RIEMANNIAN GEOMETRY AS DETERMINED BY THE VOLUMES OF SMALL GEODESIC BALLS

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1. Introduction

Let M be an n-dimensional Riemannian manifold of class C^∞. For small $r>0$ let $V_m(r)$ denote the volume of a geodesic ball with center m and radius r. This paper is concerned with the following question: *To what extent do the functions $V_m(r)$ determine the Riemannian geometry of M?* In particular we shall be concerned with the following conjecture:

(I) *Suppose*

$$V_m(r) = \omega r^n \tag{1.1}$$

for all $m \in M$ and all sufficiently small $r>0$. Then M is flat.

(Here $\omega=$ the volume of the unit ball in $\mathbf{R}^n$. The simplest expression for ω is $\omega = (1/(\frac{1}{2}n)!)\pi^{n/2}$ where $(\frac{1}{2}n)! = \Gamma(\frac{1}{2}n+1)$.)

First we make several remarks.

1. Our method for attacking the conjecture (I) will be to use the power series expansion for $V_m(r)$. This expansion will be considered in detail in section 3; however, the general facts about it are the following: (a) the first term in the series is ωr^n; (b) the coefficient of r^{n+k} vanishes provided k is odd; (c) the coefficients of r^{n+k} for k even can be expressed in terms of curvature. Unfortunately the nonzero coefficients depend on curvature in a rather complicated way, and this is what makes the resolution of the conjecture (I) an interesting problem.

2. To our knowledge the power series expansion for $V_m(r)$ was first considered in 1848 by Bertrand–Diguet–Puiseux [6]. See also [14, p. 209]. In these papers the first two terms of the expansion for $V_m(r)$ are computed for surfaces in $\mathbf{R}^3$:

$$V_m(r) = \pi r^2 \left\{1 - \frac{K}{12} r^2 + O(r^4)\right\}_m, \tag{1.2}$$

where K denotes the Gaussian curvature at m. In fact the reason why these authors obtained this expansion was to give a new proof of the famous *theorema egregium* of Gauss [16]. A new proof by Liouville [32] had appeared the preceding year. Indeed it is obvious from (1.2) that the Gaussian curvature, defined, say as the product of principal curvatures, really is intrinsic to M, and does not depend on the embedding of the surface into $\mathbf{R}^3$.

Vermeil [35] in 1917 and Hotelling [31] in 1939 generalized (1.2) to arbitrary Riemannian manifolds. (The Gaussian curvature K must be replaced by the scalar curvature.) See also [2], [37]. The third term in the expansion was computed in [20] and in section 3 of this paper we shall compute the fourth term. Furthermore we write down the fifth term for surfaces but we omit the calculation. These terms are given by complicated formulas in the invariants of the curvature operator.

3. There are many hypotheses which, when combined with the hypothesis $V_m(r)=\omega r^n$, imply that M is flat. In sections 4 and 5 we show that (I) is true in any of the following cases:

(a) $\dim M \leqslant 3$;

(b) M is Einstein, or more generally if M has nonnegative or nonpositive Ricci curvature;

(c) M is conformally flat;

(d) M is a compact oriented four-dimensional manifold whose Euler characteristic and signature satisfy $\chi(M) \geqslant -\frac{3}{2}|\tau(M)|$;

(e) M is a product of surfaces;

(f) M is a 4- or 5-dimensional manifold with parallel Ricci tensor;

(g) M is the product of symmetric spaces of classical type.

The proofs of these results utilize only the first three terms in the power series expansion of $V_m(r)$. That (I) is true when $\dim M \leqslant 3$ was first proved by P. Günther [27] by a different method.

4. Although the conjecture (I) seems quite reasonable, we have been unable to resolve it in general. In section 6 we give interesting examples for which

$$V_m(r) = \omega r^n\{1 + O(r^6)\}$$

for all points $m \in M$. One of these is a 4-dimensional positive definite metric which is a generalization of the Schwarzschild metric. Another is a homogeneous 5-dimensional metric. In section 7 we use a different technique to find a manifold of dimension 734 with

$$V_m(r) = \omega r^n\{1 + O(r^8)\}.$$

(5) There is a formal similarity between the coefficients of the power series expansion of $V_m(r)$ and the coefficients arising in the asymptotic expansion for the spectrum of the

Laplacian. We modify some of the techniques in this theory to prove (I) in certain special cases.

6. The volume functions for the symmetric spaces of rank 1 are written down in [20]. In sections 8 and 9 we consider conjectures analogous to (I) where the model spaces instead of being flat, are the symmetric spaces of rank 1.

Furthermore we show in section 11 that if M is an Einstein manifold with $\dim M \leqslant 5$ such that for each $m \in M$, $V_m(r)$ is the same as that of a symmetric Einstein space, then M is in fact a symmetric space.

7. Let $h_m(\exp_m(ru))$ denote the mean curvature of a geodesic sphere $\exp_m(S^{n-1}(r))$ in M and put

$$H_m(r) = r^{n-1} \int_{S^{n-1}(1)} h_m(\exp_m(ru))\, du.$$

In analogy with (I) we have the conjecture

(II) *Suppose* $H_m(r) = n(n-1)\omega r^{n-2}$ *for all* $m \in M$ *and all sufficiently small* r. *Then* M *is flat.*

It is remarkable that in contrast to (I), which seems difficult, (II) is true. We prove this in section 12.

8. In a series of papers [38], [30], [28], [15], [29], [26] a hypothesis similar to that of (I) was considered, namely that the volumes of tubes about all hypersurfaces be polynomials. In section 13 we consider a weaker hypothesis; we require only that the volumes of tubes about small geodesic spheres be polynomials. In this way we are able to strengthen some of the results of [38], [30], [20], [15], and [29].

We suppose all the manifolds to be connected.

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2. Curvature invariants

In this section we write down all of the scalar valued curvature invariants of order $\leqslant 6$, and we give some useful identities. The invariants of order 2 and 4 are well understood, and the 17 order 6 invariants have been written down [13], [17]. Our purpose here is to give these invariants using the notation similar to that of [11] and [7], in order to facilitate the calculations in later sections.

Let M be a Riemannian manifold. We choose the signs so that the curvature operator of M is given by $R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$, where ∇ denotes the Riemannian connection of M. The components of the curvature tensor will be denoted by R_{ijkl} where i, j, k, l are part of an orthonormal basis of the tangent space M_m for some $m \in M$. The components of the Ricci tensor will be denoted by ϱ_{ij} and the scalar curvature will be denoted by τ.

By definition a scalar valued curvature invariant is a polynomial in the components of the curvature tensor and its covariant derivatives which does not depend on the choice of basis of M_m. Such a scalar valued invariant is said to have order k if it involves a total of k derivatives of the metric tensor. (Each component of the curvature tensor contains two derivatives.) A basis for the invariants of low order has been computed using Weyl's theory of invariants [5, p. 76], [12]. (Weyl's theorem implies that the invariant polynomials are contractions in the components of the curvature tensor and its covariant derivatives.)

Let $I(k, n)$ denote the space of invariants of order $2k$ for manifolds of dimension n. The spaces $I(1, n)$ and $I(2, n)$ are well-known (see for example [5, pp. 76 and 79]). We have $\dim I(1, n) = 1$ for $n \geqslant 2$ and $\dim I(2, n) = 4$ for $n \geqslant 4$. In fact if we put

$$\tau = \sum R_{ijij},\ \|\varrho\|^2 = \sum \varrho_{ij}^2,\ \|R\|^2 = \sum R_{ijkl}^2,\ \Delta\tau = \sum \nabla_{ii}^2 \tau,$$

then $\{\tau\}$ is a basis for $I(1, n)$ and $\{\tau^2, \|\varrho\|^2, \|R\|^2, \Delta\tau\}$ is a basis for $I(2, n)$.

Furthermore the space $I(3, n)$ of order 6 invariants has dimension 17 provided $n \geqslant 6$. Using a notation similar to that of [11] and [7, chapter 6] we write down a basis for $I(3, n)$:

$$\tau^3,\ \tau\|\varrho\|^2,\ \tau\|R\|^2,$$
$$\check{\varrho} = \sum \varrho_{ij}\varrho_{jk}\varrho_{ki},$$
$$\langle \varrho, \dot{R} \rangle = \sum \varrho_{ij} R_{ipqr} R_{jpqr} \quad (\text{where } \dot{R}_{ij} = \sum_{pqr} R_{ipqr} R_{jpqr}),$$
$$\langle \varrho \otimes \varrho, \bar{R} \rangle = \sum \varrho_{ij}\varrho_{kl} R_{ikjl} \quad (\text{where } \bar{R}_{ijkl} = R_{ikjl}),$$
$$\check{R} = \sum R_{ijkl} R_{klpq} R_{pqij},$$
$$\check{\bar{R}} = \sum R_{ikjl} R_{kplq} R_{piqj},$$
$$\|\nabla\tau\|^2 = \sum (\nabla_i \tau)^2,$$
$$\|\nabla\varrho\|^2 = \sum (\nabla_i \varrho_{jk})^2,$$
$$\alpha(\varrho) = \sum \nabla_i \varrho_{jk} \nabla_k \varrho_{ij},$$
$$\|\nabla R\|^2 = \sum (\nabla_i R_{jklq})^2,$$
$$\tau\Delta\tau,$$
$$\langle \Delta\varrho, \varrho \rangle = \sum \varrho_{ij} \nabla_{kk}^2 \varrho_{ij},$$
$$\langle \nabla^2\tau, \varrho \rangle = \sum (\nabla_{ij}^2 \tau) \varrho_{ij},$$
$$\langle \Delta R, R \rangle = \sum R_{ijkl} \nabla_{pp}^2 R_{ijkl},$$
$$\Delta^2\tau.$$

Of course one must check that these 17 order 6 invariants form a basis for $I(3, n)$, $n \geq 6$. This can be carried out as follows. First one uses Weyl's theorem to show that $\dim I(3, n) \leq 17$. Then one assumes that there is a linear relation of the form

$$A_1\tau^3 + \dots + A_{17}\Delta^2\tau = 0,$$

which is valid for all manifolds of a fixed dimension $n \geq 6$. Then by carefully choosing 17 different manifolds and evaluating the linear relation on each of them, one shows that $A_1 = \dots = A_{17} = 0$, so that $\{\tau^3, \dots, \Delta^2\tau\}$ is indeed a basis for $I(3, n)$. This is not as formidable a task as it first appears, provided one makes use of certain simple 3-dimensional and 4-dimensional metrics.

Concerning the order 6 invariants see also [11], [12], [17], [33].

We shall need some identities involving the invariants. Most of these are well known or can be found in [33]. All of the identities are consequences of the symmetries of the curvature operator, including the two Bianchi identities and the Ricci identity. We write down these three identities and two of their consequences.

LEMMA 2.1. *We have*

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0, \tag{2.1}$$

$$\nabla_i R_{jklp} + \nabla_j R_{kilp} + \nabla_k R_{ijlp} = 0, \tag{2.2}$$

$$\nabla^2_{ij} - \nabla^2_{ji} = -R_{ij}, \tag{2.3}$$

where R_{ij} denotes the derivation of the tensor algebra determined by the curvature tensor.

$$\sum \nabla_i R_{iakl} = \nabla_k \varrho_{al} - \nabla_l \varrho_{ak}, \tag{2.4}$$

$$\sum \nabla_i \varrho_{ij} = \tfrac{1}{2}\nabla_j \tau. \tag{2.5}$$

The first Bianchi identity (2.1) has the following consequences.

LEMMA 2.2. *We have*

$$\sum R_{abci} R_{acbj} = \tfrac{1}{2}\sum R_{abci} R_{abcj}, \tag{2.6}$$

$$\sum R_{abcd} R_{acbd} = \tfrac{1}{2}\|R\|^2. \tag{2.7}$$

The identities (2.1)–(2.7) suffice for the theory of the order 2 and order 4 invariants. For the order 6 invariants there are many more. Many are given in [33]. For our purposes we shall write the identities in terms of the 17 order 6 invariants.

LEMMA 2.3. *We have*

$$\sum \nabla_i R_{jabc} \nabla_i R_{jbac} = \tfrac{1}{2}\|\nabla R\|^2, \tag{2.8}$$

$$\sum \nabla_i R_{jabc} \nabla_j R_{iabc} = \tfrac{1}{2}\|\nabla R\|^2, \tag{2.9}$$

$$\sum \nabla_i R_{jabc} \nabla_j R_{ibac} = \tfrac{1}{4}\|\nabla R\|^2, \tag{2.10}$$

$$\sum R_{abcd} \nabla^2_{ii} R_{acbd} = \tfrac{1}{2}\langle R, \Delta R\rangle, \tag{2.11}$$

$$\sum \varrho_{ij} R_{iabc} R_{jbac} = \tfrac{1}{2}\langle \varrho, \dot{R}\rangle, \tag{2.12}$$

$$\sum R_{ijkl} R_{klpq} R_{pqij} = \tfrac{1}{2}\check{R}, \tag{2.13}$$

$$\sum R_{ijkl} R_{kplq} R_{piqj} = \tfrac{1}{4}\check{R}, \tag{2.14}$$

$$\sum R_{ijkl} R_{jplq} R_{pkqi} = \check{\bar{R}} - \tfrac{1}{4}\check{R}, \tag{2.15}$$

$$\sum (\nabla^2_{ij} \varrho_{ik}) \varrho_{jk} = \tfrac{1}{2}\langle \nabla^2 \tau, \varrho\rangle + \check{\varrho} - \langle \varrho \otimes \varrho, \bar{R}\rangle, \tag{2.16}$$

$$\sum (\nabla^2_{ij} R_{iabc}) R_{jabc} = 2\sum (\nabla^2_{ij} R_{iabc}) R_{jbac} = \tfrac{1}{2}\langle \Delta R, R\rangle, \tag{2.17}$$

$$\sum (\nabla^2_{ij} \varrho_{kl}) R_{ikjl} = \langle \nabla^2 \varrho, \bar{R}\rangle = \tfrac{1}{4}\langle \Delta R, R\rangle - \tfrac{1}{2}\langle \varrho, \dot{R}\rangle + \check{\bar{R}} + \tfrac{1}{4}\check{R}, \tag{2.18}$$

$$\sum \nabla^4_{ijij} \tau = \sum \nabla^4_{ijji} \tau = \Delta^2 \tau + \tfrac{1}{2}\|\nabla \tau\|^2 + \langle \nabla^2 \tau, \varrho\rangle, \tag{2.19}$$

$$\begin{aligned} \sum \nabla^4_{ijki} \varrho_{jk} &= \sum \nabla^4_{ijkj} \varrho_{ik} \\ &= \tfrac{1}{2}\Delta^2 \tau + \tfrac{1}{2}\|\nabla \tau\|^2 - 2\|\nabla \varrho\|^2 + 2\langle \nabla^2 \tau, \varrho\rangle - \langle \Delta \varrho, \varrho\rangle + 3\alpha(\varrho) + 2\check{\varrho} - 2\langle \varrho \otimes \varrho, \bar{R}\rangle \\ &\quad - \tfrac{1}{4}\langle \Delta R, R\rangle + \tfrac{1}{2}\langle \varrho, \dot{R}\rangle - \check{\bar{R}} - \tfrac{1}{4}\check{R}, \end{aligned} \tag{2.20}$$

$$\begin{aligned} \sum \nabla^4_{ijkk} \varrho_{ij} &= \tfrac{1}{2}\Delta^2 \tau + \tfrac{1}{2}\|\nabla \tau\|^2 + 4\alpha(\varrho) + 2\langle \nabla^2 \tau, \varrho\rangle - 3\|\nabla \varrho\|^2 - \langle \Delta \varrho, \varrho\rangle \\ &\quad + 2\check{\varrho} - 2\langle \varrho \otimes \varrho, \bar{R}\rangle - \tfrac{1}{2}\langle \Delta R, R\rangle + \langle \varrho, \dot{R}\rangle - 2\check{\bar{R}} - \tfrac{1}{2}\check{R}. \end{aligned} \tag{2.21}$$

Proof. To prove (2.8)–(2.15) one makes repeated use of (2.1) and (2.2). The Ricci identity (2.3) is used together with (2.1), (2.2) to prove the rest of the equations.

