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Riemann zeta fractional derivative—functional equation and link with primes

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Abstract

This paper outlines further properties concerning the fractional derivative of the Riemann ζ function. The functional equation, computed by the introduction of the Grünwald–Letnikov fractional derivative, is rewritten in a simplified form that reduces the computational cost. Additionally, a quasisymmetric form of the aforementioned functional equation is derived (symmetric up to one complex multiplicative constant). The second part of the paper examines the link with the distribution of prime numbers. The Dirichlet η function suggests the introduction of a complex strip as a fractional counterpart of the critical strip. Analytic properties are shown, particularly that a Dirichlet series can be linked with this strip and expressed as a sum of the fractional derivatives of ζ . Finally, Theorem 4.3 links the fractional derivative of ζ with the distribution of prime numbers in the left half-plane.

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1 Introduction

In the last years, fractional calculus has played a significant role in many scientific fields and has been utilized for several applications (e.g., electromagnetism, signal processing). It is worth noting that the fractional derivative is not uniquely defined. This lack of uniqueness is undoubtedly the weakest point of the fractional calculus. Nevertheless, remarkable progress has been made toward addressing this and other issues in the last decade with relevant results [1–7], thus making the problem of independent interest for pure and applied mathematicians.

Fractional calculus has recently been applied to the theory of meromorphic functions (see, e.g., [8–11]). In particular, the α -order fractional derivative of the Riemann ζ function given by

$$\zeta^{(\alpha)}(s) = e^{i\pi\alpha} \sum_{n=2}^{\infty} \frac{\log^{\alpha} n}{n^s}, \quad s \in \mathbb{C}, \alpha \in \mathbb{R}_{>0} \setminus \mathbb{N}, \quad (1.1)$$

was computed for the first time in 2015 [12]. By introducing both the Hurwitz ζ function and the Dirichlet series, two coherent generalizations of (1.1) were obtained. These frac-

tional derivatives reduce to the derivative in (1.1) for $a = 1$ and $f(n) = 1$, which are the cases in which the Hurwitz ζ function and the Dirichlet series are nothing but the Riemann ζ function, respectively [13].

The main problem of fractional derivative in (1.1) is represented by the functional equation. Fortunately, the equation sought can be derived by introducing the forward Grünwald–Letnikov fractional derivative. The aforementioned functional equation was derived since this fractional derivative satisfies the generalized Leibniz rule [14, 15]. However, as often happens in mathematics, the solution of one problem gives rise to another more important problem. The right-hand side of the functional equation in (2.7) entails extremely high computational cost due to the presence of three infinite series. Unfortunately, each finite approximation of these series with different numerical simulations produces an overflow. To minimize the computational cost, we follow the approach proposed by Apostol and Spira [16, 17] for the integer-order derivatives of ζ . As a result, the aforementioned right-hand side is reduced from three to only one infinite series, with a substantial decrease in terms of computational cost (see Sect. 3). Furthermore, the functional equation of $\zeta^{(\alpha)}$ is written as the sum of sines and cosines. In 2017, Guariglia and Silvestrov [14] introduced the problem of the symmetry of (2.7). Accordingly, a quasisymmetric form of the functional equation of $\zeta^{(\alpha)}$ is derived and discussed in Sect. 3.

The second part of the paper investigates other properties of $\zeta^{(\alpha)}$. The fractional equivalent of the critical strip [18], that is, $(\alpha, 1 + \alpha)$, is presented and broadly discussed. The infinite series in $(2.9)_2$ associated with this strip is written in terms of $\zeta^{(\alpha)}$. The Riemann conjecture entails that the critical strip is linked with the nontrivial zeros of ζ , which should all lie on $\operatorname{Re} s = 1/2$ (critical line). Consequently, the distribution of prime numbers along the line $\operatorname{Re} s = 1/2 + \alpha$ is considered. Based on these considerations, Sect. 4 discusses the link with prime numbers.

The rest of this paper is organized as follows. The next section provides remarks on $\zeta^{(\alpha)}$. Section 3 reports and describes in detail equivalent forms of the functional equation of $\zeta^{(\alpha)}$. Section 4 outlines the link with prime numbers, together with the analytic properties of the strip $(\alpha, 1 + \alpha)$. Section 5 summarizes the results of this work and draws conclusions.

2 Notations and preliminary results

For the purposes of this paper α and k are always elements of \mathbb{R} and \mathbb{N}_0 , respectively, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. A generic complex variable is denoted by s and the imaginary unit by i . Moreover, for every $x \in \mathbb{R}$ and $n \in \mathbb{N}_0$, the symbol x^n indicates the so-called falling factorial [19].

2.1 Fractional calculus and $\zeta^{(\alpha)}$

In recent years, the fractional calculus of holomorphic functions has merited increasing consideration from the international mathematics community. It has several applications in mathematics, theoretical physics, and information engineering [18]. Ortigueira is currently investigating broad applications in information engineering [15] and laying the foundations of fractional linear-time invariant systems. Furthermore, he has provided a suitable generalization of the Caputo derivative to the complex plane [12, 20], that is,

$${}_c D^\alpha f(s) = \frac{e^{i(\pi-\theta)(\alpha-m)}}{\Gamma(m-\alpha)} \int_0^\infty \frac{d^m}{ds^m} \frac{f(xe^{i\theta} + s)}{x^{\alpha-m+1}} dx, \quad (2.1)$$

where f is a complex function of the complex variable s , $m - 1 < \alpha < m \in \mathbb{N}$ and $\theta = [0, 2\pi)$. The aforementioned fractional derivative can be written in terms of distribution theory [21].

The Riemann ζ function is defined as follows:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s \in \mathbb{C}.$$

Consequently, ζ converges for all complex numbers s such that $\operatorname{Re} s > 1$. Riemann showed that ζ possesses a unique analytical continuation to the entire complex plane, except for a simple pole at $s = 1$ with residue 1 [22]. The fractional derivative in (2.1) enables us to compute the α -order fractional derivative of the Riemann ζ function, that is, the derivative in (1.1). The convergence of the complex series associated is shown in [12]. In particular, it pointwise converges in the half-plane $\operatorname{Re} s > 1 + \alpha$, and thus the convergence of $\zeta^{(\alpha)}$ depends on the fractional order α . Furthermore, the k -order integer derivative of the Riemann ζ function is given [16, 17] by

$$\zeta^{(k)}(s) = e^{i\pi k} \sum_{n=2}^{\infty} \frac{\log^k n}{n^s}, \quad (2.2)$$

so that the derivative in (1.1) is a straightforward fractional generalization of the derivative in (2.2). As mentioned in the Introduction, (2.1) was applied to two generalizations of the Riemann ζ function (Hurwitz ζ function and Dirichlet series). The results achieved are in accordance with theory of the Riemann ζ function [13].

