded and that b and $\beta(s)$ vanish in a fixed neighborhood of ∞ or 0. The existence of $\theta^{\mu,a}$ as a pointwise limit and equation (45) are direct consequences of Lemmas 19 and 20.

To show that $\theta^{\mu,a}$ is a limit in $B_{R,p}$ it is convenient to use the same method as in the proof of Theorem 7. We observe first that

$$\theta_{\bar z}^{\mu,a} = \mu\theta_z^{\mu,a} + aw_z^{\mu}\ , \tag{47}$$

by direct computation. Let λ be as in the proof just referred to, and set

$$\chi = \frac{w^{\mu+sa+s\alpha(s)} - w^{\mu}}{s} - \theta^{\mu,a}\ .$$

One verifies that

$$(\lambda\chi)_{\bar z} = \mu(\lambda\chi)_z + \sigma$$

with

$$\sigma = (\lambda_{\bar z} - \mu\lambda_z)\chi + \lambda\alpha w_z^{\mu+sa+s\alpha} + \lambda a(w_z^{\mu+sa+s\alpha} - w_z^{\mu})\ .$$

It is seen by Theorem 8 that χ is bounded for $|z| \leqq 2R$, and we conclude by use of Theorems 7 and 9 that $\|\sigma\|_p \to 0$. By Lemma 5 this implies $\|\lambda\chi\|_{B_p}\to 0$ and hence $\|\chi\|_{B_{R,p}}\to 0$, which is what we wanted to prove.

5.6. We have defined θ^a in two different ways, depending on whether

a vanishes near ∞ or near 0. It follows from the two lemmas, or by direct verification, that the two definitions coincide if a satisfies both conditions. Suppose now that a is arbitrary, except for being bounded and measurable. Then we can write $a = a_1 + a_2$ where $a_1 = 0$ near 0 and $a_2 = 0$ near ∞. We set, by definition,

$$\theta^a = \theta^{a_1} + \theta^{a_2} .$$

Because the formulas (41) and (42) depend linearly on a, the result is independent of the decomposition.

The definition (45) of $\theta^{\mu,a}$ will also be carried over to arbitrary bounded a. The differential equation (47) remains in force. The following further properties will be needed:

LEMMA 22.
(i) $|\theta^a(z)| \leqq c(1 + |z|)\log(1 + |z|)\,||a||_\infty$;
(ii) $||\theta^{\mu,a}||_{B_{R,p}} \leqq c(R)\,||a||_\infty$;
(iii) *if* $\mu_n \to \mu$, $a_n \to a$ *almost everywhere and* $||a_n||_\infty \leqq c$, *then*

$$||\theta^{\mu_n,a_n} - \theta^{\mu,a}||_{B_{R,p}} \to 0 .$$

PROOF. The elementary estimate (i) is left to the reader. It follows by (45) and Theorem 7 that $||\theta^{\mu,a}||_{R,\infty} \leqq c(R)\,||a||_\infty$. From this one obtains (ii) by use of (47) and the auxiliary function λ.

The rest of the proof follows the same pattern as the proof of Lemma 18 and Theorem 9. We can choose a subsequence which satisfies $\theta^{\mu_n,a_n} \to \theta$, uniformly on each $|z| \leqq R$, and $\theta_z^{\mu_n,a_n} \to \theta_z$, $\theta_{\bar z}^{\mu_n,a_n} \to \theta_{\bar z}$, weakly in L_2 for every disk. It is concluded that $\theta_{\bar z} = \mu\theta_z + aw_z^\mu$. Hence $\theta - \theta^{\mu,a}$ is a holomorphic function of w^μ. But it follows by (i) and (45) that $\theta - \theta^{\mu,a} = O(|w^\mu|^2)$ at ∞. Because $\theta - \theta^{\mu,a}$ vanishes at 0 and 1 we conclude that $\theta = \theta^{\mu,a}$, independently of the subsequence. We leave it to the reader to complete the proof by showing that the convergence $\theta^{\mu_n,a_n} \to \theta^{\mu,a}$ takes place in $B_{R,p}$.