Remark. In dimensions $\leqslant 5$ there are certain relations between the invariants. More precisely, the situation is as follows.

Dimension 2. The spaces of invariants of order 2, 4 and 6 have dimensions 1, 2 and 4 respectively. Instead of using τ as a generator of the order 2 invariants, it is more convenient to use the Gaussian curvature K. Furthermore $\{K^2, \Delta K\}$ is a basis for $I(2, 2)$, and $\{K^3, \|dK\|^2, K\Delta K, \Delta^2 K\}$ is a basis of $I(3, 2)$. Then we have

$$\left.\begin{aligned}
&K=\tfrac{1}{2}\tau,\\
&K^2=\tfrac{1}{2}\|\varrho\|^2=\tfrac{1}{4}\|R\|^2,\\
&\Delta K=\tfrac{1}{2}\Delta\tau,\\
&K^3=\tfrac{1}{2}\check{\varrho}=\tfrac{1}{2}\langle\varrho\otimes\varrho,\bar{R}\rangle=\tfrac{1}{4}\langle\varrho,\dot{R}\rangle=\tfrac{1}{8}\check{R},\\
&\|\nabla K\|^2=\tfrac{1}{2}\|\nabla\varrho\|^2=\alpha(\varrho)=\tfrac{1}{4}\|\nabla R\|^2=\tfrac{1}{4}\|\nabla\tau\|^2,\\
&K\Delta K=\tfrac{1}{2}\langle\Delta\varrho,\varrho\rangle=\tfrac{1}{2}\langle\nabla^2\tau,\varrho\rangle=\tfrac{1}{4}\langle\Delta R,R\rangle=\tfrac{1}{4}\tau\Delta\tau,\\
&\Delta^2 K=\tfrac{1}{2}\Delta^2\tau,\\
&\check{\bar{R}}=0.
\end{aligned}\right\}\tag{2.22}$$

(2.22) can be proved by direct calculation.

Dimension 3. The spaces $I(1,3)$, $I(2,3)$ and $I(3,3)$ have dimensions 1, 3 and 10, respectively. We choose $\{\tau\}$ as a basis of $I(1,3)$, $\{\tau^2, \|\varrho\|^2, \Delta\tau\}$ as a basis of $I(2,3)$ and $\{\tau^3, \tau\|\varrho\|^2, \check{\varrho}, \|\nabla\tau\|^2, \|\nabla\varrho\|^2, \alpha(\varrho), \tau\Delta\tau, \langle\varrho,\Delta\varrho\rangle, \langle\nabla^2\tau,\varrho\rangle, \Delta^2\tau\}$ as a basis of $I(3,3)$. Then

$$\left.\begin{aligned}
&\|R\|^2=4\|\varrho\|^2-\tau^2,\\
&\langle\varrho\otimes\varrho,\bar{R}\rangle=-2\check{\varrho}+\tfrac{5}{2}\tau\|\varrho\|^2-\tfrac{1}{2}\tau^3,\\
&\langle\varrho,\dot{R}\rangle=-2\check{\varrho}+4\tau\|\varrho\|^2-\tau^3,\\
&\check{R}=-8\check{\varrho}+12\tau\|\varrho\|^2-3\tau^3,\\
&\check{\bar{R}}=-2\check{\varrho}+\tfrac{3}{2}\tau\|\varrho\|^2-\tfrac{1}{4}\tau^3,\\
&\|\nabla R\|^2=4\|\nabla\varrho\|^2-\|\nabla\tau\|^2,\\
&\langle\Delta R,R\rangle=4\langle\Delta\varrho,\varrho\rangle-\tau\Delta\tau.
\end{aligned}\right\}\tag{2.23}$$

Equations (2.23) follow from the fact that the curvature tensor of a 3-dimensional manifold is expressible in terms of the Ricci tensor and the scalar curvature. The exact formula is

$$R_{abcd}=\varrho_{ac}\delta_{bd}+\varrho_{bd}\delta_{ac}-\varrho_{ad}\delta_{bc}-\varrho_{bc}\delta_{ad}-\frac{\tau}{2}(\delta_{ac}\delta_{bd}-\delta_{ad}\delta_{bc}).$$

Dimensions 4 and 5. The spaces $I(3,4)$ and $I(3,5)$ have dimensions 15 and 16, respectively. This comes about because the 6-dimensional Gauss–Bonnet integrand must vanish for lower dimensional manifolds. Explicitly, there is the following relation between the order 6 invariants of manifolds M with $\dim M\leqslant 5$ ([11], [17]):

$$\tau^3+3\tau\|R\|^2-12\tau\|\varrho\|^2+16\check{\varrho}+4\check{R}-8\check{\bar{R}}+24\langle\varrho\otimes\varrho,\bar{R}\rangle-24\langle\varrho,\dot{R}\rangle=0.\tag{2.24}$$

For manifolds M with $\dim M \leqslant 4$ the Riemannian double form $\varrho \wedge R \wedge R$ of type (5, 5) vanishes identically. See [23], [24]. Hence the complete contraction

$$C^5(\varrho \wedge R \wedge R) = \sum_{i,j,k,l,p} (\varrho \wedge R \wedge R)(ijklp)(ijklp)$$

vanishes identically. This leads to the following relation between the order 6 invariants of manifolds with $\dim M \leqslant 4$:

$$\tau^3 - 8\tau\|\varrho\|^2 + \tau\|R\|^2 - 4\langle\varrho, \mathring{R}\rangle + 8\langle\varrho\otimes\varrho, \bar{R}\rangle + 8\check{\varrho} = 0. \tag{2.25}$$

Thus because of (2.24) and (2.25) we have $\dim I(3,4) \leqslant 15$ and $\dim I(3,5) \leqslant 16$. To show that the dimensions of $I(3,4)$ and $I(3,5)$ are actually 15 and 16, respectively, one evaluates the invariants on carefully chosen manifolds, just as with $I(3,n)$, $n \geqslant 6$, to show that there are no relations other than (2.24) and (2.25).

3. Power series expansions for volume functions

Let M be an analytic Riemannian manifold. (We could treat the C^∞ case; then all of our power series would be defined, but they might not converge.) Let $r_0 > 0$ be so small that the exponential map $\exp_m$ is defined on a ball of radius r_0 in the tangent space M_m. We put

$$S_m(r_0) = \text{volume of} \quad \{\exp_m(x) \,|\, x \in M_m,\ \|x\| = r_0\},$$
$$V_m(r_0) = \text{volume of} \quad \{\exp_m(x) \,|\, x \in M_m,\ \|x\| \leqslant r_0\}.$$

Here we mean the $(n-1)$-dimensional volume for $S_m(r_0)$ and the n-dimensional volume for $V_m(r_0)$.

Let s and σ be the functions defined on neighborhoods of $O \in M_m$ and $m \in M$ by

$$s(x) = \text{the Euclidean distance from } O \text{ to } x,$$
$$\sigma(p) = \text{the distance in } M \text{ from } m \text{ to } p.$$

If $\exp_m$ denotes the exponential map then $\sigma = s \circ \exp_m^{-1}$. The functions s and σ are differentiable in deleted neighborhoods of O and m respectively. Finally, let $(x_1, \ldots, x_n)$ be a system of normal coordinates on M at m. Write

$$\omega_{1\ldots n} = \omega\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right).$$

In [20] the following power series expansion is given for $\omega_{1\ldots n}$:

LEMMA 3.1.

$$
\begin{aligned}
\omega_{1\ldots n} = \Bigg\{ 1 &- \tfrac{1}{6} \sum_{i,j=1}^{n} \varrho_{ij} x_i x_j - \tfrac{1}{12} \sum_{i,j,k=1}^{n} \nabla_i \varrho_{jk} x_i x_j x_k \\
&+ \tfrac{1}{24} \sum_{i,j,k,l=1}^{n} \Big(-\tfrac{3}{5} \nabla_{ij}^2 \varrho_{kl} + \tfrac{1}{3} \varrho_{ij} \varrho_{kl} - \tfrac{2}{15} \sum_{a,b=1}^{n} R_{iajb} R_{kalb} \Big) x_i x_j x_k x_l \\
&+ \tfrac{1}{120} \sum_{i,j,k,l,h=1}^{n} \Big(-\tfrac{2}{3} \nabla_{ijk}^3 \varrho_{lh} + \tfrac{5}{3} (\nabla_i \varrho_{jk}) \varrho_{lh} - \tfrac{2}{3} \sum_{a,b=1}^{n} (\nabla_i R_{jakb}) R_{lahb} \Big) x_i x_j x_k x_l x_h \\
&+ \tfrac{1}{720} \sum_{i,j,k,l,h,g=1}^{n} \Big(-\tfrac{5}{7} \nabla_{ijkl}^4 \varrho_{hg} + 3 (\nabla_{ij}^2 \varrho_{kl}) \varrho_{hg} + \tfrac{5}{2} (\nabla_i \varrho_{jk}) (\nabla_l \varrho_{hg}) \\
&- \tfrac{8}{7} \sum_{a,b=1}^{n} (\nabla_{ij}^2 R_{kalb}) R_{hagb} - \tfrac{5}{9} \varrho_{ij} \varrho_{kl} \varrho_{hg} - \tfrac{15}{14} \sum_{a,b=1}^{n} (\nabla_i R_{jakb}) (\nabla_l R_{hagb}) \\
&- \tfrac{16}{63} \sum_{a,b,c=1}^{n} R_{iajb} R_{kblc} R_{hcga} + \tfrac{2}{3} \varrho_{ij} \sum_{a,b=1}^{n} R_{kalb} R_{hagb} \Big) x_i x_j x_k x_l x_h x_g \Bigg\}_m + \ldots
\end{aligned} \tag{3.1}
$$

Using (3.1) we compute the power series expansion of $V_m(r)$ and $S_m(r)$ where $r>0$ is sufficiently small. In doing so we clarify the exposition of [20]. First we prove

LEMMA 3.2. *We have*

$$S_m(r) = r^{n-1} \int_{S^{n-1}(1)} \omega_{1\ldots n}(\exp_m ru)\, du$$

where u varies on $S^{n-1}(1)$.

Proof. $*ds$ is the volume element of any sphere in M_m. Furthermore by the Gauss lemma $*d\sigma$ is the volume element of any small geodesic sphere in M with center at m. Moreover, let $(u_1, \ldots, u_n)$ be the natural coordinates in M_m corresponding to the normal coordinates (so that $x_i = u_i \circ \exp_m^{-1}$). Then

$$
\begin{aligned}
\exp_m^*(\omega) &= \omega\left((\exp_m)_* \left(\frac{\partial}{\partial u_1}\right), \ldots, (\exp_m)_* \left(\frac{\partial}{\partial u_n}\right) \right) du_1 \wedge \ldots \wedge du_n \\
&= \omega_{1\ldots n}\, du_1 \wedge \ldots \wedge du_n = \omega_{1\ldots n}\, ds \wedge * ds.
\end{aligned}
$$

On the other hand we have

$$\exp_m^*(\omega) = \exp_m^*(d\sigma \wedge * d\sigma) = \exp_m^*(d\sigma) \wedge \exp_m^*(*d\sigma) = ds \wedge \exp_m^*(*d\sigma).$$

It follows that $\exp_m^*(*d\sigma) = \omega_{1\ldots n} * ds$.

Next let $h\colon S^{n-1}(1) \to S^{n-1}(r)$ be defined by $h(x) = rx$ on $S^{n-1}(1)$. Then $h^*(ds) = r\,ds$ and

$$h^*(du_1 \wedge \ldots \wedge du_n) = r^n du_1 \wedge \ldots \wedge du_n.$$

Thus $h^*(\ast ds) = r^{n-1} \ast ds$. Hence we get

$$
\begin{aligned}
S_m(r) &= \int_{\exp_m(S^{n-1}(r))} \ast\, d\sigma = \int_{S^{n-1}(r)} \exp_m^*(\ast\, d\sigma) \\
&= \int_{S^{n-1}(r)} \omega_{1\ldots n} \ast ds = r^{n-1} \int_{S^{n-1}(1)} \omega_{1\ldots n} \ast ds \\
&= r^{n-1} \int_{S^{n-1}(1)} \omega_{1\ldots n}(\exp_m ru)\, du.
\end{aligned}
$$

Next we compute the power series expansions. We write down the formulas for Euclidean space in a way that is especially easy to remember. Let $(\frac{1}{2}n)! = \Gamma(\frac{1}{2}n+1)$. Then

$$
V_m(r) = \frac{(\pi r^2)^{n/2}}{\left(\dfrac{n}{2}\right)!},
$$

$$
S_m(r) = \frac{d}{dr} V_m(r) = \frac{(\pi r^2)^{(n/2)-1}(2\pi r)}{\left(\dfrac{n}{2}-1\right)!} = \frac{2\pi^{n/2} r^{n-1}}{\left(\dfrac{n}{2}-1\right)!}.
$$

For a general Riemannian manifold it will turn out that the power series expansion for $V_m(r)$ is of the form

$$
V_m(r) = \frac{(\pi r^2)^{n/2}}{\left(\dfrac{n}{2}\right)!} \{1 + Ar^2 + Br^4 + Cr^6 + \ldots\}.
$$

(The coefficient of r^{n+k} vanishes for k odd.) Here

A = a multiple of the scalar curvature;

B = a linear combination of the order 4 invariants of the curvature operator;

C = a linear combination of the order 6 invariants of the curvature operator.

Next we determine A, B and C precisely.