2.2 Functional equation of $\zeta^{(\alpha)}$

The Riemann ζ function, introduced in 1859 [22], is characterized by the following functional equation:

$$\zeta(s) = 2(2\pi)^{s-1} \Gamma(1-s) \zeta(1-s) \sin \frac{\pi s}{2}, \quad \text{for any } s \in \mathbb{C}, \quad (2.3)$$

that is,

$$\zeta(1-s) = 2(2\pi)^{-s} \Gamma(s) \zeta(s) \cos \frac{\pi s}{2}. \quad (2.4)$$

The literature calls (2.3) and (2.4) asymmetric equations. In fact, the functional equation of ζ can also be written symmetrically as follows:

$$\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-(1-s)/2} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s). \quad (2.5)$$

Introducing a variant of the Riemann ζ function, the Riemann Xi function, defined by

$$\xi(s) = \frac{1}{2} s(1-s) \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s),$$

(2.5) becomes $\xi(s) = \xi(1-s)$. The function ξ is an entire function of s since $\pi^{-s/2} \Gamma(-s/2) \zeta(s)$ has simple poles at $s = 0$ and $s = 1$.

Unfortunately, the fractional derivative in (2.1) is unsuitable for deriving a functional equation of $\zeta^{(\alpha)}$. In fact, the integral in (2.1) entails computational problems. Hence, the fractional operator in (2.1) is substituted with the forward Grünwald–Letnikov fractional derivative.

Definition 2.1 (see [15]) Let $\alpha \in \mathbb{R}$, $h \in \mathbb{R}_{\geq 0}$, and f be a complex-valued function of the complex variable s . The α -order forward Grünwald–Letnikov fractional derivative of f is defined by

$$D_f^\alpha f(s) = \lim_{h \rightarrow 0^+} \frac{\sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k f(s - kh)}{h^\alpha}. \quad (2.6)$$

Unlike the Caputo derivative in (2.1), Definition 2.1 satisfies the generalized Leibniz rule. Thus, the fractional derivative $\zeta^{(\alpha)}$ can be recomputed by (2.6). The result achieved is in accordance with (1.1). The proof provided in [14] holds for $\alpha \in \mathbb{R}_{>0} \setminus \mathbb{N}$. Since Definition 2.1 makes sense for any $\alpha \in \mathbb{R}$, (1.1) becomes

$$\zeta^{(\alpha)}(s) = e^{i\pi\alpha} \sum_{n=2}^{\infty} \frac{\log^\alpha n}{n^s}, \quad s \in \mathbb{C}, \alpha \in \mathbb{R}.$$

The same reasoning fails for the functional equation of $\zeta^{(\alpha)}$ due to the presence of the generalized Leibniz rule (see [15] for more details). Accordingly, for any $\alpha \in \mathbb{R}_{>0} \setminus \mathbb{N}$, it is

$$\begin{aligned} \zeta^{(\alpha)}(s) &= 2(2\pi)^{s-1} e^{i\pi\alpha} \sum_{h=0}^{\infty} \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} A_{h,j,n}^\alpha \zeta^{(n)}(1-s) \left(-\frac{\pi}{2}\right)^j \\ &\quad \times \frac{\Gamma^{(h)}(1-s)}{\log^{h+j+n-\alpha} 2\pi} \sin \frac{\pi(s+j)}{2}, \end{aligned} \quad (2.7)$$

where $A_{h,j,n}^\alpha = \frac{\alpha^{h+j+n}}{h!j!n!}$. The complete proof of (2.7) can be found in [14]. As previously mentioned, this equation suffers from problems of computational complexity. In fact, each numerical approximation of (2.7) produces a buffer overflow. Section 3 discusses such a computational issue.

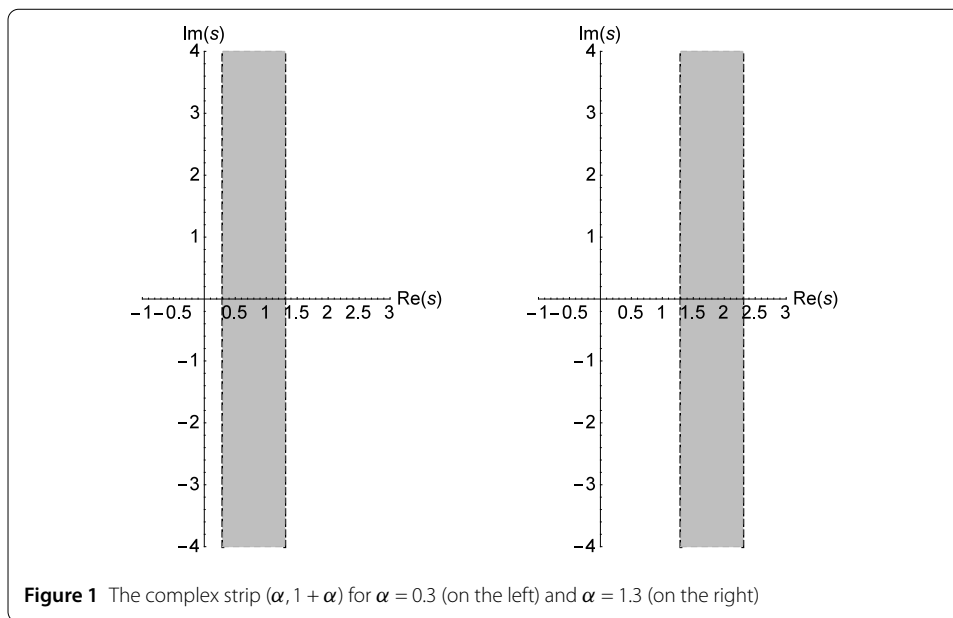
2.3 Critical strip and $\zeta^{(\alpha)}$

Prime numbers have played a central role in mathematics since the age of Euclid. In recent years, the distribution of prime numbers has been widely applied in science as well as engineering. The Riemann ζ function and the distribution of prime numbers are linked through the following representation:

$$\zeta(s) = \prod_{p \in \mathbb{P}} \frac{1}{1 - p^{-s}}, \quad (2.8)$$

where \mathbb{P} is the set of prime numbers [22]. $\zeta(s) = 0$ for any negative even integers, that is, for $s = -2, -4, 6, \dots$ (trivial zeros). As a consequence of (2.8), there are no zeros in the half-plane $\operatorname{Re} s > 1$, so that all the nontrivial zeros must belong to the (critical) strip $(0, 1)$.