5.7. We are now ready to prove the differentiability theorem in full generality. In this theorem $t = (t_1, \cdots, t_n)$ and $s = (s_1, \cdots, s_n)$ denote real vectors.

THEOREM 10. *For all t in some open set Δ, suppose that*

$$\mu(t + s) = \mu(t) + \textstyle\sum_1^n a_i(t)\, s_i + |s|\,\alpha(t, s)$$

with $||\alpha(t, s)||_\infty \leqq c$ *and* $\alpha(t, s) \to 0$ *almost everywhere as* $s \to 0$. *Suppose further that the norms* $||a_i(t + s)||_\infty$ *are bounded and that* $a_i(t+s) \to a_i(t)$ *almost everywhere for* $s \to 0$. *Then* $w^{\mu(t)}$ *has a development*

$$(48) \qquad w^{\mu(t+s)} = w^{\mu(t)} + \sum_{i=1}^{n} \theta^{\mu(t),a_i(t)} s_i + |s| \gamma(t, s)$$

with $\|\gamma(t, s)\|_{B_{R,p}} \to 0$ for $s \to 0$.

PROOF. Let g be the characteristic function of $|z| \leqq 1$, and set

$$\mu(t, u) = g\mu(t) + (1 - g)\mu(u)$$

where $(t, u) = (t_1, \cdots, t_n, u_1, \cdots, u_n) \in \Delta \times \Delta$. By Lemma 21 the function

$$w(t, u) = w^{\mu(t,u)}$$

considered as an element of $B_{R,p}$, has partial derivatives

$$\frac{\partial w}{\partial t_j} = \theta^{\mu(t,u), g a_j(t)}$$

$$\frac{\partial w}{\partial u_j} = \theta^{\mu(t,u), (1-g) a_j(u)} .$$

It follows from (iii) of Lemma 22 that these derivatives are continuous as functions of t and u. Therefore we can write

$$w(t + s, u + v) = w(t, u) + \sum_1^n \theta^{\mu(t,u), g a_j(t)} s_j$$

$$+ \sum_1^n \theta^{\mu(t,u), (1-g) a_j(u)} v_j + \gamma(t, u, s, v)$$

where $\|\gamma(t, u, s, v)\|_{B_{R,p}} \to 0$ as $|s| + |v| \to 0$. We need merely set $u = t$, $v = s$ to obtain the development (48).

COROLLARY. *Let* μ *be as in Theorem* 10, *except that it is defined for* $|z| < 1$. *Then* W^μ *is a continuously differentiable function of* t *with values in* $B_{1,p}$. *The partial derivative*

$$\Omega_j = \frac{\partial W^\mu}{\partial t^j}$$

is the solution of the non-homogeneous Beltrami equation

$$\Omega_{\bar z} = \mu \Omega_z + a_j w_z^\mu$$

which satisfies the conditions

$$W^\mu(0) = W^\mu(1) = 0$$

and

$$\mathrm{Re}\,[\overline{\Omega(z)}\, W^\mu(z)] = 0 \qquad \textit{for } |z| = 1 .$$

5.7. THEOREM 11. *If* μ *depends holomorphically on complex parameters* $(\tau_1, \cdots, \tau_n)$, *as an element of* $L_\infty(\Omega)$, *then* w^μ *is a holomorphic function of these parameters, as an element of the Banach space* $B_{R,p}$.

PROOF. Set $\tau_j = \tau_j' + i\tau_j''$. By Theorem 10 w^μ is a continuously differentiable function of $(\tau_1', \tau_1'', \cdots, \tau_n'')$. Setting $\partial\mu/\partial\tau_j' = a_j$, $\partial\mu/\partial\tau_j'' = b_j$ we have $a_j + ib_j = 0$ and, by Theorem 10,

$$\frac{\partial w^{\mu}}{\partial \tau_j'} - i\frac{\partial w^{\mu}}{\partial \tau_j''} = \theta^{\mu, a_j - ib_j} = 0 \ .$$

COROLLARY. *If $\mu \in L_\infty(|z|<1)$ depends analytically on real parameters $(t_1, \cdots, t_n)$, so does W^{μ}, as an element of the Banach space $B_{1,p}$.*

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