THEOREM 3.3. *We have for any Riemannian manifold M and any $m \in M$*

$$
\begin{aligned}
V_m(r) = \frac{(\pi r^2)^{n/2}}{\left(\dfrac{n}{2}\right)!} \Bigg\{ & 1 - \frac{\tau}{6(n+2)} r^2 + \frac{1}{360(n+2)(n+4)} (-3\|R\|^2 + 8\|\varrho\|^2 + 5\tau^2 - 18\Delta\tau) r^4 \\
& + \frac{1}{720(n+2)(n+4)(n+6)} (-\tfrac{5}{9}\tau^3 - \tfrac{8}{3}\tau\|\varrho\|^2 + \tau\|R\|^2 + \tfrac{64}{63}\check{\varrho} - \tfrac{64}{21}\langle \varrho\otimes\varrho, \bar{R}\rangle \\
& + \tfrac{32}{7}\langle \varrho, \dot{R}\rangle - \tfrac{110}{63}\dot{\check{R}} - \tfrac{200}{63}\bar{\check{R}} + \tfrac{45}{7}\|\nabla\tau\|^2 + \tfrac{45}{14}\|\nabla\varrho\|^2 + \tfrac{45}{7}\alpha(\varrho) - \tfrac{45}{14}\|\nabla R\|^2 \\
& + 6\tau\Delta\tau + \tfrac{48}{7}\langle\Delta\varrho, \varrho\rangle + \tfrac{54}{7}\langle\nabla^2\tau, \varrho\rangle - \tfrac{30}{7}\langle\Delta R, R\rangle - \tfrac{45}{7}\Delta^2\tau) r^6 + O(r^8) \Bigg\}_m .
\end{aligned}
\tag{3.2}
$$

Proof. It suffices to compute $S_m(r)$, because we can compute $V_m(r)$ from the formula

$$V_m(r) = \int_0^r S_m(t)\,dt.$$

We write

$$\omega_{1\ldots n}(\exp_m(ru)) = \sum_{p=0}^{\infty} \frac{\gamma_p}{p!} r^p.$$

If $u = \sum a_i e_i$ where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of M_m, then γ_p is a homogeneous polynomial of degree p in the a_i's. From (3.1) we have

$$\gamma_0 = 1, \quad \gamma_1 = 0, \quad \gamma_2 = -\tfrac{1}{3} \sum_{i,j=1}^{n} \varrho_{ij} a_i a_j, \quad \gamma_3 = -\tfrac{1}{2} \sum_{i,j,k=1}^{n} \nabla_i \varrho_{jk} a_i a_j a_k,$$

$$\gamma_4 = \sum_{i,j,k,l=1}^{n} \left\{ -\tfrac{3}{5} \nabla_{ij}^2 \varrho_{kl} + \tfrac{1}{3} \varrho_{ij} \varrho_{kl} - \tfrac{2}{15} \sum_{a,b=1}^{n} R_{iajb} R_{kalb} \right\} a_i a_j a_k a_l,$$

$\gamma_5 =$ something irrelevant,

$$\gamma_6 = \sum_{i,j,k,l,h,g=1}^{n} \Bigg\{ -\tfrac{5}{7} \nabla_{ijkl}^4 \varrho_{hg} + 3(\nabla_{ij}^2 \varrho_{kl}) \varrho_{hg} + \tfrac{5}{2} (\nabla_i \varrho_{jk})(\nabla_l \varrho_{hg})$$

$$- \tfrac{8}{7} \sum_{a,b=1}^{n} (\nabla_{ij}^2 R_{kalb}) R_{hagb} - \tfrac{5}{9} \varrho_{ij} \varrho_{kl} \varrho_{hg} - \tfrac{15}{14} \sum_{a,b=1}^{n} (\nabla_i R_{jakb})(\nabla_l R_{hagb})$$

$$- \tfrac{16}{63} \sum_{a,b,c=1}^{n} R_{iajb} R_{kblc} R_{hcga} + \tfrac{2}{3} \varrho_{ij} \sum_{a,b=1}^{n} R_{kalb} R_{hagb} \Bigg\} a_i a_j a_k a_l a_h a_g.$$

From Lemma 3.2 it follows that

$$S_m(r) = r^{n-1} \sum_{p=0}^{\infty} \frac{r^p}{p!} \int_{S^{n-1}(1)} \gamma_p\, du.$$

By symmetry on the sphere $\int_{S^{n-1}(1)} \gamma_p\, du = 0$ when p is odd.

Furthermore

$$\int_{S^{n-1}(1)} \gamma_0\, du = \int_{S^{n-1}(1)} du = \frac{2\pi^{n/2}}{\left(\frac{n}{2} - 1\right)!}.$$

Next

$$\int_{S^{n-1}(1)} \gamma_2\, du = -\tfrac{1}{3} \sum_{i,j} \int_{S^{n-1}(1)} \varrho_{ij} a_i a_j\, du$$

$$= -\tfrac{1}{3} \sum_{i=1}^{n} \varrho_{ii} \int_{S^{n-1}(1)} a_i^2\, du = -\frac{\tau}{3n} \int_{S^{n-1}(1)} du = -\frac{\pi^{n/2} \tau}{3\left(\frac{n}{2}\right)!}.$$

Here we have used the fact that $\sum a_i^2 = 1$ and $\int a_i^2\, du = \int a_j^2\, du$.

To compute $\int_{S^{n-1}(1)} \gamma_4\, du$ we first note that

$$\int_{S^{n-1}(1)} a_i^4\, du = \frac{3\pi^{n/2}}{(n+2)\left(\frac{n}{2}\right)!},$$

$$\int_{S^{n-1}(1)} a_i^2 a_j^2\, du = \frac{\pi^{n/2}}{(n+2)\left(\frac{n}{2}\right)!}, \quad i \neq j.$$

(These formulas can be proved by making appropriate choices of the orthonormal basis $\{e_1, \ldots, e_n\}$. All other integrals of degree 4 vanish. See also [37].) Put

$$\lambda_{ijkl} = -\tfrac{3}{5}\nabla_{ij}^2 \varrho_{kl} + \tfrac{1}{3}\varrho_{ij}\varrho_{kl} - \tfrac{2}{15}\sum_{a,b=1}^{n} R_{iajb} R_{kalb}.$$

Then using (2.5) and (2.7) we obtain

$$\begin{aligned}
\int_{S^{n-1}(1)} \gamma_4\, du &= \sum_{i,j,k,l=1}^{n} \lambda_{ijkl} \int_{S^{n-1}(1)} a_i a_j a_k a_l\, du \\
&= \frac{\pi^{n/2}}{(n+2)\left(\frac{n}{2}\right)!}\left\{3\sum_{i=1}^{n}\lambda_{iiii} + \sum_{i\neq j}(\lambda_{iijj} + \lambda_{ijij} + \lambda_{ijji})\right\} \\
&= \frac{\pi^{n/2}}{(n+2)\left(\frac{n}{2}\right)!}\sum_{i,j=1}^{n}(\lambda_{iijj} + \lambda_{ijij} + \lambda_{ijji}) \\
&= \frac{\pi^{n/2}}{(n+2)\left(\frac{n}{2}\right)!}\sum_{i,j=1}^{n}\Big\{-\tfrac{3}{5}\nabla_{ii}^2\varrho_{jj} - \tfrac{6}{5}\nabla_{ij}^2\varrho_{ij} + \tfrac{1}{3}\varrho_{ii}\varrho_{jj} + \tfrac{2}{3}\varrho_{ij}^2 \\
&\qquad - \tfrac{2}{15}\sum_{a,b=1}^{n}(R_{iaib}R_{jajb} + R_{iajb}^2 + R_{iajb}R_{ibja})\Big\} \\
&= \frac{\pi^{n/2}}{15(n+2)\left(\frac{n}{2}\right)!}\{5\tau^2 + 8\|\varrho\|^2 - 3\|R\|^2 - 18\Delta\tau\}.
\end{aligned}$$

Next we compute $\int_{S^{n-1}(1)} \gamma_6\, du$. We need the following formulas:

$$\int_{S^{n-1}(1)} a_i^6\, du = \frac{15\pi^{n/2}}{(n+2)(n+4)\left(\frac{n}{2}\right)!},$$

$$\int_{S^{n-1}(1)} a_i^2 a_j^4\, du = \frac{3\pi^{n/2}}{(n+2)(n+4)\left(\frac{n}{2}\right)!},$$

$$\int_{S^{n-1}(1)} a_i^2 a_j^2 a_k^2\, du = \frac{\pi^{n/2}}{(n+2)(n+4)\left(\frac{n}{2}\right)!}.$$

Put

$$\mu_{ijklhg} = -\tfrac{5}{7}\nabla^4_{ijkl}\varrho_{hg} + 3(\nabla^2_{ij}\varrho_{kl})\varrho_{hg} + \tfrac{5}{2}(\nabla_i\varrho_{jk})(\nabla_l\varrho_{hg})$$
$$-\tfrac{8}{7}\sum_{a,b=1}^{n}(\nabla^2_{ij}R_{kalb})R_{hagb} - \tfrac{5}{9}\varrho_{ij}\varrho_{kl}\varrho_{hg} - \tfrac{15}{14}\sum_{a,b=1}^{n}(\nabla_i R_{jakb})(\nabla_l R_{hagb})$$
$$-\tfrac{16}{63}\sum_{a,b,c=1}^{n}R_{iajb}R_{kblc}R_{hcga} + \tfrac{2}{3}\varrho_{ij}\sum_{a,b=1}^{n}R_{kalb}R_{hagb}.$$

Then

$$\int_{S^{n-1}(1)}\gamma_6\,du = \sum_{i,j,k,l,h,g=1}^{n}\mu_{ijklhg}\int_{S^{n-1}(1)}a_i a_j a_k a_l a_h a_g\,du$$
$$= \frac{\pi^{n/2}}{(n+2)(n+4)\left(\frac{n}{2}\right)!}\left\{15\sum_{i=1}^{n}\mu_{iiiiii} + 3\sum_{i\neq j}(\mu_{iiiijj} + \ldots + \mu_{jjiiii}) + \sum_{i,j,k\neq}(\mu_{iijjkk} + \ldots)\right\}$$
$$= \frac{\pi^{n/2}}{(n+2)(n+4)\left(\frac{n}{2}\right)!}\sum_{i,j,k=1}^{n}\{\mu_{iijjkk} + \mu_{iijkjk} + \mu_{iijkkj} + \mu_{ijijkk} + \mu_{ijikjk} + \mu_{ijikkj}$$
$$+ \mu_{ijjikk} + \mu_{ijkijk} + \mu_{ijkikj} + \mu_{ijjkik} + \mu_{ijkjik} + \mu_{ijkkij} + \mu_{ijjkki} + \mu_{ijkjki} + \mu_{ijkkji}\}$$
$$= \frac{\pi^{n/2}}{(n+2)(n+4)\left(\frac{n}{2}\right)!}\{A_1 + \ldots + A_8\}.$$

Here the eight A_i's correspond to the eight terms in the expression for γ_6. Each A_i is a sum of 15 different types of terms. We now compute each A_i. We start with A_3 which is the easiest. Extensive use will be made of Lemmas 2.1–2.3.

First we use (2.5) to find

$$A_3 = \tfrac{5}{2}\sum_{i,j,k=1}^{n}\left\{2\nabla_i\varrho_{ij}\nabla_j\varrho_{kk} + 4\nabla_i\varrho_{ij}\nabla_k\varrho_{jk} + \nabla_i\varrho_{jj}\nabla_i\varrho_{kk} + 2\nabla_i\varrho_{jj}\nabla_k\varrho_{ki} + 2\nabla_i\varrho_{jk}\nabla_i\varrho_{jk} + 4\nabla_i\varrho_{jk}\nabla_j\varrho_{ik}\right\}$$
$$= 5\|\nabla\varrho\|^2 + 10\|\nabla\tau\|^2 + 10\alpha(\varrho).$$

Next we have

$$A_5 = -\tfrac{5}{9}\sum_{i,j,k=1}^{n}\{\varrho_{ii}\varrho_{jj}\varrho_{kk} + 6\varrho_{ii}\varrho^2_{jk} + 8\varrho_{ij}\varrho_{jk}\varrho_{ki}\}$$
$$= -\tfrac{5}{9}\tau^3 - \tfrac{10}{3}\tau\|\varrho\|^2 - \tfrac{40}{9}\check{\varrho}.$$

Using (2.6) we obtain

$$A_8 = \tfrac{2}{3}\sum_{i,j,k,a,b=1}^{n}\{\varrho_{ii}(R_{jajb}R_{kakb} + R^2_{jakb} + R_{jakb}R_{jbka})$$
$$+ \varrho_{ij}(2R_{iajb}R_{kakb} + 4R_{iakb}R_{jakb} + 4R_{iakb}R_{kajb} + 2R_{kakb}R_{iajb})\}$$
$$= \tfrac{2}{3}\tau\|\varrho\|^2 + \tau\|R\|^2 + \tfrac{8}{3}\langle\varrho\otimes\varrho, \bar{R}\rangle + 4\langle\varrho, \dot{R}\rangle.$$

To obtain A_6 we use (2.4), (2.8), (2.9), (2.10) and get

$$A_6 = -\tfrac{15}{14} \sum_{i,j,k,a,b=1}^{n} \{2(\nabla_i R_{iajb})(\nabla_j R_{kakb}) + 2(\nabla_i R_{iajb})(\nabla_k R_{jakb}) + 2(\nabla_i R_{iajb})(\nabla_k R_{kajb})$$
$$+ (\nabla_i R_{jajb})(\nabla_i R_{kakb}) + 2(\nabla_i R_{jajb})(\nabla_k R_{iakb}) + (\nabla_i R_{jakb})^2 + (\nabla_i R_{jakb})(\nabla_i R_{kajb})$$
$$+ 2(\nabla_i R_{jakb})(\nabla_j R_{iakb}) + 2(\nabla_i R_{jakb})(\nabla_j R_{kaib})\}$$
$$= -\tfrac{15}{14}\Big\{2 \sum_{j,a,b=1}^{n} (\nabla_j \varrho_{ab} - \nabla_b \varrho_{aj}) \nabla_j \varrho_{ab} + 2 \sum_{j,a,b=1}^{n} (\nabla_j \varrho_{ab} - \nabla_b \varrho_{aj})(\nabla_j \varrho_{ba} - \nabla_a \varrho_{bj})$$
$$+ 2 \sum_{j,a,b=1}^{n} (\nabla_j \varrho_{ab} - \nabla_b \varrho_{aj})^2 + \sum_{i,a,b=1}^{n} (\nabla_i \varrho_{ab})^2 + 2 \sum_{i,a,b=1}^{n} (\nabla_i \varrho_{ab})(\nabla_i \varrho_{ba} - \nabla_a \varrho_{bi})^2$$
$$+ \|\nabla R\|^2 + \tfrac{1}{2}\|\nabla R\|^2 + \|\nabla R\|^2 + \tfrac{1}{2}\|\nabla R\|^2\Big\}$$
$$= -\tfrac{165}{14}\|\nabla \varrho\|^2 + \tfrac{75}{7}\alpha(\varrho) - \tfrac{45}{14}\|\nabla R\|^2.$$

To calculate A_2 we must use (2.16). We find

$$A_2 = 3 \sum_{i,j,k=1}^{n} \{(\nabla^2_{ii} \varrho_{jj}) \varrho_{kk} + 2(\nabla^2_{ii} \varrho_{jk}) \varrho_{jk} + 2(\nabla^2_{ij} \varrho_{ij}) \varrho_{kk} + 4(\nabla^2_{ij} \varrho_{ik}) \varrho_{jk} + 4(\nabla^2_{ij} \varrho_{jk}) \varrho_{ik} + 2(\nabla^2_{ij} \varrho_{kk}) \varrho_{ij}\}$$
$$= 6\tau\Delta\tau + 6\langle \Delta\varrho, \varrho\rangle + 18\langle \nabla^2\tau, \varrho\rangle + 12\check{\varrho} - 12\langle \varrho\otimes\varrho, \bar{R}\rangle.$$