The critical strip has great importance in the theory of prime numbers. Investigating $\zeta^{(\alpha)}$ leads to the introduction of the complex strip $(\alpha, 1 + \alpha)$. Note that (2.1) and (2.6) allow



the computation of the derivative $\eta^{(\alpha)}$, where η is the Dirichlet η function. Both are given [18] by

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \quad s \in \mathbb{C},$$

and

$$\eta^{(\alpha)}(s) = e^{i\pi\alpha} \sum_{n=2}^{\infty} (-1)^{n-1} \frac{\log^{\alpha} n}{n^s}, \quad s \in \mathbb{C}, \alpha \in \mathbb{R}.$$

Moreover, it can be shown [18] that

$$\begin{cases} \zeta(s) = \frac{\eta(s)}{1-2^{1-s}}, \\ \eta^{(\alpha)}(s) = \zeta^{(\alpha)}(s) - e^{i\pi\alpha} \cdot 2^{1-s} \sum_{n=1}^{\infty} \frac{\log^{\alpha} 2n}{n^s}. \end{cases} \quad (2.9)$$

$\zeta^{(\alpha)}$ and $\eta^{(\alpha)}$ converge for $\operatorname{Re} s > 1 + \alpha$ and $\operatorname{Re} s > \alpha$, respectively [18]. Consequently, for each fixed $\alpha \in \mathbb{R}$, the fractional derivative $\zeta^{(\alpha)}$ is associated with a unique complex strip $(\alpha, 1 + \alpha)$. Being

$$(\alpha, 1 + \alpha) \xrightarrow{\alpha \rightarrow 0} (0, 1),$$

this complex strip can be considered the fractional counterpart of the critical strip. Figure 1 shows the shift by α in the half-plane of convergence (due to the action of α -order fractional derivative on ζ). The strip $(\alpha, 1 + \alpha)$ can open up new scenarios in mathematics. In fact, some properties of the critical strip can be invariant under the transformation $s \rightarrow s + \alpha$ (e.g., zero-free regions, Riemann conjecture).

3 Equivalent forms of the functional equation

In the literature, Spira and Apostol were the first to investigate the higher integer derivatives of the Riemann ζ function [16, 17]. In particular, Spira determined both the zero-free regions associated with the k -order integer derivative $\zeta^{(k)}$ and the functional equation, that is,

$$\begin{aligned} (-1)^k \zeta^{(k)}(1-s) &= 2(2\pi)^{-s} \sum_{h=0}^k \sum_{n=0}^k \left(a_{hkn} \cos \frac{\pi s}{2} + b_{hkn} \sin \frac{\pi s}{2} \right) \\ &\times \Gamma^{(h)}(s) \zeta^{(n)}(s). \end{aligned} \quad (3.1)$$

Note that the terms a_{hkn} and b_{hkn} are constants [23]. Replacing s by $1-s$ yields

$$\begin{aligned} \zeta^{(k)}(s) &= 2(2\pi)^{s-1} e^{i\pi k} \sum_{h=0}^k \sum_{n=0}^k \left(a_{hkn} \cos \frac{\pi(1-s)}{2} + b_{hkn} \sin \frac{\pi(1-s)}{2} \right) \\ &\times \Gamma^{(h)}(1-s) \zeta^{(n)}(1-s). \end{aligned} \quad (3.2)$$

3.1 Simplified form

The undertaken study aims to repurpose the outcomes contained in [16] to write the functional equation of $\zeta^{(\alpha)}$ in a simplified form. In particular, it allows us to reduce the computational cost of (2.7) to only one infinite series (see Theorem 3.3). Furthermore, (3.1) and (3.2) express the functional equation of $\zeta^{(k)}$ in terms of sines and cosines, that is, of complex exponentials. The next result, reported below, is due to Apostol [16].

Proposition 3.1 *Let $k \in \mathbb{N}$. For any $s \in \mathbb{C}$, it is*

$$\begin{aligned} \zeta^{(k)}(s) &= \sum_{h=0}^k \binom{k}{h} e^{i\pi(k-h)} \left(e^{(1-s)w} w^{k-h} - e^{(1-s)\bar{w}-i\pi} (\bar{w})^{k-h} \right) \\ &\times \left(\Gamma(1-s) \zeta(1-s) \right)^{(h)}, \end{aligned} \quad (3.3)$$

where $w = -\log 2\pi - i\pi/2$.

Proof The main idea of the proof is that (2.3) can be written as follows:

$$\begin{aligned} \zeta(s) &= \Gamma(1-s) \zeta(1-s) (2\pi)^{s-1} 2 \sin \frac{\pi s}{2} \\ &= \Gamma(1-s) \zeta(1-s) \left(e^{(1-s)[- \log 2\pi - i\pi/2]} - e^{(1-s)[- \log 2\pi + i\pi/2] - i\pi} \right), \end{aligned}$$

being

$$\begin{cases} (2\pi)^{s-1} = e^{(s-1)\log 2\pi}, \\ 2 \sin \frac{\pi s}{2} = e^{i\frac{\pi}{2}(s-1)} - e^{-i\frac{\pi}{2}(s+1)}, \\ e^{-i\frac{\pi}{2}(s+1)} = e^{i\frac{\pi}{2}(1-s)-i\pi}. \end{cases}$$

By introducing the function ψ defined by

$$\psi(s, w, z) = \Gamma(s) \zeta(s) e^{sw+z}, \quad z \in \mathbb{C},$$

it is

$$\zeta(s) = \psi(1-s, w) - \psi(1-s, \bar{w}, -i\pi), \quad (3.4)$$

with $\psi(1-s, w) = \psi(1-s, w, 0)$. Differentiating (3.4) k times gives

$$\zeta^{(k)}(s) = \frac{d^k}{ds^k}(\psi(1-s, w)) - \frac{d^k}{ds^k}(\psi(1-s, \bar{w}, -i\pi)).$$