Using (2.4), (2.16), (2.17), (2.18) we have

$$A_4 = -\tfrac{8}{7} \sum_{i,j,k,a,b=1}^{n} \{(\nabla^2_{ii} R_{jajb}) R_{kakb} + (\nabla^2_{ii} R_{jakb}) R_{jakb} + (\nabla^2_{ii} R_{jakb}) R_{kajb} + 2(\nabla^2_{ij} R_{iajb}) R_{kakb}$$
$$+ 2(\nabla^2_{ij} R_{iakb}) R_{jakb} + 2(\nabla^2_{ij} R_{iakb}) R_{jbka} + 2(\nabla^2_{ij} R_{jakb}) R_{iakb} + 2(\nabla^2_{ij} R_{jakb}) R_{kaib} + 2(\nabla^2_{ij} R_{kakb}) R_{jaib}\}$$
$$= -\tfrac{8}{7}\{\langle \Delta\varrho, \varrho\rangle + \tfrac{3}{2}\langle \Delta R, R\rangle + 2 \sum_{i,a,b} (\nabla^2_{ii} \varrho_{ab} - \nabla^2_{ia} \varrho_{ib}) \varrho_{ab} + 3\sum (\nabla^2_{ij} R_{iakb}) R_{jakb}$$
$$+ 2\sum (\nabla^2_{ik} \varrho_{ab} - \nabla^2_{ib} \varrho_{ak}) R_{iakb} + 2\sum (\nabla^2_{ik} \varrho_{ab} - \nabla^2_{ib} \varrho_{ak}) R_{kaib} + 2\sum (\nabla^2_{ij} \varrho_{ab}) R_{jaib}\}$$
$$= -\tfrac{8}{7}\{3\langle \Delta\varrho, \varrho\rangle + \tfrac{3}{2}\langle \Delta R, R\rangle - 2\sum (\nabla^2_{ia} \varrho_{ib}) \varrho_{ab} + 3\sum (\nabla^2_{ij} R_{iakb}) R_{jakb} + 8\sum (\nabla^2_{ij} \varrho_{ab}) R_{iajb}\}$$
$$= -\tfrac{8}{7}\{3\langle \Delta\varrho, \varrho\rangle + 3\langle \Delta R, R\rangle - \langle \nabla^2\tau, \varrho\rangle - 2\check{\varrho} + 2\langle \varrho\otimes\varrho, \bar{R}\rangle$$
$$+ 2\langle \Delta R, R\rangle - 4\langle \varrho, \mathring{R}\rangle + 8\check{\bar{R}} + 2\check{R}\}$$
$$= -\tfrac{8}{7}\{3\langle \Delta\varrho, \varrho\rangle + 5\langle \Delta R, R\rangle - \langle \nabla^2\tau, \varrho\rangle - 2\check{\varrho} + 2\langle \varrho\otimes\varrho, \bar{R}\rangle - 4\langle \varrho, \mathring{R}\rangle + 8\check{\bar{R}} + 2\check{R}\}.$$

Further using (2.12)–(2.15) we have

$$A_7 = -\tfrac{16}{63} \sum_{i,j,k,a,b,c=1}^{n} \{R_{iaib} R_{jbjc} R_{kcka} + R_{iaib} R_{jbkc} R_{jcka} + R_{iaib} R_{jbkc} R_{kcja} + R_{iajb} R_{ibjc} R_{kcka}$$
$$+ R_{iajb} R_{ibkc} R_{jcka} + R_{iajb} R_{ibkc} R_{kcja} + R_{iajb} R_{jbic} R_{kcka} + R_{iajb} R_{jbkc} R_{icka} + R_{iajb} R_{jbkc} R_{kcia}$$
$$+ R_{iajb} R_{kbic} R_{jcka} + R_{iajb} R_{kbic} R_{kcja} + R_{iajb} R_{kbjc} R_{icka} + R_{iajb} R_{kbjc} R_{kcia} + R_{iajb} R_{kbkc} R_{icja}$$
$$+ R_{iajb} R_{kbkc} R_{jcia}\}$$

$$
\begin{aligned}
&= -\tfrac{16}{63}\{\check{\varrho} + \textstyle\sum (\varrho_{ab} R_{jbkc} R_{jcka} + \varrho_{ab} R_{jbkc} R_{kcja} + \varrho_{ca} R_{iajb} R_{ibjc} + \varrho_{ca} R_{iajb} R_{jbic} \\
&\quad + \varrho_{bc} R_{iajb} R_{icja} + \varrho_{bc} R_{iajb} R_{jcia}) + \check{R} \\
&\quad + \textstyle\sum (R_{iajb} R_{ibkc} R_{jcka} + R_{iajb} R_{ibkc} R_{kcja} + R_{iajb} R_{jbkc} R_{icka} + R_{iajb} R_{kbic} R_{jcka} \\
&\quad + R_{iajb} R_{kbic} R_{kcja} + R_{iajb} R_{kbjc} R_{icka} + R_{iajb} R_{kbjc} R_{kcia})\} \\
&= -\tfrac{16}{63}\{\check{\varrho} + \check{R} + \tfrac{9}{2}\langle \varrho, \dot{R}\rangle + \textstyle\sum (R_{iajb} R_{ibkc} R_{jcka} + R_{iajb} R_{ibkc} R_{kcja} + R_{iajb} R_{jbkc} R_{icka} \\
&\quad + R_{iajb} R_{kbic} R_{jcka} + R_{iajb} R_{kbic} R_{kcja} + R_{iajb} R_{kbjc} R_{icka} + R_{iajb} R_{kbjc} R_{kcia})\} \\
&= -\tfrac{16}{63}\{\check{\varrho} + \check{R} + \tfrac{9}{2}\langle \varrho, \dot{R}\rangle + \tfrac{1}{2}\textstyle\sum (R_{iajb} - R_{ijab}) R_{kcja} R_{ibkc} + \tfrac{1}{2}\textstyle\sum R_{iajb} R_{jbkc}(R_{icka} - R_{ikca}) \\
&\quad + \tfrac{1}{2}\textstyle\sum R_{iajb}(R_{kbjc} - R_{kjbc}) R_{kcia} + \tfrac{1}{4}\textstyle\sum (R_{aibj} - R_{abij}) R_{ibkc}(R_{jcka} - R_{jkca}) \\
&\quad + \tfrac{1}{4}\textstyle\sum (R_{iajb} - R_{ijab}) R_{kcja}(R_{kbic} - R_{cbik}) + \tfrac{1}{4}\textstyle\sum R_{iajb}(R_{kbjc} - R_{kjbc})(R_{icka} - R_{acki}) \\
&\quad + \textstyle\sum R_{iajb} R_{kbic} R_{jcka}\} \\
&= -\tfrac{16}{63}\{\check{\varrho} + \check{R} + \tfrac{9}{2}\langle \varrho, \dot{R}\rangle + \tfrac{3}{2}\textstyle\sum R_{jaib} R_{kcib} R_{kcja} \\
&\quad + \tfrac{3}{4}\textstyle\sum R_{biaj} R_{ibkc} R_{kcja} - \textstyle\sum R_{iajb} R_{ickb}(R_{ckja} + R_{kjca})\} \\
&= -\tfrac{16}{63}\{\check{\varrho} + \check{R} + \tfrac{9}{2}\langle \varrho, \dot{R}\rangle + \tfrac{3}{2}\check{R} + \tfrac{3}{4}\check{R} - \textstyle\sum R_{aijb} R_{icbk} R_{acjk} \\
&\quad + \tfrac{1}{4}\textstyle\sum (R_{iajb} - R_{ijab})(R_{ickb} - R_{ikcb}) R_{ajck}\} \\
&= -\tfrac{16}{63}\{\check{\varrho} + \tfrac{13}{4}\check{R} + \tfrac{9}{2}\langle \varrho, \dot{R}\rangle - \check{\bar{R}} + \tfrac{1}{4}\textstyle\sum R_{jaib} R_{kcib} R_{ajck}\} \\
&= -\tfrac{16}{63}\{\check{\varrho} + \tfrac{13}{4}\check{R} + \tfrac{9}{2}\langle \varrho, \dot{R}\rangle - \check{\bar{R}} + \tfrac{1}{4}\check{R}\} \\
&= -\tfrac{16}{63}\check{\varrho} - \tfrac{8}{7}\langle \varrho, \dot{R}\rangle + \tfrac{16}{63}\check{\bar{R}} - \tfrac{8}{9}\check{R}.
\end{aligned}
$$

Finally, using (2.19), (2.20) and (2.21) we have

$$
\begin{aligned}
A_1 &= -\tfrac{5}{7}\sum_{i,j,k=1}^{n} \{\nabla^4_{iijj}\varrho_{kk} + 2\nabla^4_{iijk}\varrho_{jk} + \nabla^4_{ijij}\varrho_{kk} + 2\nabla^4_{ijik}\varrho_{jk} + \nabla^4_{ijji}\varrho_{kk} + 2\nabla^4_{ijki}\varrho_{jk} + 2\nabla^4_{ijkj}\varrho_{ik} \\
&\quad + 2\nabla^4_{ijkk}\varrho_{ij} + 2\nabla^4_{ijjk}\varrho_{ik}\} \\
&= -\tfrac{5}{7}\{2\Delta^2\tau + 2\textstyle\sum \nabla^4_{ijij}\tau + 2\textstyle\sum \nabla^4_{ijji}\tau + 2\textstyle\sum \nabla^4_{ijki}\varrho_{jk} + 2\textstyle\sum \nabla^4_{ijkj}\varrho_{ik} + 2\textstyle\sum \nabla^4_{ijkk}\varrho_{ij}\} \\
&= -\tfrac{5}{7}\{6\Delta^2\tau + 2\|\nabla\tau\|^2 + 4\langle \nabla^2\tau, \varrho\rangle + 2\Delta^2\tau + 2\|\nabla\tau\|^2 - 8\|\nabla\varrho\|^2 + 8\langle \nabla^2\tau, \varrho\rangle \\
&\quad - 4\langle \Delta\varrho, \varrho\rangle + 12\alpha(\varrho) + 8\check{\varrho} - 8\langle \varrho\otimes\varrho, \bar{R}\rangle - \langle \Delta R, R\rangle + 2\langle \varrho, \dot{R}\rangle - 4\check{\bar{R}} - \check{R} \\
&\quad + \Delta^2\tau + \|\nabla\tau\|^2 + 8\alpha(\varrho) + 4\langle \nabla^2\tau, \varrho\rangle - 6\|\nabla\varrho\|^2 - 2\langle \Delta\varrho, \varrho\rangle + 4\check{\varrho} - 4\langle \varrho\otimes\varrho, \bar{R}\rangle \\
&\quad - \langle \Delta R, R\rangle + 2\langle \varrho, \dot{R}\rangle - 4\check{\bar{R}} - \check{R}\} \\
&= -\tfrac{5}{7}\{9\Delta^2\tau + 5\|\nabla\tau\|^2 + 16\langle \nabla^2\tau, \varrho\rangle - 14\|\nabla\varrho\|^2 - 6\langle \Delta\varrho, \varrho\rangle + 20\alpha(\varrho) + 12\check{\varrho} \\
&\quad - 12\langle \varrho\otimes\varrho, \bar{R}\rangle - 2\langle \Delta R, R\rangle + 4\langle \varrho, \dot{R}\rangle - 8\check{\bar{R}} - 2\check{R}\}.
\end{aligned}
$$

Adding up all the terms we get the expansion for $S_m(r)$. Then integrating from 0 to r we find that $V_m(r)$ is given by (3.2), completing the proof.

For convenience we write down the expansions for $V_m(r)$ in dimensions 2 and 3. Our computation makes use of the fact that there are fewer invariants of order 4 and 6 in these dimensions. Furthermore we give the fifth term in the expansion of $V_m(r)$ for a surface. This is an expression using the order 8 invariants. In fact [18] we have $I(4,2)=8$. We omit the long tedious calculations.

COROLLARY 3.4. *If* $\dim M=2$, *then for each* $m\in M$,

$$\begin{aligned} V_m(r) = \pi r^2 \Big\{ 1 - \frac{K}{12} r^2 + \frac{1}{720}(2K^2 - 3\Delta K) r^4 \\ + \frac{1}{161280}(-8K^3 + 30\|\nabla K\|^2 + 42K\Delta K - 15\Delta^2 K) r^6 \\ + \frac{1}{29030400}(16K^4 - 60K\|\nabla K\|^2 - 168K^2\Delta K + 112(\Delta K)^2 + 168\|\nabla^2 K\|^2 \\ + 420\langle\nabla(\Delta K), \nabla K\rangle + 170K\Delta^2 K - 35\Delta^3 K) r^8 + O(r^{10}) \Big\}_m . \end{aligned} \tag{3.3}$$

COROLLARY 3.5. *If* $\dim M=3$, *then for each* $m\in M$,

$$\begin{aligned} V_m(r) = \frac{4\pi r^3}{3} \Big\{ 1 - \frac{\tau}{30} r^2 + \frac{1}{6300}(4\tau^2 - 2\|\varrho\|^2 - 9\Delta\tau) r^4 \\ + \frac{1}{1587600}\Big(10\tau^3 - 96\tau\|\varrho\|^2 + 128\check{\varrho} - \frac{135}{2}\|\nabla\varrho\|^2 - 72\langle\Delta\varrho, \varrho\rangle \\ + 45\alpha(\varrho) + \frac{135}{2}\|\nabla\tau\|^2 + 72\tau\Delta\tau + 54\langle\nabla^2\tau, \varrho\rangle - 45\Delta^2\tau\Big) r^6 + O(r^8) \Big\}_m . \end{aligned} \tag{3.4}$$

These expansions follow upon substituting (2.22) and (2.23) into (3.2).

4. Proof of the conjecture in some particular cases

First we note that (1.1) implies

$$\tau = 0, \tag{4.1}$$

$$3\|R\|^2 = 8\|\varrho\|^2. \tag{4.2}$$

In fact (4.1) and (4.2) are equivalent to

$$V_m(r) = \omega r^n \{1 + O(r^6)\}. \tag{4.3}$$

In section 6 we shall show that (4.3) is weaker than (1.1).

In the present section we prove (I) in several cases. Actually what we prove is slightly stronger than (I) because we make use of (4.3) instead of (1.1).

THEOREM 4.1. (I) *is true provided any of the following additional hypotheses are made:*

(i) $\dim M \leqslant 3$;

(ii) *M has nonpositive or nonnegative Ricci curvature (in particular if M is Einstein);*

(iii) *M is conformally flat;*

(iv) *M is a Bochner flat Kähler manifold;*

(v) *M is a product of surfaces;*

(vi) *M is a 4- or 5-dimensional manifold with parallel Ricci tensor;*

(vii) *M is compact and the Laplacian of M has the same spectrum on functions as that of a compact flat manifold.*

Proof. (i) For a surface M we have $\|R\|^2=2\|\varrho\|^2$. Combining this with (4.2) we get at once $R=0$.