Since

$$\begin{aligned} \frac{d^k}{ds^k}(\psi(1-s, w, z)) &= \frac{d^k}{ds^k}(\Gamma(1-s)\zeta(1-s)e^{(1-s)w+z}) \\ &= \sum_{h=0}^k \binom{k}{h} (\Gamma(1-s)\zeta(1-s))^{(h)} (e^{(1-s)w+z})^{(k-h)}, \end{aligned}$$

and

$$\frac{d^{k-h}}{ds^{k-h}}(e^{(1-s)w+z}) = e^{i\pi(k-h)} w^{k-h} e^{(1-s)w+z},$$

we have

$$\frac{d^k}{ds^k}(\psi(1-s, w, z)) = \sum_{h=0}^k \binom{k}{h} e^{i\pi(k-h)} e^{(1-s)w+z} w^{k-h} (\Gamma(1-s)\zeta(1-s))^{(h)}.$$

Therefore

$$\zeta^{(k)}(s) = \sum_{h=0}^k \binom{k}{h} e^{i\pi(k-h)} (e^{(1-s)w} w^{k-h} - e^{(1-s)\bar{w}-i\pi} (\bar{w})^{k-h}) (\Gamma(1-s)\zeta(1-s))^{(h)},$$

taking into account the $2\pi i$ -periodicity of the complex exponential. \square

The fractional generalization of the functional equation in (3.3) is based on the following result.

Lemma 3.2 *Let $\alpha \in \mathbb{R}$ and $w \in \mathbb{C}$ such that $\operatorname{Re} w < 0$. For any $s \in \mathbb{C}$, we get*

$$D_f^\alpha e^{(1-s)w} = e^{i\pi\alpha} e^{(1-s)w} w^\alpha.$$

Proof Definition 2.1, for $f(s) = e^{(1-s)w}$, gives

$$D_f^\alpha e^{(1-s)w} = e^{(1-s)w} \lim_{h \rightarrow 0^+} \frac{\sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k e^{wkh}}{h^\alpha}.$$

The series above converges to $g(w) = (1 - e^{wh})^\alpha$ if $|e^{wh}| < 1$, that is, $\operatorname{Re} w < 0$. From L'Hôpital's rule it follows

$$D_f^\alpha e^{(1-s)w} = e^{(1-s)w} \lim_{h \rightarrow 0^+} \frac{(1 - e^{wh})^\alpha}{h^\alpha} = e^{(1-s)w} \left(\lim_{h \rightarrow 0^+} \frac{1 - e^{wh}}{h} \right)^\alpha = e^{i\pi\alpha} e^{(1-s)w} w^\alpha. \quad \square$$

Relying on Lemma 3.2, we are able to give the fractional counterpart of (3.3).

Theorem 3.3 *Let $\alpha \in \mathbb{R}_{>0} \setminus \mathbb{N}$ and $w = -\log 2\pi - i\pi/2$. For any $s \in \mathbb{C}$, it is*

$$\begin{aligned} \zeta^{(\alpha)}(s) &= \sum_{h=0}^{\infty} \binom{\alpha}{h} e^{i\pi(\alpha-h)} \left(e^{(1-s)w} w^{\alpha-h} - e^{(1-s)\bar{w}-i\pi} (\bar{w})^{\alpha-h} \right) \\ &\quad \times (\Gamma(1-s)\zeta(1-s))^{(h)}. \end{aligned} \quad (3.5)$$

Proof From (3.4) we get

$$\zeta^{(\alpha)}(s) = D_f^\alpha \psi(1-s, w) - D_f^\alpha \psi(1-s, \bar{w}, -i\pi). \quad (3.6)$$

The generalized Leibniz rule gives

$$D_f^\alpha \psi(1-s, w, z) = \sum_{h=0}^{\infty} \binom{\alpha}{h} (\Gamma(1-s)\zeta(1-s))^{(h)} e^z (e^{(1-s)w})^{(\alpha-h)},$$

and taking into account Lemma 3.2 it follows

$$\begin{aligned} D_f^\alpha \psi(1-s, w, z) &= \sum_{h=0}^{\infty} \binom{\alpha}{h} e^{i\pi(\alpha-h)} e^{(1-s)w+z} w^{\alpha-h} \\ &\quad \times (\Gamma(1-s)\zeta(1-s))^{(h)}. \end{aligned} \quad (3.7)$$

ξ is an entire function of s , thus substituting (3.7) into (3.6), the proof follows. \square

Theorem 3.3 shows that (3.5) has less computational cost than (2.7). Moreover, (3.2) can be generalized as the next result points out.

Theorem 3.4 *Let $\alpha \in \mathbb{R}_{>0} \setminus \mathbb{N}$. For any $s \in \mathbb{C}$, it is*

$$\begin{aligned} \zeta^{(\alpha)}(s) &= 2(2\pi)^{s-1} e^{i\pi\alpha} \sum_{h=0}^{\infty} \sum_{n=0}^{\infty} \left(a_{h\alpha n} \sin \frac{\pi s}{2} + b_{h\alpha n} \cos \frac{\pi s}{2} \right) \\ &\quad \times \Gamma^{(h)}(1-s) \zeta^{(n)}(1-s), \end{aligned} \quad (3.8)$$

where the coefficients $a_{h\alpha n}$ and $b_{h\alpha n}$ are given by

$$\begin{cases} a_{h\alpha n} = \sum_{j=0}^{\infty} \frac{A_{h,j,n}^\alpha}{\log^{h+j+n-\alpha} 2\pi} \left(-\frac{\pi}{2}\right)^j \cos \frac{\pi j}{2}, \\ b_{h\alpha n} = \sum_{j=0}^{\infty} \frac{A_{h,j,n}^\alpha}{\log^{h+j+n-\alpha} 2\pi} \left(-\frac{\pi}{2}\right)^j \sin \frac{\pi j}{2}. \end{cases}$$

Proof The trigonometric identity

$$\sin \frac{\pi(s+j)}{2} = \sin \frac{\pi s}{2} \cos \frac{\pi j}{2} + \cos \frac{\pi s}{2} \sin \frac{\pi j}{2}$$

gives (3.8) when substituted in (2.7). \square

Theorem 3.4 takes on particular relevance in harmonic analysis. In fact, (3.8) expresses the functional equation of $\zeta^{(\alpha)}$ as a sum of sines and cosines. Additionally, (3.5) and (3.8), which are two different forms of (2.7), represent a step forward in our investigation of $\zeta^{(\alpha)}$.

3.2 Symmetric form

The problem concerning the symmetric form of (2.7) is dealt with here.

Lemma 3.5 *Let $\alpha \in \mathbb{R}$. For any $s \in \mathbb{C}$, it is*

$$D_f^\alpha \zeta(1-s) = e^{-i\pi\alpha} \zeta^{(\alpha)}(1-s).$$

Proof The fractional derivative in (2.6), for $f(s) = \zeta(1-s)$, gives

$$\begin{aligned} D_f^\alpha \zeta(1-s) &= \lim_{h \rightarrow 0^+} \frac{\sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k \zeta(1-s+kh)}{h^\alpha} \\ &= \sum_{n=1}^{\infty} \frac{1}{n^{1-s}} \lim_{h \rightarrow 0^+} \frac{1}{h^\alpha} \sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k n^{-kh}. \end{aligned} \quad (3.9)$$

The binomial series now yields

$$\sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k n^{-kh} = (1 - n^{-h})^\alpha. \quad (3.10)$$

From L'Hôpital's rule it follows

$$\lim_{h \rightarrow 0^+} \left(\frac{1 - n^{-h}}{h} \right)^\alpha = \left(\lim_{h \rightarrow 0^+} \frac{1 - n^{-h}}{h} \right)^\alpha = \log^\alpha n. \quad (3.11)$$

Combining (3.9), (3.10), and (3.11), the proof is straightforward. \square

Theorem 3.6 *Let $\alpha \in \mathbb{R}_{>0} \setminus \mathbb{N}$. Moreover, we set T_α as follows:*

$$T_\alpha(s) = \pi^{-s/2} \sum_{h=0}^{\infty} \sum_{j=0}^h B_{h,j}^\alpha \frac{e^{i\pi j}}{2^h} \log^j \pi \Gamma^{(h-j)} \left(\frac{s}{2} \right) \zeta^{(\alpha-h)}(s).$$

For any $s \in \mathbb{C}$, it is

$$T_\alpha(s) \doteq T_\alpha(1-s), \quad (3.12)$$

where the symbol \doteq denotes equality up to a multiplicative constant and $B_{h,j}^\alpha = \frac{\alpha^h}{h!(h-j)!}$.

Proof Differentiating both members of (2.5) α times, it is

$$\begin{aligned} &\sum_{h=0}^{\infty} \binom{\alpha}{h} \left(\pi^{-s/2} \Gamma \left(\frac{s}{2} \right) \right)^{(h)} \zeta^{(\alpha-h)}(s) \\ &= \sum_{k=0}^{\infty} \binom{\alpha}{k} \left(\pi^{-(1-s)/2} \Gamma \left(\frac{1-s}{2} \right) \right)^{(k)} (\zeta(1-s))^{(\alpha-k)}. \end{aligned} \quad (3.13)$$

Direct calculation leads to

$$\left(\pi^{-s/2} \Gamma\left(\frac{s}{2}\right)\right)^{(h)} = \sum_{j=0}^h \binom{h}{j} (\pi^{-s/2})^{(j)} \left(\Gamma\left(\frac{s}{2}\right)\right)^{(h-j)}, \quad (3.14)$$

with

$$\begin{cases} (\pi^{-s/2})^{(j)} = e^{i\pi j} \pi^{-s/2} \frac{\log^j \pi}{2^j}, \\ \left(\Gamma\left(\frac{s}{2}\right)\right)^{(h-j)} = \frac{\Gamma^{(h-j)}(s/2)}{2^{h-j}}. \end{cases} \quad (3.15)$$

Substituting (3.14) and (3.15) into the left-hand side of (3.13) yields

$$\begin{aligned} & \sum_{h=0}^{\infty} \binom{\alpha}{h} \left(\pi^{-s/2} \Gamma\left(\frac{s}{2}\right)\right)^{(h)} \zeta^{(\alpha-h)}(s) \\ &= \sum_{h=0}^{\infty} \binom{\alpha}{h} \zeta^{(\alpha-h)}(s) \sum_{j=0}^h \binom{h}{j} e^{i\pi j} \pi^{-s/2} \frac{\log^j \pi}{2^j} \frac{\Gamma^{(h-j)}(s/2)}{2^{h-j}} \\ &= \pi^{-s/2} \sum_{h=0}^{\infty} \sum_{j=0}^h B_{h,j}^{\alpha} \frac{e^{i\pi j}}{2^h} \log^j \pi \Gamma^{(h-j)}\left(\frac{s}{2}\right) \zeta^{(\alpha-h)}(s). \end{aligned} \quad (3.16)$$

The same holds for the right-hand side of (3.13). In fact, we get

$$\left(\pi^{-(1-s)/2} \Gamma\left(\frac{1-s}{2}\right)\right)^{(h)} = \sum_{j=0}^h \binom{h}{j} (\pi^{-(1-s)/2})^{(j)} \left(\Gamma\left(\frac{1-s}{2}\right)\right)^{(h-j)},$$

with

$$\begin{cases} (\pi^{-(1-s)/2})^{(j)} = \pi^{-(1-s)/2} \frac{\log^j \pi}{2^j}, \\ \left(\Gamma\left(\frac{1-s}{2}\right)\right)^{(h-j)} = \frac{e^{i\pi(h-j)}}{2^{h-j}} \Gamma^{(h-j)}\left(\frac{1-s}{2}\right). \end{cases}$$

Thus, Lemma 3.5 gives

$$\begin{aligned} & \sum_{h=0}^{\infty} \binom{\alpha}{h} \left(\pi^{-(1-s)/2} \Gamma\left(\frac{1-s}{2}\right)\right)^{(h)} (\zeta(1-s))^{(\alpha-h)} \\ &= \sum_{h=0}^{\infty} \binom{\alpha}{h} \cdot e^{-i\pi(\alpha-h)} \zeta^{(\alpha-h)}(1-s) \sum_{j=0}^h \binom{h}{j} \pi^{-(1-s)/2} \frac{\log^j \pi}{2^j} \\ & \quad \times \frac{e^{i\pi(h-j)}}{2^{h-j}} \Gamma^{(h-j)}\left(\frac{1-s}{2}\right) \\ &= \pi^{-(1-s)/2} \sum_{h=0}^{\infty} \sum_{j=0}^h B_{h,j}^{\alpha} \cdot \frac{e^{i\pi(j-\alpha)}}{2^h} \log^j \pi \Gamma^{(h-j)}\left(\frac{1-s}{2}\right) \zeta^{(\alpha-h)}(1-s), \end{aligned} \quad (3.17)$$

taking into account the $2\pi i$ -periodicity of the complex exponential. Substituting (3.16) and (3.17) into (3.13), the proof follows. \square

Note that (3.12) is the symmetric form of (2.7) up to the multiplicative constant $e^{-i\pi\alpha}$, that is, the fractional equivalent of (2.5). Therefore, Theorem 3.6 provides a quasisymmetric functional equation of $\zeta^{(\alpha)}$ by opening up new scenarios in functional fractional analysis.