Further let $\dim M \geqslant 3$. Then the Weyl curvature tensor C of M satisfies

$$\|C\|^2=\|R\|^2-\frac{4}{n-2}\|\varrho\|^2+\frac{2}{(n-1)(n-2)}\tau^2. \tag{4.4}$$

Since $C=0$ on 3-dimensional manifolds we get from (4.4) that $\|R\|^2=4\|\varrho\|^2$. Using (4.2) we obtain (I).

(ii) If M has nonnegative or nonpositive Ricci curvature, then (I) follows at once from (4.1) and (4.2). In particular (I) is true for Einstein manifolds and for M with nonnegative or nonpositive sectional curvature.

(iii) This case is a consequence of (4.1), (4.2), and (4.4) with $C=0$.

(iv) Let B be the Bochner curvature tensor for a $2n$-dimensional Kähler manifold $(n>1)$. Then we have

$$\|B\|^2=\|R\|^2-\frac{8}{n+2}\|\varrho\|^2+\frac{2}{(n+1)(n+2)}\tau^2. \tag{4.5}$$

M is Bochner flat if and only if $B=0$. Then $(n+2)\|R\|^2=8\|\varrho\|^2$ which implies the required result.

(v) Let M be the Riemannian product of the surfaces M_i, $i=1, ..., p$. Then we have $\|R\|^2=\sum\|R_i\|^2$ and $\|\varrho\|^2=\sum\|\varrho_i\|^2$. Hence $\|R\|^2=2\|\varrho\|^2$, just as for surfaces. As before (I) is true.

(vi) Suppose M is a 4-dimensional manifold with parallel Ricci tensor. If M is reducible then locally $M=M_1^2\times M_2^2$ or $M=M^3\times M^1$ and (I) follows from (i) and (v). If M is irreducible then M is an Einstein manifold and (ii) implies (I).

The proof when the dimension of M is 5 is the same, except that one must also take care of the case when locally $M = M^2 \times M^3$. But then both M^2 and M^3 must have constant curvatures a and b, respectively, because they are Einstein manifolds. An easy calculation using (4.1) and (4.2) shows that in fact $a = b = 0$, and so (I) holds.

(vii) Finally, let (M^n, g) be a compact n-dimensional Riemannian manifold and Δ its Laplacian on functions. If $\{\lambda_k\}$ is the spectrum of (M, g) we have the following asymptotic expansion [5, p. 215]:

$$t > 0: \sum_{k\geqslant 0} e^{-\lambda_k t} \sim (4\pi t)^{-n/2} \sum_{i\geqslant 0} a_i t^i$$

where the first three coefficients are given by

$$a_0 = \mathrm{vol}(M, g), \quad a_1 = \frac{1}{6}\int_M \tau\, dV, \quad a_2 = \frac{1}{360}\int_M \{2\|R\|^2 - 2\|\varrho\|^2 + 5\tau^2\}\, dV.$$

Hence (I) is true if $a_2 \leqslant 0$. In particular we obtain (vii).

5. The conjecture (I) for locally symmetric spaces

It seems quite probable that the conjecture (I) is true for all locally symmetric spaces. Of course, this could be verified if one knew explicitly the curvature of all the irreducible symmetric spaces. To our knowledge this has been done for the Hermitian symmetric spaces [10], [8], the symmetric spaces of rank 1 (see for example [25]) and a few others, but not in general.

Therefore we proceed in the following way. We introduce a class $\mathcal{A}$ of Riemannian manifolds which contains all the nonflat symmetric spaces of classical type, and also a few exceptional symmetric spaces such as the Cayley plane. The relation (1.1) turns out to be impossible for any manifold in $\mathcal{A}$. This implies that (I) holds for manifolds of the form $\mathbf{R}^k \times M$ where $M \in \mathcal{A}$. In particular (I) holds for all of the classical symmetric spaces.

Definition. $\mathcal{A}$ is the class of Riemannian manifolds M for which

$$3\|R\|^2 - 8\|\varrho\|^2 < 0. \tag{5.1}$$

LEMMA 5.1. *If M_1 and M_2 are in $\mathcal{A}$ then so is the product Riemannian manifold $M_1 \times M_2$.*

Proof. Let $\|R\|^2$, $\|R_1\|^2$ and $\|R_2\|^2$ denote the length of the curvature operator for $M_1 \times M_2$, M_1 and M_2, respectively. Similarly let $\|\varrho\|^2$, $\|\varrho_1\|^2$ and $\|\varrho_2\|^2$ be the corresponding lengths of the Ricci tensors. Then one checks that

$$\|R\|^2 = \|R_1\|^2 + \|R_2\|^2 \quad \text{and} \quad \|\varrho\|^2 = \|\varrho_1\|^2 + \|\varrho_2\|^2.$$

Hence it is clear that if both M_1 and M_2 satisfy (5.1), then so does $M_1 \times M_2$.

LEMMA 5.2. *Let M be a symmetric space and suppose that $M \in \mathcal{A}$. Then the dual symmetric space M^* is also in $\mathcal{A}$.*

Proof. The curvature operator of M^* is just the negative of the curvature operator of M. Thus the quantities $3\|R\|^2-8\|\varrho\|^2$ are the same for both M and M^*.

LEMMA 5.3. *If $M \in \mathcal{A}$, then*

$$V_m(r) \neq \frac{(\pi r^2)^{n/2}}{\left(\frac{n}{2}\right)!}$$

for all $m \in M$ and for a sequence of r tending to 0.

Proof. Let $M \in \mathcal{A}$ and suppose

$$V_m(r) = \frac{(\pi r^2)^{n/2}}{\left(\frac{n}{2}\right)!}$$

for some $m \in M$ and all sufficiently small r. Then in particular equation (4.2) holds for m. But this contradicts 5.1.

COROLLARY 5.4. *The conjecture* (I) *holds for all manifolds of the form $M \times \mathbf{R}^k$ with $M \in \mathcal{A}$.*

Next we show that most (nonflat) symmetric spaces belong to $\mathcal{A}$.

LEMMA 5.5. *The class $\mathcal{A}$ contains the following manifolds:*

(1) *all symmetric spaces of classical type;*

(2) *all Hermitian symmetric spaces;*

(3) *all symmetric spaces of rank* 1.

Proof. To verify that all Hermitian symmetric spaces are in $\mathcal{A}$, it suffices (using Lemmas 5.1 and 5.2) to check that all irreducible Hermitian symmetric spaces of noncompact type lie in $\mathcal{A}$. The curvature operators of the six types of irreducible Hermitian symmetric spaces have been computed in [10] and [8]. Using the results in these two papers, the quantities τ, $\|\varrho\|^2$, $\|R\|^2$, and $3\|R\|^2-8\|\varrho\|^2$ can be computed. The results are given in Table I.

Next we consider the symmetric spaces of rank 1. The complete power series expansions for the volume functions of these spaces have been given in [20]. From this the first four terms in each of the power series are given as follows:

Table I. *Hermitian symmetric spaces*

Type	Complex dimension	τ	$\|\varrho\|^2$	$\|R\|^2$	$3\|R\|^2-8\|\varrho\|^2$
$A\text{-}I_{m,m'}$ $1\leqslant m\leqslant m'$	mm'	$-2mm'(m+m')\alpha$	$2mm'(m+m')\alpha^2$	$8mm'(1+mm')\alpha^2$	$-8mm'(2m^2+2m'^2+mm'-3)\alpha^2$
$A\text{-}II_m$ $m\geqslant 2$	$\frac{1}{2}m(m-1)$	$-2m(m-1)^2\alpha$	$4m(m-1)^3\alpha^2$	$4m(m-1)\times(m^2-3m+4)\alpha^2$	$-4m(m-1)\times(5m^2-7m-4)\alpha^2$
$A\text{-}III_m$ $m\geqslant 1$	$\frac{1}{2}m(m+1)$	$-2m(m+1)^2\alpha$	$4m(m+1)^3\alpha^2$	$4m(m+1)\times(m^2+3m+4)\alpha^2$	$-4m(m+1)\times(5m^2+7m-4)\alpha^2$
$A\text{-}IV_m$ $m\geqslant 3$	m	$-2m^2\alpha$	$2m^3\alpha^2$	$4m(3m-2)\alpha^2$	$-4m(4m^2-9m+6)\alpha^2$
$A\text{-}V$	16	-192α	$1\,152\alpha^2$	$864\alpha^2$	$-6\,624\alpha^2$
$A\text{-}VI$	27	-486α	$4\,374\alpha^2$	$3\,132\alpha^2$	$-25\,596\alpha^2$

The sphere $S^n(\lambda)$ (with constant sectional curvature λ), or its dual:

$$V_m(r)=\frac{(\pi r^2)^{n/2}}{\left(\frac{n}{2}\right)!}\times\left\{1-\frac{n(n-1)\lambda}{6(n+2)}r^2+\frac{n(n-1)(5n-7)\lambda^2}{360(n+4)}r^4-\frac{n(n-1)(35n^2-112n+93)\lambda^3}{45360(n+6)}r^6+O(r^8)\right\}. \tag{5.2}$$

The complex projective space $CP^n(\mu)$ (with constant holomorphic sectional curvature μ), or its dual:

$$V_m(r)=\frac{(\pi r^2)^n}{n!}\left\{1-\frac{n\mu}{12}r^2+\frac{n(5n-1)\mu^2}{1440}r^4-\frac{n(35n^2-21n+4)\mu^3}{362280}r^6+O(r^8)\right\}. \tag{5.3}$$

The quaternionic projective space $QP^n(\nu)$ (with maximum sectional curvature ν), or its dual:

$$V_m(r)=\frac{(\pi r^2)^{2n}}{(2n)!}\times\left\{1-\frac{n(n+2)\nu}{3(2n+1)}r^2+\frac{n(20n^2+68n+29)\nu^2}{720(2n+1)}-\frac{n(70n^3+329n^2+275n+64)\nu^3}{45760}r^6+O(r^8)\right\}. \tag{5.4}$$

Table II. *Symmetric spaces of rank 1*

Type	Real dimension	τ	$\lVert\varrho\rVert^2$	$\lVert R\rVert^2$	$3\lVert R\rVert^2-8\lVert\varrho\rVert^2$
$S^n(\lambda)$	n	$n(n-1)\lambda$	$n(n-1)^2\lambda^2$	$2n(n-1)\lambda^2$	$-2n(n-1)(4n-7)\lambda^2$
$CP^n(\mu)$	$2n$	$n(n+1)\mu$	$\frac{1}{2}n(n+1)^2\mu^2$	$2n(n+1)\mu^2$	$-2n(n+1)(2n-1)\mu^2$
$QP^n(\nu)$	$4n$	$4n(n+2)\nu$	$4n(n+2)^2\nu^2$	$4n(5n+1)\nu^2$	$-4n(8n^2+17n+29)\nu^2$
Cay $P^2(\zeta)$	16	$16(36)\zeta$	$16(36)^2\zeta^2$	$(16)^2 36\zeta^2$	$-240(16)(36)\zeta^2$

The Cayley plane Cay $P^2(\zeta)$ (with maximum sectional curvature ζ), or its dual:

$$V_m(r)=\frac{(\pi r^2)^8}{8!}\left\{1-\frac{4\zeta}{3}r^2+\frac{13\zeta^2}{15}r^4-\frac{2747\zeta^3}{7560}r^6+O(r^8)\right\}. \tag{5.5}$$

The quantities τ, $\lVert\varrho\rVert^2$, and $3\lVert R\rVert^2-8\lVert\varrho\rVert^2$ for each of the symmetric spaces of rank 1 can be computed by comparing the power series (5.2)–(5.5) with (3.2) and using the fact that each symmetric space of rank 1 is an Einstein manifold. The results are given in table II above.

Now we consider the classical symmetric spaces. Each of the classical compact simple Lie groups can be realized as a group of orthogonal matrices. A biinvariant inner product on the corresponding Lie algebra is given by

$$\langle X, Y\rangle=\alpha\operatorname{tr}(XY^*).$$

This, in the standard way, induces a metric on each classical symmetric space of compact type considered as a coset space. Then the curvature tensor of $\langle\ ,\ \rangle$ is given by $R_{WXYZ}=\frac{1}{4}\langle[W,X],[Y,Z]\rangle$. From the formula for the curvature operator the quantities τ, $\lVert\varrho\rVert^2$, $\lVert R\rVert^2$, and $3\lVert R\rVert^2-8\lVert\varrho\rVert^2$ can be computed by brute force. The results are given in Table III.

Thus in all cases we see that $3\lVert R\rVert^2-8\lVert\varrho\rVert^2$ is negative. Hence the lemma follows.

Combining Lemmas 5.1–5.5 we have

THEOREM 5.6. *Let M be a Riemannian product of classical symmetric spaces, Hermitian symmetric spaces, symmetric spaces of rank* 1, *and* $\mathbf{R}^k$. *If* (1.1) *holds for M, then M is flat.*

COROLLARY 5.7. *Let M be a locally symmetric space with* $\dim M\leqslant 9$ *and suppose* (1.1) *holds for M. Then M is flat.*

Proof. The only nonclassical symmetric spaces of dimension less or equal 8 are $G_2/SO(4)$ and its noncompact dual. However these spaces are Einstein and so (I) holds for them, as well as their products with $\mathbf{R}^1$.

For general symmetric spaces we have also the following result:

Table III. *Classical symmetric spaces*

Type	Real dimension	τ	$\lVert\varrho\rVert^2$	$\lVert R\rVert^2$	$3\lVert R\rVert^2-8\lVert\varrho\rVert^2$
$SO(n)$	$\frac{1}{2}n(n-1)$	$n(n-1)(n-2)\beta$	$2n(n-1)\times(n-2)^2\beta^2$	$2n(n-1)(n-2)^2\beta^2$	$-10n(n-1)(n-2)^2\beta^2$
$SU(n)$	n^2-1	$4n(n^2-1)\beta$	$16n^2(n^2-1)\beta^2$	$8n(n-1)\times(n^2+8n-8)\beta^2$	$-8n(n-1)(13n^2-8n+24)\beta^2$
$Sp(n)$	$n(2n+1)$	$4n(n+1)\times(2n+1)\beta$	$16n(2n+1)\times(n+1)^2\beta^2$	$8n(n^3+70n^2-151n+104)\beta^2$	$-8n(29n^3-130n^2+517n-296)\beta^2$
$\dfrac{SO(p+q)}{SO(p)\times SO(q)}$	pq	$pq(p+q-2)\beta$	$pq(p+q-2)^2\beta^2$	$2pq(2pq-p-q)\beta^2$	$-2pq(4p^2+4q^2+2pq-13p-13q+16)\beta^2$
$\dfrac{U(p+q)}{U(p)\times U(q)}$	$2pq$	$2pq(p+q)\beta$	$2(p+q)pq\beta^2$	$8pq(pq+1)\beta^2$	$-8pq(2p^2+2q^2+pq-3)\beta^2$
$\dfrac{Sp(p+q)}{Sp(p)\times Sp(q)}$	$4pq$	$8pq(p+q+1)\beta$	$16pq(p+q+1)^2\beta^2$	$16pq(pq+6p+6q-7)\beta^2$	$-16pq(8p^2+8q^2+13pq-2p-2q+29)\beta^2$
$\dfrac{SU(n)}{SO(n)}$	$\frac{1}{2}(n-1)\times(n+2)$	$n(n-1)(n+2)\beta$	$2n^2(n-1)\times(n+2)\beta^2$	$2n(n-1)(n+2)^2\beta^2$	$-2n(n-1)(n+2)\times(5n-6)\beta^2$
$\dfrac{SO(2n)}{U(n)}$	$n(n-1)$	$2n(n-1)^2\beta$	$4n(n-1)^3\beta^2$	$4n(n-1)\times(n^2-3n+4)\beta^2$	$-4n(n-1)(5n^2-7n-4)\beta^2$
$\dfrac{SU(2n)}{Sp(n)}$	$(n-1)\times(2n+1)$	$4n(n-1)\times(2n+1)\beta$	$16n^2(n-1)\times(2n+1)\beta^2$	$8n(n-1)\times(n^2+20n-34)\beta^2$	$-8n(n-1)\times(29n^2-44n+102)\beta^2$
$\dfrac{Sp(n)}{U(n)}$	$n(n+1)$	$2n(n+1)^2\beta$	$4n(n+1)^3\beta^2$	$4n(n+1)\times(n^2+3n+4)\beta^2$	$-4n(n+1)\times(5n^2+7n-4)\beta^2$

THEOREM 5.8. *Let M be a symmetric space and M^* the dual symmetric space of M. Then the volume function of $M\times M^*$ satisfies*

$$V_m(r)=\frac{(\pi r^2)^n}{n!}\sum_{k=0}^{\infty}a_{4k}r^{4k}$$

where $\dim M=n$.