4 Distribution of prime numbers and complex strip $(\alpha, 1 + \alpha)$

This section is devoted to investigation of the strip $(\alpha, 1 + \alpha)$. First and foremost, $\eta^{(\alpha)}$ is written in terms of $\zeta^{(\alpha)}$. The second part discusses the link between $\zeta^{(\alpha)}$ and the distribution of prime numbers.

4.1 Representation of $\eta^{(\alpha)}$ in terms of $\zeta^{(\alpha)}$

Theorem 4.1 *Let $\alpha \in \mathbb{R}_{>0} \setminus \mathbb{N}$. For any $s \in \mathbb{C}$ such that $\operatorname{Re} s > \alpha$, the functions $\eta^{(\alpha)}$ and $\zeta^{(\alpha)}$ are linked by*

$$\eta^{(\alpha)}(s) = \zeta^{(\alpha)}(s) - 2^{1-s} \sum_{k=0}^{\infty} \binom{\alpha}{k} e^{i\pi k} \log^k 2 \zeta^{(\alpha-k)}(s). \quad (4.1)$$

Furthermore, the series in (2.9)₂ can be written in terms of $\zeta^{(\alpha)}$ as follows:

$$\sum_{n=1}^{\infty} \frac{\log^{\alpha} 2n}{n^s} = \sum_{k=0}^{\infty} \binom{\alpha}{k} e^{i\pi(k-\alpha)} \log^k 2 \zeta^{(\alpha-k)}(s). \quad (4.2)$$

Proof From (2.9)₁ it follows

$$\eta^{(\alpha)}(s) = \zeta^{(\alpha)}(s) - D_f^{\alpha} (2^{1-s} \zeta(s)). \quad (4.3)$$

The generalized Leibniz rule implies

$$D_f^{\alpha} (2^{1-s} \zeta(s)) = 2^{1-s} \sum_{k=0}^{\infty} \binom{\alpha}{k} e^{i\pi k} \log^k 2 \zeta^{(\alpha-k)}(s). \quad (4.4)$$

Combining (4.3) and (4.4) gives the first part of Theorem 4.1. Finally, comparison of (2.9)₂ and (4.1) completes the proof. \square

As mentioned in Sect. 2, the condition $\operatorname{Re} s > \alpha$ in Theorem 4.1 assures convergence of $\eta^{(\alpha)}$. Additionally, (4.1) is the fractional generalization of (2.9)₁. In fact, with some caution it is

$$\begin{aligned} \zeta^{(\alpha)}(s) - 2^{1-s} \sum_{k=0}^{\infty} \binom{\alpha}{k} e^{i\pi(k-\alpha)} \log^k 2 \zeta^{(\alpha-k)}(s) &\xrightarrow{\alpha \rightarrow 0^+} \zeta(s) - 2^{1-s} \zeta(s) \\ &= \zeta(s)(1 - 2^{1-s}). \end{aligned}$$

Corollary 4.2 *Let $\alpha \in \mathbb{R}_{>0}$. The series in (2.9)₂ can be expressed in terms of $\zeta^{(\alpha)}$ by*

$$\sum_{n=1}^{\infty} \frac{\log^{\alpha} 2n}{n^s} = \sum_{k=0}^{\tilde{k}} \binom{\alpha}{k} e^{i\pi(k-\alpha)} \log^k 2 \zeta^{(\alpha-k)}(s),$$

where

$$\tilde{k} = \begin{cases} \alpha, & \alpha \in \mathbb{N}, \\ \infty, & \alpha \in \mathbb{R}_{>0} \setminus \mathbb{N}. \end{cases}$$

Proof The generalized Leibniz rule reduces to the Leibniz rule for $\alpha \in \mathbb{N}$. Accordingly, the proof follows from (4.2). \square

Corollary 4.2 enables us to write the series in (2.9)₂ in terms of $\zeta^{(\alpha)}$. Note that this result is in accordance with theory. Fix $m \in \mathbb{N}$. Therefore

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\log 2n}{n^s} &= \log 2\zeta(s) - \zeta'(s) = \sum_{k=0}^1 \binom{1}{k} (-1)^{1-k} \log^k 2\zeta^{(1-k)}(s), \\ \sum_{n=1}^{\infty} \frac{\log^2 2n}{n^s} &= \log^2 2\zeta(s) - 2\log 2\zeta'(s) + \zeta''(s) \\ &= \sum_{k=0}^2 \binom{2}{k} (-1)^{2-k} \log^k 2\zeta^{(2-k)}(s), \\ &\vdots \\ \sum_{n=1}^{\infty} \frac{\log^m 2n}{n^s} &= \sum_{k=0}^m \binom{m}{k} (-1)^{m-k} \log^k 2\zeta^{(m-k)}(s) \\ &= \sum_{k=0}^m \binom{m}{k} e^{i\pi(m-k)} \log^k 2\zeta^{(m-k)}(s), \end{aligned}$$

which is Corollary 4.2 for $\alpha = m$.

4.2 The role of $\zeta^{(\alpha)}$ in the distribution of prime numbers

The Euler product in (2.8) cannot be used to investigate the link between $\zeta^{(\alpha)}$ and the distribution of prime numbers due to the nonmultiplicativity of \log^α . In addition, the introduction of the strip $(\alpha, 1 + \alpha)$ raises more delicate problems, such as the fractional generalization of both the Riemann hypothesis and the critical line $\operatorname{Re} s = 1/2$. According to the approach given in [18], the principal candidate for this role is the line $\operatorname{Re} s = 1/2 + \alpha$.