Proof. Let R be the curvature operator of M (at any point) and R^* the curvature operator of M^*. Then $R^*=-R$. Hence if we compute the volume expansion for $M\times M^*$ we see that the coefficients of $r^{2n+4k+2}$ vanish.

6. Manifolds with $V_m(r)=\omega r^n\{1+O(r^6)\}$

In the previous section we showed that (I) is true if $\dim M\leqslant 3$ using only the nullity of the second and third term in the power series expansion of $V_m(r)$. (I) need not be true when dim $M \geqslant 4$. We show this by giving two examples of nonflat manifolds for which

$$V_m(r) = \omega r^n\{1 + O(r^6)\} \tag{6.1}$$

for all $m \in M$ and sufficiently small $r > 0$.

6.a. A generalization of the Schwarzschild metric

The Schwarzschild metric in relativity is a spherically symmetric metric which is Ricci flat but not flat. Specifically it is given in spherical coordinates by

$$ds_1^2 = \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2 - \left(1 - \frac{2m}{r}\right) dt^2 \tag{6.2}$$

(assuming the speed of light to be unity). There are generalizations of (6.2) which are Einstein metrics or have scalar curvature 0.

If one changes the sign of the coefficient of dt^2 in (6.2) then one obtains a positive definite metric. Just as before this metric is Ricci flat but not flat. In fact let us consider the metric

$$ds^2 = e^{\lambda(r)} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2 + e^{\nu(r)} dt^2. \tag{6.3}$$

We shall compute the curvature of this metric for general λ and ν. Then we determine λ and ν so that (6.1) is satisfied.

The simplest method to compute the curvature of (6.3) is to utilize the Cartan structure equations:

$$d\theta_i = \sum_j \omega_{ij} \wedge \theta_j, \quad \omega_{ij} + \omega_{ji} = 0,$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} + \Omega_{ij}. \tag{6.4}$$

Then the sectional curvatures are the only nonvanishing components of the curvature tensor of M and are given by

$$\Omega_{ij} = -K_{ij}\theta_i \wedge \theta_j.$$

Specifically we take

$$\theta_1 = e^{\frac{1}{2}\lambda} dr, \quad \theta_2 = r\,d\theta, \quad \theta_3 = r \sin\theta \, d\varphi \quad \text{and} \quad \theta_4 = e^{\frac{1}{2}\nu} dt.$$

Then we obtain the following expressions for the sectional curvatures K_{ij}:

$$K_{12} = K_{13} = \frac{1}{2r}\lambda' e^{-\lambda}, \quad K_{24} = K_{34} = -\frac{1}{2r}\nu' e^{-\lambda},$$

$$K_{23} = \frac{1}{r^2}(1 - e^{-\lambda}), \qquad K_{14} = \tfrac{1}{4}e^{-\lambda}(\nu'\lambda' - \nu'^2 - 2\nu''). \tag{6.5}$$

It is easy to see that (6.3) is Einstein if and only if $\lambda' + \nu' = 0$ and

$$\lambda'' - \lambda'^2 = \frac{2}{r^2}(e^\lambda - 1).$$

However we need a weaker condition, so we choose to set

$$\lambda' + \nu' = \frac{b}{r}, \quad b \text{ a constant.} \tag{6.6}$$

From (6.5) and (6.6) it follows that the scalar curvature τ of the metric (6.3) is given by

$$\tau = -u'' - \frac{1}{2r}(8+3b)\,u' - \frac{1}{2r^2}(b^2+2b+4)\,u + \frac{2}{r^2}$$

where $u = e^{-\lambda}$. Then

$$u = \frac{4}{b^2+2b+4}$$

is a particular solution of the differential equation $\tau = 0$ and for this solution we obtain

$$K_{12} = K_{13} = 0, \qquad K_{24} = K_{34} = -\frac{2b}{r^2(b^2+2b+4)},$$

$$K_{23} = \frac{2b+b^2}{r^2(b^2+2b+4)}, \quad K_{14} = \frac{2b-b^2}{r^2(b^2+2b+4)}.$$

Hence

$$\|\varrho\|^2 = \frac{4b^2(b^2+2)}{r^4(b^2+2b+4)^2}, \quad \|R\|^2 = \frac{8b^2(b^2+8)}{r^4(b^2+2b+4)^2}.$$

It follows easily that $3\|R\|^2 - 8\|\varrho\|^2 = 0$ if and only if $b^2 = 16$ or $b = 0$. Taking $b = \pm 4$ we obtain the two following 4-dimensional metrics:

$$\begin{aligned} ds_1^2 &= 7dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) + (cr)^4 dt^2, \\ ds_2^2 &= 3dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) + (cr)^{-4} dt^2. \end{aligned} \tag{6.7}$$

Here c is an arbitrary constant. For each of these metrics (6.1) holds at all points, but neither of the metrics is flat.

A lengthy calculation, which we omit, shows that for the metrics (6.7)

$$V_m(r) = \frac{\pi^2 r^4}{2}\{1 + A_m r^6 + O(r^8)\}$$

where $A_m \neq 0$.

6.b. A 5-dimensional manifold with $V_m(r)=\frac{8\pi^2 r^5}{15}\{1+O(r^6)\}$

Next we consider the manifold $M=S^3\times H^2(c)$, where $H^2(c)$ is a surface with constant negative curvature c and S^3 is the 3-dimensional sphere with a homogeneous metric constructed as follows. Let N denote the unit outward normal to the unit sphere S^3 in $\mathbf{R}^4$. Regarding $\mathbf{R}^4$ as the quaternions we obtain tangent vector fields IN, JN, KN tangent to S^3. Let φ_I, φ_J and φ_K be the 1-forms on S^3 given by $\varphi_I(X)=\langle X, IN\rangle$, etc. We consider the metrics of the form

$$(\ ,\ )=\alpha^2\varphi_I^2+\beta^2\varphi_J^2+\gamma^2\varphi_K^2, \tag{6.8}$$

α, β, γ being constant.

The curvature of the metric (6.8) can be computed using the Cartan structure equations (6.4) together with the relations

$$d\varphi_I=2\varphi_J\wedge\varphi_K,\quad d\varphi_J=2\varphi_K\wedge\varphi_I,\quad d\varphi_K=2\varphi_I\wedge\varphi_J.$$

We obtain for the sectional curvatures of S^3:

$$\left.\begin{aligned}
K_{IJ}&=\frac{1}{\alpha^2\beta^2\gamma^2}(\alpha^4+\beta^4-3\gamma^4-2\alpha^2\beta^2+2\alpha^2\gamma^2+2\beta^2\gamma^2),\\
K_{IK}&=\frac{1}{\alpha^2\beta^2\gamma^2}(\alpha^4-3\beta^4+\gamma^4+2\alpha^2\beta^2-2\alpha^2\gamma^2+2\beta^2\gamma^2),\\
K_{JK}&=\frac{1}{\alpha^2\beta^2\gamma^2}(-3\alpha^4+\beta^4+\gamma^4+2\alpha^2\beta^2+2\alpha^2\gamma^2-2\beta^2\gamma^2).
\end{aligned}\right\} \tag{6.9}$$

All other curvature components vanish.

Let $S^3(\alpha,\beta,\gamma)$ denote S^3 with the metric (6.8) and let

$$\gamma^2=\beta^2+\alpha^2,\quad c=-\frac{4}{\alpha^2+\beta^2}.$$

Put $M=S^3(\alpha,\beta,\gamma)\times H^2(c)$. It is not difficult using (6.9) to check that for each $m\in M$ the volume $V_m(r)$ satisfies (4.1).

Actually this construction yields a 1-parameter family of (normalized) metrics for which

$$V_m(r)=\frac{8\pi^2r^5}{15}\{1+O(r^6)\}.$$

7. Manifolds with $V_m(r) = \omega r^n\{1 + O(r^8)\}$

In order to find a manifold M such that $V_m(r) = \omega r^n\{1 + O(r^8)\}$ at every point we shall first consider certain manifolds of dimension 2 and 3. Let M_1 be the set of matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

with the left invariant metric $ds^2 = dx^2 + dy^2 + (x\,dy - dz)^2$. Also let M_2 be a space of constant curvature $-a$, and M_3 be a space of constant curvature b. Here we require that $\dim M_2 = 2$, $\dim M_3 = 3$, and that $a, b > 0$. Let

$$\begin{aligned} A_i &= \tau_{M_i}, \\ B_i &= (-3\|R\|^2 + 8\|\varrho\|^2)_{M_i}, \\ C_i &= (64\check{\varrho} - 192\langle \varrho \otimes \varrho, \bar{R}\rangle + 288\langle \varrho, \dot{R}\rangle - 110\check{R} - 200\check{\bar{R}} - \tfrac{459}{2}\|\nabla\varrho\|^2 + 405\alpha(\varrho) + \tfrac{135}{2}\|\nabla R\|^2)_{M_i}. \end{aligned}$$

Note that up to a constant factor B_i is that part of the coefficient of r^{n+4} in the expansion of $V_m(r)$ which does not involve τ. A similar remark applies to C_i.

Now let $M = M_1^\alpha \times M_1^\beta \times M_3^\gamma$ where α, β, γ are integers. It is easily seen that $V_m(r) = \omega r^n\{1 + O(r^8)\}$ at each point of M if and only if

$$\begin{aligned} \alpha A_1 + \beta A_2 + \gamma A_3 &= 0, \\ \alpha B_1 + \beta B_2 + \gamma B_3 &= 0, \\ \alpha C_1 + \beta C_2 + \gamma C_3 &= 0. \end{aligned} \tag{7.1}$$

Thus we must find nontrivial solutions of (7.1) for which α, β, γ are positive integers.

First we compute

$$\begin{aligned} &A_1 = -\tfrac{1}{2}, && B_1 = -\tfrac{9}{4}, && C_1 = 26, \\ &A_2 = -2a, && B_2 = 4a^2, && C_2 = -16a^3, \\ &A_3 = 6b, && B_3 = 60b^2, && C_3 = 0. \end{aligned}$$

Thus, if

$$\det\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix} = 0$$

we must have

$$b = \frac{a}{10}\left(\frac{9a-26}{2a^2+13}\right).$$

Assuming this, we solve (7.1) and obtain

$$\alpha = \frac{8a^3}{13}\beta, \quad \gamma = \frac{10(2a^2+13)^2}{39(9a-26)}\beta.$$

Now let β be an arbitrary positive integer and take $a = 13/2$. Then $b = 13/60$, $\alpha = 13^2\beta$ and $\gamma = 75\beta$. Thus

$$M = M_1^{13^2\beta} \times M_2^{\beta} \times M_3^{75\beta}$$

is our required manifold. When $\beta = 1$, M has dimension 734.

It is an interesting problem to determine if there are 4-dimensional manifolds with $V_m(r) = \omega r^n\{1 + O(r^8)\}$.

8. Characterizations of spaces of constant curvature by volume functions

Let $M(\lambda)$ be an n-dimensional manifold of constant sectional curvature $\lambda \neq 0$. The n-dimensional volume of a geodesic ball is given by

$$V_p(r) = \int_0^r S_p(t)\,dt$$

for $p \in M(\lambda)$, where

$$S_p(r) = \frac{2\pi^{n/2}}{\left(\frac{n}{2}-1\right)!}\left\{\frac{\sin\sqrt{\lambda}\,r}{\sqrt{\lambda}}\right\}^{n-1}$$

if $\lambda > 0$. If $\lambda < 0$, sin must be replaced by sinh and λ by $|\lambda|$. See for example [20]. ($S_p(r)$ is just the $(n-1)$-dimensional volume of the sphere of radius r and center p in $M(\lambda)$.)

We state the following conjecture:

(III) *Let M be an n-dimensional Riemannian manifold and suppose that for all $m \in M$ and all sufficiently small $r > 0$, $V_m(r)$ is the same as that of an n-dimensional manifold of constant sectional curvature λ. Then M is also a space of constant sectional curvature λ.*

We prove this conjecture in some particular cases.

THEOREM 8.1. (III) *is true in the following cases:*

(i) $\dim M \leq 3$;
(ii) *M is conformally flat;*
(iii) *M is an Einstein manifold.*

Proof. Let τ_λ, $\|\varrho_\lambda\|^2$ and $\|R_\lambda\|^2$ be the appropriate functions for a space of constant sectional curvature λ. In fact

$$\tau_\lambda = n(n-1)\lambda, \quad \|\varrho_\lambda\|^2 = n(n-1)^2\lambda^2, \quad \|R_\lambda\|^2 = 2n(n-1)\lambda^2. \tag{8.1}$$

The hypotheses of (III) imply that τ, $\|\varrho\|^2$ and $\|R\|^2$ for M satisfy

$$\tau = n(n-1)\lambda, \quad 3\|R\|^2 - 8\|\varrho\|^2 = -2n(n-1)(4n-7)\lambda^2. \tag{8.2}$$

For dim $M=2$ the result follows at once from $\tau = 2\lambda$. If dim $M=3$ or if M is conformally flat, (8.2) and the vanishing of the Weyl tensor imply that

$$\|\varrho\|^2 = n(n-1)^2\lambda^2, \quad \|R\|^2 = 2n(n-1)\lambda^2. \tag{8.3}$$

Hence

$$\|R\|^2 = \frac{2}{n-1}\|\varrho\|^2.$$

So the required result follows immediately from a result of [4].

Finally, let M be an Einstein manifold; then $\tau^2 = n\|\varrho\|^2$. This and (8.2) give again (8.3), proving the result.

9. Characterizations of the other rank 1 symmetric spaces

Let $N(\mu)$ be a Kähler manifold with complex dimension n and constant holomorphic sectional curvature $\mu \neq 0$. Then the volume function for $N(\mu)$ is given by:

$$V_p(r,\mu) = \frac{(4\pi)^n}{n!\,\mu^n}\left\{\sin\frac{\sqrt{\mu}}{2}r\right\}^{2n}$$

or

$$V_p(r,\mu) = \frac{(4\pi)^n}{n!\,|\mu|^n}\left\{\sinh\frac{\sqrt{|\mu|}}{2}r\right\}^{2n}$$

according to whether $\mu > 0$ or $\mu < 0$. See for example [20]. We state

(IV) *Let M be a Kähler manifold with complex dimension n and suppose that for all $m \in M$ and all sufficiently small $r > 0$, $V_m(r)$ is the same as that of an n-dimensional Kähler manifold with constant holomorphic sectional curvature μ. Then M has constant holomorphic sectional curvature μ.*

There are two cases where we can prove this conjecture.

THEOREM 9.1. (IV) *is true in the following cases*

(i) *M is Bochner flat* $(n \geq 2)$;

(ii) *M is an Einstein Kähler manifold.*

Proof. Let τ_μ, $\|\varrho_\mu\|^2$ and $\|R_\mu\|^2$ denote the appropriate functions for a space of constant holomorphic sectional curvature μ. Thus

$$\tau_\mu = n(n+1)\mu, \quad \|\varrho_\mu\|^2 = \tfrac{1}{2}n(n+1)^2\mu^2, \quad \|R_\mu\|^2 = 2n(n+1)\mu^2. \tag{9.1}$$

The hypotheses of the conjecture imply that

$$\tau = n(n+1)\mu, \quad 3\|R\|^2 - 8\|\varrho\|^2 = -2n(n+1)(2n-1)\mu^2. \tag{9.2}$$

First suppose M is Bochner flat. Then (4.5) and (9.2) imply

$$\|\varrho\|^2 = \tfrac{1}{2}n(n+1)^2\mu^2, \quad \|R\|^2 = 2n(n+1)\mu^2.$$

Hence

$$\|R\|^2 = \frac{4}{n+1}\|\varrho\|^2,$$

and according to [13] M has constant holomorphic sectional curvature μ.

For an Einstein Kähler manifold M we have $\tau^2 = 2n\|\varrho\|^2$. This fact together with (9.2) implies again

$$\|R\|^2 = \frac{4}{n+1}\|\varrho\|^2.$$

Hence M must have constant holomorphic sectional curvature μ.

Next let $Q(\nu)$ be a $4n$-dimensional Riemannian manifold locally isometric to quaternionic projective space or its noncompact dual, where $\nu \neq 0$ denotes the maximum of the sectional curvatures in the positive curvature case and the minimum of the sectional curvatures in the negative curvature case. Then the volume function for $Q(\nu)$ is given by:

$$V_p(r,\nu) = \frac{(4\pi)^{2n}}{(2n+1)!\,\nu^{2n}} \sin^{4n}(\tfrac{1}{2}\sqrt{\nu}\,r)\,(2n\cos^2(\tfrac{1}{2}\sqrt{\nu}\,r)+1)$$

or

$$V_p(r,\nu) = \frac{(4\pi)^{2n}}{(2n+1)!\,|\nu|^{2n}} \sinh^{4n}(\tfrac{1}{2}\sqrt{|\nu|}\,r)\,(2n\cosh^2(\tfrac{1}{2}\sqrt{|\nu|}\,r)+1)$$

according to whether $\nu > 0$ or $\nu < 0$ [20].

The following question naturally arises: is $Q(\nu)$ characterized by its volume function among manifolds with holonomy group contained in $Sp(n)\cdot Sp(1)$, $n>1$? The answer is yes, in contrast to the characterizations of $S^n(\lambda)$ and $CP^n(\mu)$.

THEOREM 9.2. *Let M be a Riemannian manifold whose holonomy group is a subgroup of $Sp(n)\cdot Sp(1)$, $n>1$. Further, suppose that for all $m\in M$ and all sufficiently small $r>0$, $V_m(r)$ is the same as that of $Q(\nu)$. Then M is locally isometric to $Q(\nu)$.*

Proof. The key fact is that for $n>1$, a manifold whose holonomy group is contained in $Sp(n)\cdot Sp(1)$ is automatically Einstein [3]. See also [22]. Hence if $V_m(r)$ coincides with $V_p(r,\nu)$, then a computation shows that

$$\tau = 4n(n+2)\nu, \quad \|\varrho\|^2 = \frac{\tau^2}{4n}$$

and so

$$\|R\|^2 = \frac{\tau^2(5n+1)}{4n(n+2)^2}.$$

In other words, the τ, $\|\varrho\|^2$, and $\|R\|^2$ of M are the same as the corresponding functions on $Q(\nu)$. That M is locally isometric to $Q(\nu)$ then follows from a result of [36].

Finally, we need not formulate a similar theorem for manifolds with holonomy group contained in Spin (9), because such manifolds are automatically flat or are locally isometric to the Cayley plane or its noncompact dual [1], [9].

10. Topological characterizations of compact 4-dimensional manifolds

In this section we consider some topological implications of the different conjectures and give some characterizations of 4-dimensional compact manifolds.

First we consider a compact oriented 4-dimensional manifold M such that

$$V_m(r) = \frac{\pi^2 r^4}{2}\{1+O(r^6)\} \tag{10.1}$$

for all $m\in M$ and for sufficiently small $r>0$. Then we have

THEOREM 10.1. *Let M be a compact oriented 4-dimensional manifold such that $V_m(r)$ satisfies* (10.1) *for all $m\in M$ and sufficiently small $r>0$. Then*

$$\chi(M) \leqslant -\tfrac{3}{2}|\tau(M)|,$$

where $\chi(M)$ and $\tau(M)$ denote the Euler characteristic and signature of M. If $\chi(M) = -\frac{3}{2}|\tau(M)|$ then M is flat.

Proof. It is well-known [5, p. 82] that

$$\chi(M) = \frac{1}{32\pi^2}\int_M \{\|R\|^2 - 4\|\varrho\|^2 + \tau^2\}\,dV. \tag{10.2}$$

Hence it follows from (10.1) and (10.2) that

$$\chi(M) = -\frac{1}{64\pi^2}\int_M \|R\|^2 dV. \tag{10.3}$$

Further, the Hirzebruch index formula for the signature $\tau(M)$ of M states that

$$\tau(M) = \frac{1}{12\pi^2}\int_M (\Omega_{12}^2 + \ldots + \Omega_{34}^2).$$

By evaluation of the integrand on the oriented orthonormal frame $\{e_1, e_2, e_3, e_4\}$ we obtain

$$\tau(M) = \frac{1}{96\pi^2}\sum_{i,j,k,l}\int_M R_{ijkl}R_{ij*(kl)}\,dV = \frac{1}{96\pi^2}\int_M \langle R, R*\rangle\, dV, \tag{10.4}$$

$*$ being the Hodge $*$-operator determined by the given orientation.

According to the decomposition for curvature tensors (see for example [19], [34]) we have

$$R = R_1 + R_2 + R_w$$

with

$$R_1* = R_1, \quad R_2* = -R_2, \quad R_w* = -*R_w.$$

Hence

$$R* = R_1 - R_2 + R_w*$$

and

$$\langle R, R*\rangle = \|R_1\|^2 - \|R_2\|^2 + \langle R_w, R_w*\rangle.$$

Now $\langle R_w, R_w*\rangle = 4\,\mathrm{tr}\,(R_w^2*) = 4\,\mathrm{tr}\,(R_w*R_w) = -4\,\mathrm{tr}\,(R_w^2*) = 0$. Hence $\langle R, R*\rangle = \|R_1\|^2 - \|R_2\|^2$, and so $\|R\|^2 \geqslant \langle R, R*\rangle$. The result follows from this, (10.3) and (10.4).

Next suppose $\chi(M) = -\frac{3}{2}|\tau(M)|$. This is equivalent to

$$\int_M \{\langle R, R*\rangle + \|R\|^2\}\, dV \leqslant 0$$

or

$$\int_M \{2\|R_1\|^2 + \|R_w\|^2\}\, dV \leqslant 0.$$

Hence $R_w = 0$ and M is an Einstein manifold. But then M is flat.

The proof of Theorem 10.1 yields also the following

COROLLARY 10.2. *Let M be a manifold satisfying the same hypotheses as in Theorem 10.1. Then $\chi(M) \leqslant 0$ and $\chi(M) = 0$ if and only if M is flat.*

Furthermore we have

COROLLARY 10.3. *Let M be a compact 4-dimensional manifold which admits a metric such that the associated volume function satisfies* (10.1). *If M also admits an Einstein metric then all Einstein metrics are flat.*

Proof. $\chi(M)\geqslant 0$ for any 4-dimensional manifold which admits an Einstein metric [3]. The result follows now from Corollary 10.2 since $\chi(M)=0$.

We have a stronger result for compact 4-dimensional Kähler manifolds.

THEOREM 10.4. *Let M be a compact 4-dimensional Kähler manifold such that $V_m(r)$ satisfies* (10.1) *for all $m\in M$ and sufficiently small $r>0$. Then*

$$\alpha(M)=\tau(M)=\tfrac{1}{3}\chi(M)\leqslant 0$$

where $\alpha(M)$ denotes the arithmetic genus of M. The equality sign holds if and only if M is flat.

Proof. This follows easily from the following formulas (see for example [13])

$$\tau(M)=-\frac{1}{48\pi^2}\int_M\{\|R\|^2-2\|\varrho\|^2\}\,dV,$$

$$\alpha(M)=\frac{1}{384\pi^2}\int_M\{\|R\|^2-8\|\varrho\|^2+3\tau^2\}\,dV. \tag{10.5}$$

Here M has the orientation induced by the almost complex structure.

Next we consider compact oriented 4-dimensional manifolds such that

$$V_m(r)=\frac{\pi^2r^4}{2}\{1+\alpha r^2+\beta r^4+O(r^6)\} \tag{10.6}$$

for all $m\in M$ and sufficiently small $r>0$, where α and β are the same as in the volume function of a 4-dimensional space of constant sectional curvature λ, that is

$$\alpha=-\frac{\lambda}{3}\quad\text{and}\quad\beta=\frac{13}{240}\lambda^2.$$

THEOREM 10.5. *Let M be a compact oriented 4-dimensional manifold whose volume function satisfies* (10.6). *Then*

$$\chi(M)\leqslant-\tfrac{3}{2}|\tau(M)|+\frac{3\lambda^2}{4\pi^2}\operatorname{vol}(M).$$

The equality sign holds if and only if M is a space of constant sectional curvature λ.

Proof. Using (8.1), (8.2), (10.2) and (10.6) we obtain

$$\chi(M) = -\frac{1}{64\pi^2}\int_M \|R\|^2 dV + \frac{9\lambda^2}{8\pi^2}\operatorname{vol}(M).$$

Further we let **1** denote the identity transformation and put

$$R = \tilde{R} + \frac{\tau}{12}\mathbf{1}.$$

(Thus the scalar curvature of $\tilde{R}$ is 0 [19].) Then we have

$$\chi(M) = -\frac{1}{64\pi^2}\int_M \|\tilde{R}\|^2 dV + \frac{3\lambda^2}{4\pi^2}\operatorname{vol}(M)$$

since $\langle \tilde{R}, \mathbf{1}\rangle = 0$, $\langle \mathbf{1}, \mathbf{1}\rangle = 24$ and $\tau^2 = 144\lambda^2$. Now we get the required result by proceeding in the same way as in the proof of Theorem 10.1 using $\tilde{R}$ in place of R.

COROLLARY 10.6. *Let M be a manifold satisfying the hypotheses of Theorem 10.5. Then*

$$\chi(M) \leqslant \frac{3\lambda^2}{4\pi^2}\operatorname{vol}(M),$$

with equality sign if and only if M is a space of constant sectional curvature λ.

It is also easy to prove the following

COROLLARY 10.7. *Let M be a compact 4-dimensional manifold which admits a metric such that the associated volume function satisfies* (10.6). *If M admits an Einstein metric then all Einstein metrics have constant sectional curvature.*

Finally we consider a compact 4-dimensional Kähler manifold with volume function

$$V_m(r) = \frac{\pi^2 r^4}{2}\{1 + \tilde{\alpha} r^2 + \tilde{\beta} r^4 + O(r^6)\} \tag{10.7}$$

for all $m \in M$ and all sufficiently small $r > 0$, where $\tilde{\alpha}$ and $\tilde{\beta}$ are the same as in the volume function for a 4-dimensional space of constant holomorphic sectional curvature μ, that is

$$\tilde{\alpha} = -\frac{\mu}{6} \quad \text{and} \quad \tilde{\beta} = \frac{\mu^2}{80}.$$

THEOREM 10.8. *Let M be a compact Kähler manifold whose volume function satisfies* (10.7). *Then*

$$\alpha(M)=\tau(M)=\tfrac{1}{3}\chi(M)\leqslant\frac{\mu^2}{8\pi^2}\operatorname{vol}(M).$$

The equality sign holds if and only if M is a space of constant holomorphic sectional curvature μ.

Proof. The result follows from (9.1) and (10.4).

Using a result of [13] we obtain

COROLLARY 10.9. *Let M be a compact complex analytic manifold of complex dimension two which admits a Kähler metric such that the associated volume function satisfies (10.7). If M admits an Einstein Kähler metric, then all Einstein Kähler metrics have constant holomorphic sectional curvature.*

Finally we prove a theorem concerning $2n$-dimensional Kähler manifolds.

THEOREM 10.10. *Let M be a $2n$-dimensional compact Kähler manifold with nonnegative generalized Chern number $c_2[F]^{n-2}(M)$. (F is the Kähler form.) Then the conjecture* (I) *is true.*

Proof. Let F denote the Kähler form and γ_2 the second Chern class of M. Then [5]

$$c_2[F]^{n-2}(M)=\int_M\gamma_2\wedge F^{n-2}=\frac{(n-2)!}{32\pi^2}\int_M\{\|R\|^2-4\|\varrho\|^2+\tau^2\}\,dV. \tag{10.8}$$

Suppose $c_2[F]^{n-2}(M)$ is nonnegative. Then we obtain from (4.1), (4.2), and (10.8) that

$$\int_M\|R\|^2dV\leqslant 0$$

and hence $R=0$.

11. Characterizations of locally symmetric spaces by volume functions

In this section we prove some results concerning volume functions and locally symmetric spaces. This is the first time we make use of the coefficient of r^{n+6} in the power series expansion of $V_m(r)$. Our theorems are analogous to those of [33] for the spectrum of the Laplacian.

THEOREM 11.1. *Let M be an n-dimensional Einstein manifold with $n=4$ or 5 and suppose M has for all $m\in M$ and all sufficiently small $r>0$ the same volume function as an n-dimensional locally symmetric Einstein space M'. Then M is locally symmetric.*

Proof. For an Einstein manifold we have

$$\varrho = \frac{\tau}{n} g, \quad \tau^2 = n\|\varrho\|^2 \tag{11.1}$$

and so

$$\left.\begin{aligned} \check{\varrho} &= \frac{\tau^3}{n^2}, \\ \langle \varrho, \check{R} \rangle &= \frac{\tau}{n} \|R\|^2, \\ \langle \varrho \otimes \varrho, \bar{R} \rangle &= \frac{\tau^3}{n^2}. \end{aligned}\right\} \tag{11.2}$$

Then (11.2) and (2.18) imply

$$\langle \Delta R, R \rangle = \frac{2\tau}{n} \|R\|^2 - \check{R} - 4\check{\bar{R}}. \tag{11.3}$$

Further on an n-dimensional manifold with $n \leqslant 5$ the 6-dimensional Gauss–Bonnet integrand vanishes and so (2.24) holds.