Unfortunately, the zeros of $\zeta^{(\alpha)}$ represent a major challenge. In fact, even now the zeros of the integer derivative $\zeta^{(k)}$ remain an open problem. In 1974, assuming the Riemann hypothesis and $k > 1$, Levinson and Montgomery [24] proved that ζ' has no zeros in the half-plane $0 < \operatorname{Re} s < 1/2$ and $\zeta^{(k)}$ has at most finitely many zeros in $\operatorname{Re} s < 1/2$. Spira showed that most of these zeros lie in $0 \leq \operatorname{Re} s \leq 1/2 + \delta$ for $\delta > 0$ [23]. Hence, they are located close to the critical line. In particular, many scholars agree that nontrivial zeros of $\zeta^{(k)}$ are located randomly to the right of $\operatorname{Re} s = 1/2$. However, the left half-plane $\operatorname{Re} s < 0$ is not zero-free since ζ'' has one pair of nontrivial zeros near $-0.355 \pm 3.591i$ [23]. Thus, the current state of the art does not facilitate any investigation into zeros of $\zeta^{(\alpha)}$, that is, the localization of prime numbers near the line $\operatorname{Re} s = 1/2 + \alpha$. Nevertheless, $\zeta^{(\alpha)}$ can be linked with the distribution of prime numbers, as shown below.

Theorem 4.3 Let $\alpha \in \mathbb{R}$ and \mathbb{P} be the set of prime numbers. For any $s \in \mathbb{C}$ such that $\operatorname{Re} s < 0$, it is

$$\zeta^{(\alpha)}(s) \sim \sum_{p \in \mathbb{P}} \sum_{t=0}^{\infty} \frac{\log^{\alpha} p^t}{p^{-st}}, \quad (4.5)$$

where the symbol \sim means that both sides above converge or diverge together.

Proof From (2.8) we get

$$\begin{aligned} \zeta^{(\alpha)}(s) &= \lim_{h \rightarrow 0^+} \frac{\sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k \zeta(s - hk)}{h^{\alpha}} \\ &= \lim_{h \rightarrow 0^+} \frac{1}{h^{\alpha}} \sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k \prod_{p \in \mathbb{P}} \frac{1}{1 - p^{-s+kh}}. \end{aligned}$$

The link between series and infinite products [25] gives

$$\prod_{p \in \mathbb{P}} \frac{1}{1 - p^{-s+kh}} = \prod_{p \in \mathbb{P}} \left(1 + \frac{p^{-s+kh}}{1 - p^{-s+kh}} \right) \sim \sum_{p \in \mathbb{P}} \frac{p^{-s+kh}}{1 - p^{-s+kh}},$$

so that

$$\begin{aligned} \zeta^{(\alpha)}(s) &\sim \lim_{h \rightarrow 0^+} \frac{1}{h^{\alpha}} \sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k \sum_{p \in \mathbb{P}} \frac{p^{-s+kh}}{1 - p^{-s+kh}} \\ &= \sum_{p \in \mathbb{P}} \lim_{h \rightarrow 0^+} \frac{1}{h^{\alpha}} \sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k \frac{p^{-s+kh}}{1 - p^{-s+kh}}. \end{aligned}$$

The proof is straightforward by showing that

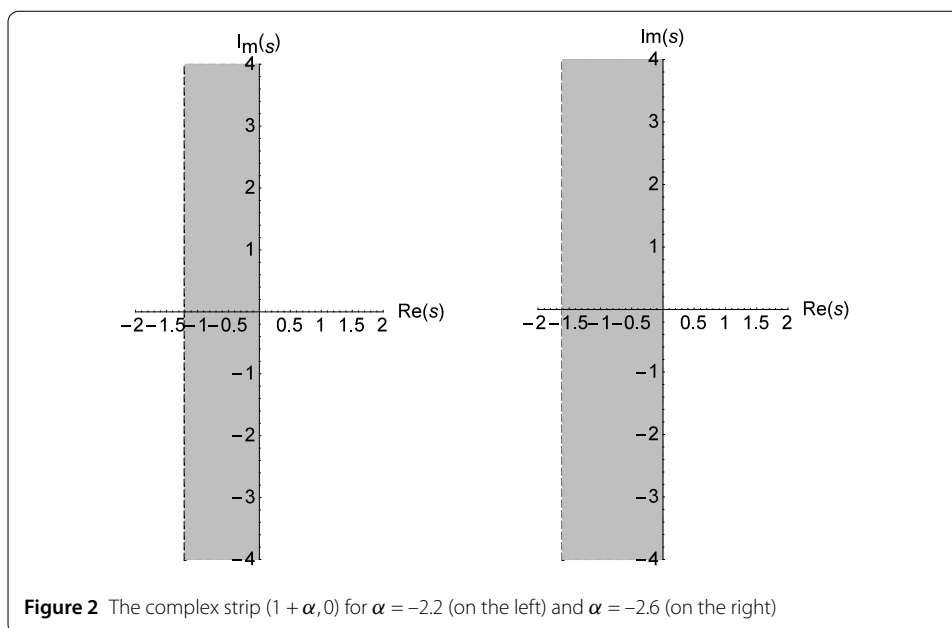
$$\sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k \frac{p^{-s+kh}}{1 - p^{-s+kh}} = - \sum_{t=0}^{\infty} p^{st} (1 - p^{-h})^{\alpha}.$$

Note that $|p^{s-kh}| = |p^{\operatorname{Re} s}| |p^{-kh}|$ and $|p^{\operatorname{Re} s}| < 1$, and thus

$$\frac{1}{1 - p^{s-kh}} = \sum_{t=0}^{\infty} p^{(s-kh)t}. \quad (4.6)$$

The series expansion in (4.6) implies

$$\begin{aligned} \sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k \frac{p^{-s+kh}}{1 - p^{-s+kh}} &= - \sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k \frac{1}{1 - p^{s-kh}} \\ &= - \sum_{t=0}^{\infty} p^{st} (1 - p^{-ht})^{\alpha}. \end{aligned}$$