On a 4-dimensional Einstein manifold (2.24) reduces to

$$\frac{\tau^3}{2} - 3\tau\|R\|^2 + 4\check{R} - 8\check{\bar{R}} = 0. \tag{11.4}$$

(11.3) combined with (11.4) gives

$$\left.\begin{aligned} \check{R} &= -\frac{\tau^3}{12} + \frac{2}{3}\tau\|R\|^2 - \frac{1}{3}\langle \Delta R, R \rangle, \\ \check{\bar{R}} &= \frac{\tau^3}{48} - \frac{1}{24}\tau\|R\|^2 - \frac{1}{6}\langle \Delta R, R \rangle. \end{aligned}\right\} \tag{11.5}$$

For a 5-dimensional Einstein manifold we proceed in the same way and obtain

$$\left.\begin{aligned} &\frac{\tau^3}{5} - \frac{9}{5}\tau\|R\|^2 + 4\check{R} - 8\check{\bar{R}} = 0, \\ &\check{R} = -\frac{\tau^3}{30} + \frac{13}{30}\tau\|R\|^2 - \frac{1}{3}\langle \Delta R, R \rangle, \\ &\check{\bar{R}} = \frac{\tau^3}{120} - \frac{1}{120}\tau\|R\|^2 - \frac{1}{6}\langle \Delta R, R \rangle. \end{aligned}\right\} \tag{11.6}$$

Now let M' be a locally symmetric Einstein space of dimension 4 or 5 and suppose that for all $m \in M$ and sufficiently small r, M has the same volume function as M'. Then the equalities of the corresponding coefficients of r^{n+2} and r^{n+4} in the two expansions imply that

$$\tau=\tau', \quad \|\varrho\|^2=\|\varrho'\|^2, \quad \|R\|^2=\|R'\|^2. \tag{11.7}$$

In particular $\|R\|^2$ is constant and so

$$\langle \Delta R, R\rangle = -\|\nabla R\|^2. \tag{11.8}$$

Then, using (11.7), (11.2), (11.5), (11.6), (11.8) and the equality of the coefficients of r^{n+6} in both expansions, we obtain

$$\|\nabla R\|^2=0.$$

Hence the result follows.

Of course a 3-dimensional Einstein manifold has constant curvature and so it is automatically symmetric, which is the reason that we considered only 4- and 5-dimensional manifolds in Theorem 11.1. However, for 3-dimensional manifolds we have a stronger result:

THEOREM 11.2. *Let M be a 3-dimensional manifold with the same volume function as a locally symmetric 3-dimensional manifold M'. Assume also that $\alpha(\varrho)\geqslant 0$ and that $\breve{\varrho}\geqslant\breve{\varrho}'$. Then M is also locally symmetric.*

Proof. We use the special expansion (3.4) for 3-dimensional manifolds. Equality of the coefficients of r^5 and r^7 in the two expansions implies that

$$\tau=\tau' \quad \text{and} \quad \|\varrho\|^2=\|\varrho'\|^2. \tag{11.9}$$

Since M' is locally symmetric, (11.9) implies that τ and $\|\varrho\|^2$ are constant. In particular

$$\|\nabla\tau\|^2=0, \tag{11.10}$$

$$\langle\Delta\varrho,\varrho\rangle+\|\nabla\varrho\|^2=\tfrac{1}{2}\Delta\|\varrho\|^2=0. \tag{11.11}$$

Next (11.9), (11.10), (11.11) and the equality of the coefficients of r^9 in the two expansions imply that

$$128\breve{\varrho}+\tfrac{9}{2}\|\nabla\varrho\|^2+45\alpha(\varrho)=128\breve{\varrho}'. \tag{11.12}$$

Finally assume $\breve{\varrho}\geqslant\breve{\varrho}'$ and $\alpha(\varrho)\geqslant 0$. From (11.12) it follows that M is locally symmetric.

12. Mean curvature and geodesic spheres

Let $h_m(\exp_m ru)$ denote the mean curvature of a geodesic sphere of radius $r>0$ and center m with respect to the outward normal. Put $H_m(r)=r^{n-1}\int_{\|u\|=1}h_m(\exp_m ru)\,du$. Here

$H_m(r)r^{1-n}$ is the integral over the unit sphere in the tangent space of the mean curvature of a geodesic sphere of radius $r>0$. It should be remarked that this is not the same as the integral over the geodesic sphere itself of the mean curvature. The latter integral is just

$$\frac{d^2}{dr^2}V_m(r).$$

See [26].

We compute the first four terms in the power series expansion of $H_m(r)$.

LEMMA 12.1. *Let $m \in M$ and let $\exp_m(ru)$ be a point of the geodesic sphere with center m and radius r. Then the mean curvature h_m of this geodesic sphere at the point $\exp_m(ru)$ is given by*

$$h_m(\exp_m(ru)) = \frac{n-1}{r} + \left(\frac{\theta'}{\theta}\right)(\exp_m(ru))$$

where $\theta = \omega_{1\ldots n}$ and θ' is the radial derivative of the function $r \to \theta(\exp_m(ru))$.

Proof. See for example [5, p. 134]. Note that $h_m(\exp_m(ru))$ is essentially the Laplacian of the distance function.

LEMMA 12.2. *We have*

$$h_m(\exp_m(ru)) = \frac{n-1}{r} + \alpha_1 r + \alpha_2 r^2 + \alpha_3 r^3 + \alpha_4 r^4 + \alpha_5 r^5 + O(r^6)$$

where

$$\alpha_1 = -\tfrac{1}{3}\sum_{i,j=1}^{n} \varrho_{ij}(m)\, a_i a_j,$$

$$\alpha_2 = -\tfrac{1}{4}\sum_{i,j,k=1}^{n} \nabla_i \varrho_{jk}(m)\, a_i a_j a_k,$$

$$\alpha_3 = -\tfrac{1}{90}\sum_{i,j,k,l=1}^{n} \left\{9\nabla^2_{ij}\varrho_{kl} + 2\sum_{a,b=1}^{n} R_{iajb}R_{kalb}\right\}_m a_i a_j a_k a_l,$$

$\alpha_4 =$ *something irrelevant,*

$$\alpha_5 = -\tfrac{1}{120}\sum_{i,j,k,l,g,h=1}^{n} \left\{\tfrac{5}{7}\nabla^4_{ijkl}\varrho_{hg} + \tfrac{8}{7}\sum_{a,b=1}^{n}\nabla^2_{ij}R_{kalb}R_{hagb} + \tfrac{15}{14}\sum_{a,b=1}^{n}\nabla_i R_{jakb}\nabla_l R_{hagb} + \tfrac{16}{63}\sum_{a,b,c=1}^{n} R_{iajb}R_{kblc}R_{hcga}\right\}_m a_i a_j a_k a_l a_g a_h.$$

Proof. Put $\theta = 1 + \beta_2 r^2 + \beta_3 r^3 + \ldots$. Then

$$\frac{\theta'}{\theta} = 2\beta_2 r + 3\beta_3 r^2 + (4\beta_4 - 2\beta_2^2)r^3 + 5(\beta_5 - \beta_2\beta_3)r^4 + (6\beta_6 - 6\beta_2\beta_4 + 2\beta_2^3 - 3\beta_3^2)r^5 + O(r^6).$$

The result follows from (3.1) and Lemma 12.1.

THEOREM 12.3. *We have*

$$
\begin{aligned}
H_m(r) &= r^{n-1} \int_{S^{n-1}(1)} h_m(\exp_m(ru))\, du \\
&= n\omega r^{n-2} \bigg\{ n-1-\frac{\tau}{3n} r^2 - \frac{1}{90n(n+2)} (3\|R\|^2 + 2\|\varrho\|^2 + 18\Delta\tau) r^4 \\
&\quad + \frac{1}{840n(n+2)(n+4)} \big(-\tfrac{412}{9}\check{\varrho} + 44\langle \varrho \otimes \varrho, \bar{R}\rangle + 4\langle \varrho, \dot{R}\rangle - \tfrac{110}{9}\check{R} - \tfrac{200}{9}\check{\bar{R}} \\
&\quad - 25\alpha(\varrho) - 45\Delta^2\tau - 25\|\nabla\tau\|^2 - 72\langle \nabla^2\tau, \varrho\rangle - \tfrac{25}{2}\|\nabla\varrho\|^2 + 6\langle \Delta\varrho, \varrho\rangle \\
&\quad - 30\langle \Delta R, R\rangle - \tfrac{45}{2}\|\nabla R\|^2 \big) r^6 + O(r^8) \bigg\}_m .
\end{aligned} \tag{12.1}
$$

Proof. This follows immediately using Lemma 12.2 and the formulas in the proof of Theorem 3.3.

We have at once

THEOREM 12.4. *For sufficiently small $r>0$ we have $H_m(r)>0$. If $\tau \geqslant 0$, then $H_m(r) \leqslant n(n-1)\omega r^{n-2}$, for sufficiently small $r>0$.*

THEOREM 12.5. *Let M be a compact n-dimensional manifold such that*

$$
H_m(r) = n\omega r^{n-2} \left\{ n-1-\frac{\tau}{3n} r^2 + O(r^6) \right\}_m \tag{12.2}
$$

for all $m \in M$ and all sufficiently small $r>0$. Then M is flat.

Proof. From (12.1) and (12.2) it follows that

$$
18\Delta\tau = -(3\|R\|^2 + 2\|\varrho\|^2), \tag{12.3}
$$

and so the result follows at once from the maximum principle.

THEOREM 12.6. *Let M be an n-dimensional manifold such that*

$$
H_m(r) = n\omega r^{n-2} \left\{ n-1-\frac{\tau}{3n} r^2 + O(r^6) \right\}_m
$$

for all $m \in M$ and all sufficiently small $r>0$ and suppose τ is constant. Then M is flat.

Proof. This is immediate from (12.3) with $\Delta\tau = 0$.

COROLLARY 12.7. *The conjecture* (II) *is true.*

In the same way as in sections 8 and 9 we deduce from (12.1):

THEOREM 12.8. *Let M be an n-dimensional manifold such that*

$$H_m(r) = n\omega r^{n-2}\{n-1+\alpha r^2+\beta r^4+O(r^6)\}$$

for all m and all sufficiently small $r>0$, where α and β are the same as for a space of constant (constant holomorphic) sectional curvature λ. Then M has constant (constant holomorphic) sectional curvature.

We also remark that the coefficient of r^{n+4} in the power series expansion of $H_m(r)$ may be used to obtain theorems for locally symmetric spaces analogous to those of section 11.

13. Growth functions of hypersurfaces

Let M be a compact orientable hypersurface of an n-dimensional Riemannian manifold $\bar{M}$. For small $s \geqslant 0$ denote by $A(s)$ the $(n-1)$-dimensional volume of the hypersurface at a distance s from M, in the direction of a chosen normal. In [30], [38] the following results have been proved:

THEOREM 13.1. *Suppose that for any compact orientable hypersurface M of $\bar{M}$ the function $A(s)$ is linear for small $s \geqslant 0$. Then $\bar{M}$ is flat and* $\dim \bar{M} = 2$.

THEOREM 13.2. *Suppose that for any compact orientable hypersurface M of $\bar{M}$ we have $A''(s)+cA(s)=0$ for small $s \geqslant 0$. Then $\bar{M}$ has constant curvature c and* $\dim \bar{M} = 2$.

See also [15], [26], [28], [29] for related results. In each of these theorems a condition on the function A is required for *every* hypersurface. We shall show that it is only necessary to assume that A satisfy a differential equation for hypersurfaces of the form $G_m(r) = \{p \in \bar{M} \mid d(p, m) = r\}$ for $m \in \bar{M}$ and small $r>0$.

The point is that once one has the power series expansion for the volume function $V_m(r)$ the results of [15], [38], [30] can be strengthened and the proofs simplified.

THEOREM 13.3. *Suppose that for all $m \in \bar{M}$ and all small $r>0$ the growth function $A(s)$ of each hypersurface $G_m(r)$ satisfies $A''(s)+cA(s)=0$ for small $s \geqslant 0$, where c is a constant. Then $\bar{M}$ has constant curvature c, and* $\dim \bar{M} = 2$.

Proof. We do the case when $c>0$. The proofs for $c=0$ and $c<0$ are similar.

The $(n-1)$-dimensional volume of $G_m(r)$ is

$$S_m(r) = \frac{d}{dr} V_m(r).$$

Moreover $A(s) = S_m(r+s)$. Suppose now that $A''+cA=0$ where $c>0$. Then

$$A(s) = a_m(r) \cos \sqrt{c}\, s + b_m(r) \sin \sqrt{c}\, s.$$

Using the fact that $A(0)=S_m(r)$ and $A(-r)=0$ we can determine $a_m(r)$ and $b_m(r)$. Thus

$$A(s)=\frac{S_m(r)}{\sin\sqrt{c}\,r}\sin\sqrt{c}\,(s+r).$$

On the other hand reversing the roles of r and s we have

$$A(s)=S_m(r+s)=\frac{S_m(s)}{\sin\sqrt{c}\,s}\sin\sqrt{c}\,(s+r).$$

It follows that

$$\frac{S_m(r)}{\sin\sqrt{c}\,r}=\frac{S_m(s)}{\sin\sqrt{c}\,s}=c_m,$$

a constant. Therefore

$$S_m(r)=c_m\sin\sqrt{c}\,r.$$

From the power series expansion of $S_m(r)$ we see that

$$\dim\tilde{M}=2,\quad c_m=\frac{2\pi}{\sqrt{c}}$$

for all m, and so $\tilde{M}$ has constant curvature c.

In the same way we strengthen another result of [15].

THEOREM 13.4. *Let M be an n-dimensional Riemannian manifold ($n\geqslant 2$) such that for all $m\in M$ and all sufficiently small $r>0$ the growth function $A(s)$ of each geodesic sphere $G_m(r)$ satisfies the differential equation*

$$A'''+c_2A''+c_1A'+c_0A=0,\tag{13.1}$$

where the c_i's are functions of s. Then the dimension of M must be 2 or 3 and M is a space of constant curvature.

Proof. Since $A(s)=S_m(r+s)$ we have by Theorem 3.3 that

$$A(s)=\omega\{n(r+s)^{n-1}+A(r+s)^{n+1}+B(r+s)^{n+3}+O((r+s)^{n+5})\},\tag{13.2}$$

where

$$A=-\frac{\tau(m)}{6}\quad\text{and}\quad B=\frac{1}{360(n+2)}(-3\|R\|^2+8\|\varrho\|^2+5\tau^2-18\Delta\tau)_m.$$

We differentiate (13.2) with respect to s, use (13.1) and set $s=0$. In this way we obtain a power series expansion in r which must be identically zero. Setting the coefficients of this power series equal to zero, we obtain certain relations. The first such relation implies that the dimension of M is 2 or 3. The next five conditions imply that M has constant curvature.

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