Therefore

$$\begin{aligned} -\sum_{p \in \mathbb{P}} \lim_{h \rightarrow 0^+} \frac{1}{h^\alpha} \sum_{t=0}^{\infty} p^{st} (1 - p^{-ht})^\alpha &= -\sum_{p \in \mathbb{P}} \sum_{t=0}^{\infty} p^{st} \left(\lim_{h \rightarrow 0^+} \frac{1 - p^{-ht}}{h} \right)^\alpha \\ &= -\sum_{p \in \mathbb{P}} \sum_{t=0}^{\infty} \frac{\log^\alpha p^t}{p^{-st}}, \end{aligned}$$

being $\frac{1 - p^{-ht}}{h} \xrightarrow{h \rightarrow 0^+} \log p^t$. This completes the proof, as desired. \square

The aforementioned proof is strongly dependent on the assumption that $\operatorname{Re} s < 0$. In particular, the set of prime numbers and $\zeta^{(\alpha)}$ are linked by (4.5) in the left half-plane. As mentioned in Sect. 2, $\zeta^{(\alpha)}$ converges for $\operatorname{Re} s > 1 + \alpha$, thus making this theorem true for any $\alpha < -1$. Accordingly, Theorem 4.3 holds in the strip $1 + \alpha < \operatorname{Re} s < 0$ with $\alpha < -1$ (Fig. 2).

5 Conclusions

This paper analyzed the fractional derivative of the Riemann ζ function. In particular, the functional equation and the link with the distribution of prime numbers have been discussed. Equivalent forms of the functional equation in (2.7) are given. More precisely, such a functional equation is written in a quasisymmetric form. The introduction of the strip $(\alpha, 1 + \alpha)$ has allowed investigation of the link with prime numbers. Analytic properties are given and treated. These results are far from being conclusive. However, Theorem 4.3 represents a step forward towards a final representation of $\zeta^{(\alpha)}$ in terms of prime numbers.

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References

1. Atangana, A., Baleanu, D.: New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model. *Therm. Sci.* **20**(2), 763–769 (2016)
2. Chen, Y., Yan, Y., Zhang, K.: On the local fractional derivative. *J. Math. Anal. Appl.* **362**(1), 17–33 (2010)
3. Wu, G.C., Lee, E.W.M.: Fractional variational iteration method and its application. *Phys. Lett. A* **374**(25), 2506–2509 (2010)
4. Hasan, S., Al-Smadi, M., Freihet, A., Momani, S.: Two computational approaches for solving a fractional obstacle system in Hilbert space. *Adv. Differ. Equ.* **2019**(1), 55 (2019)
5. Morales-Delgado, V.F., Gómez-Aguilar, J.F., Saad, K.M., et al.: Analytic solution for oxygen diffusion from capillary to tissues involving external force effects: a fractional calculus approach. *Physica A* **523**, 48–65 (2019)
6. Jain, S., Agarwal, P., Kilicman, A.: Pathway fractional integral operator associated with 3m-parametric Mittag-Leffler functions. *Int. J. Appl. Comput. Math.* **4**(5), 115 (2018)
7. Jain, S., Agarwal, P., Kıymaz, I.O., Çetinkaya, A.: Some composition formulae for the M-S-M fractional integral operator with the multi-index Mittag-Leffler functions. *AIP Conf. Proc.* **1926**, 020020 (2018)
8. Abadias, L., De León-Contreras, M., Torrea, J.L.: Non-local fractional derivatives. Discrete and continuous. *J. Math. Anal. Appl.* **449**(1), 734–755 (2017)
9. Guariglia, E.: Harmonic symmetry of the Riemann zeta fractional derivative. *AIP Conf. Proc.* **2046**, 020035 (2018)
10. Ragusa, M.A.: Commutators of fractional integral operators in Vanishing-Morrey spaces. *J. Glob. Optim.* **40**(1–3), 361–368 (2008)
11. Sawangtong, W., Sawangtong, P.: A single quenching point for a fractional heat equation based on the Riemann-Liouville fractional derivative with a nonlinear concentrate source. *Bound. Value Probl.* **2017**(1), 97 (2017)
12. Guariglia, E.: Fractional derivative of the Riemann zeta function. In: Cattani, C., Srivastava, H.M., Yang, X.-Y. (eds.) *Fractional Dynamics*, pp. 357–368. De Gruyter, Berlin (2015)
13. Cattani, C., Guariglia, E.: Fractional derivative of the Hurwitz ζ -function and chaotic decay to zero. *J. King Saud Univ., Sci.* **28**(1), 75–81 (2016)
14. Guariglia, E., Silvestrov, S.: A functional equation for the Riemann zeta fractional derivative. *AIP Conf. Proc.* **1798**, 020063 (2017)
15. Ortigueira, M.D., Coito, F.J.: From differences to derivatives. *Fract. Calc. Appl. Anal.* **7**(4), 459–471 (2004)
16. Apostol, T.M.: Formulas for higher derivatives of the Riemann zeta function. *Math. Comput.* **44**(169), 223–232 (1985)
17. Choudhury, B.K.: The Riemann zeta-function and its derivatives. *Proc. R. Soc. Lond. A* **450**(1940), 477–499 (1995)
18. Cattani, C., Guariglia, E., Wang, S.: On the critical strip of the Riemann zeta fractional derivative. *Fundam. Inform.* **151**(1–4), 459–472 (2017)
19. Graham, R.L., Knuth, D.E., Patashnik, O.: *Concrete Mathematics: A Foundation for Computer Science*. Addison-Wesley, Reading (1994)
20. Li, C., Dao, X., Guo, P.: Fractional derivatives in complex planes. *Nonlinear Anal.* **71**(5–6), 1857–1869 (2009)
21. Guariglia, E., Silvestrov, S.: Fractional-wavelet analysis of positive definite distributions and wavelets on $\mathcal{D}'(\mathbb{C})$. In: Silvestrov, S., Rančić, M. (eds.) *Engineering Mathematics II*, pp. 337–353. Springer, Berlin (2016)
22. Edwards, H.M.: *Riemann's Zeta Function*. Dover Publications, New York (2001)
23. Suriajaya, A.I.: On the zeros of the k th derivative of the Riemann zeta function under the Riemann hypothesis. *Funct. Approx. Comment. Math.* **53**(1), 69–95 (2015)
24. Lee, Y.: Zeros of the derivatives of the Riemann zeta function on $\text{Re } s < 1/2$. *J. Number Theory* **134**(1), 38–48 (2014)
25. Trench, W.F.: Conditional convergence of infinite products. *Am. Math. Mon.* **106**(7), 646–651 (1999)