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RIEMANNIAN MANIFOLDS ADMITTING A CERTAIN CONFORMAL TRANSFORMATION GROUP

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1. Introduction

Several authors have studied compact Riemannian manifolds admitting a conformal non-Killing vector field. The main results are as follows.

Let M be a connected n-dimensional Riemannian manifold admitting a conformal non-Killing vector field.

(1) *If M is a complete Einstein space of dimension $n \geq 3$, then M is isometric to a sphere* (Nagano-Yano [8]).

(2) *If M is a complete Riemannian manifold of dimension $n \geq 3$ with parallel Ricci tensor, then M is isometric to a sphere* (Nagano [5]).

(3) *If M is compact and homogeneous, then M is isometric to a sphere provided $n > 3$* (Goldberg-Kobayashi [2]).

(4) *M can not be a compact Riemannian manifold with constant nonpositive scalar curvature* (Yano [7], Lichnerowicz [4]).

Recently S. Tanno and W. C. Weber [6] investigated compact connected Riemannian manifolds which have constant scalar curvature and admit a closed conformal vector field with certain conditions. The purpose of this paper is to prove the following theorems.

Theorem I. *If a compact connected Riemannian manifold M admits a closed conformal non-Killing vector field, then M is diffeomorphic to a generalized twisted torus or a sphere.*

Theorem 2. *If a compact Riemannian manifold M with finite fundamental group admits a closed conformal non-Killing vector field, then M is diffeomorphic to a sphere.*

Theorem 3. *If a compact connected Riemannian manifold M admits a closed conformal non-Killing vector field which vanishes at some point of M, then M is diffeomorphic to a sphere.*

Theorem 2 is an immediate consequence of Theorem 1, and Theorem 3 follows from the proof of Theorem 1.

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2. Preliminaries

Let M be a compact connected n-dimensional Riemannian manifold with metric g. A vector field X on M is *conformal* if and only if

$$L_X g = 2\lambda g , \tag{2.1}$$

where L_X denotes the Lie derivation with respect to X, and λ is a differentiable function on M which is called the characteristic function of X. If X is a conformal non-Killing vector field, then λ is a non-constant function. Since M is compact, X generates a global 1-parameter group of transformations φ_t of M. Then condition (2.1) is equivalent to

$$(\varphi_t^* g) = f_t \cdot g , \tag{2.2}$$

where

$$f_t(p) = \exp\left(2 \int_0^t \lambda(\varphi_u(p))du\right) , \qquad p \in M .$$

If we put $X = \sum_{i=1}^n \xi^i \partial/\partial x^i$ in a coordinate neighborhood of M with local coordinate $(x^1, \cdots, x^n)$, (2.1) is equivalent to

$$\xi_{i;j} + \xi_{j;i} = 2\lambda g_{ij} , \tag{2.3}$$

where g_{ij} are the components of g with respect to the coordinate system $(x^1, \cdots, x^n)$, $\xi_i = \sum_{j=1}^n g_{ij}\xi^j$, and ";" denotes the covariant derivative with respect to the coordinates system $(x^1, \cdots, x^n)$. From now on, we assume that X is closed, that is to say,

$$\xi_{i;j} = \xi_{j;i} . \tag{2.4}$$

By (2.3) and (2.4) we have

$$\xi_{i;j} = \lambda g_{ij} . \tag{2.5}$$

so that

$$\xi^i{}_{;j} = \lambda \delta^i{}_j , \tag{2.6}$$

where

$$\delta^i{}_j = \begin{cases} 1 & (i = j) , \\ 0 & (i \neq j) . \end{cases}$$

If we denote the divergence of X by div X, from (2.6) follows immediately

$$\operatorname{div} X = \sum_{i=1}^{n} \xi^i{}_{;i} = n\lambda .$$

Let $\bar{M}$ be an $(n-1)$-dimensional differentiable manifold, and φ be a diffeomorphism of $\bar{M}$, and consider $\bar{M} \times [0, a], a > 0$. If M is a differentiable manifold obtained by identifying $\bar{M} \times \{0\}$ and $\bar{M} \times \{a\}$ in $\bar{M} \times [0, a]$ by using the map φ, then we call it a generalized twisted torus.

Let N be a compact submanifold of M, and c be a geodesic starting from $p \in N$ such that c is perpendicular to N at p. If the point q on c is the last point such that the subarc $\overline{pq}$ of c is the shortest geodesic between q and N, then the point q is called *the cut point* of N along c.

3. Proof of Theorem I

Setting $M' \underset{\leftarrow}{=} \{p \in M \mid X_p \neq 0\}, M'$ is an open subset of M so that M' is an open submanifold of M. Then there exists a distribution D of dimension $n-1$ on M' such that for all $p \in M'$ we have

$$D_p \underset{\leftarrow}{=} \{Z \in M_p \mid g(Z, X) = 0\} .$$

Lemma 3.1. *The distribution D is differentiable involutive.*

Proof. Since $X_p \neq 0$ for all $p \in M'$ there exists a coordinate system $(x^1, \cdots, x^n)$ around p such that X coincides with the vector field $\partial/\partial x^1$ in this coordinate neighborhood W (cf. Chevalley [1]). Setting

$$Y_i = \partial/\partial x^i - \frac{g(\partial/\partial x^1, \partial/\partial x^i)}{\| \partial/\partial x^1 \|^2} \frac{\partial}{\partial x^1} \qquad \text{for } i = 2, \cdots, n ,$$

the set $Y_2, \cdots, Y_n$ is a local basis for the distribution D in W. Thus D is differentiable and also involutive. In fact, for any two vector fields Z, Z' belonging to D we have

$$g([Z, Z'], X) = g(\nabla_z Z', X) - g(\nabla_{z'} Z, X) . \tag{3.1}$$

By (2.6) we obtain

$$\begin{aligned} 0 &= Z \cdot g(Z', X) = g(\nabla_z Z', X) + g(Z', \nabla_z X) \\ &= g(\nabla_z Z', X) + \lambda g(Z', Z) , \end{aligned} \tag{3.2}$$

$$0 = Z' \cdot g(Z, X) = g(\nabla_{z'} Z, X) + \lambda g(Z', Z) , \tag{3.3}$$

from which and (3.1) follows immediately $g([Z, Z'], X) = 0$. So $[Z, Z']$ belongs to D, and D is involutive. q.e.d.

Hence there exists an integral manifold of D passing through each point of M'.

Lemma 3.2. *There exists a point p on M such that $\lambda(p) < 0$ and $X_p \neq 0$.*

Proof. Let $\tilde{M}$ be an oriented 2-fold covering manifold of M, and $\tilde{X}$ a lift of X by the covering map. Then $\tilde{X}$ is a conformal vector field on $\tilde{M}$. Let $\tilde{\lambda}$ be a characteristic function of $\tilde{X}$. Then we have $\operatorname{div} \tilde{X} = n\tilde{\lambda}$ and

$$0 = \frac{1}{n}\int_{\tilde{M}} \operatorname{div} \tilde{X} = \int_{\tilde{M}} \tilde{\lambda} . \tag{3.4}$$

Since $\tilde{\lambda}$ is a non-constant function on $\tilde{M}$, two sets $\{p \in \tilde{M} \mid \tilde{\lambda}(p) > 0\}$ and $\{p \in \tilde{M} \mid \tilde{\lambda}(p) < 0\}$ are non-empty, and therefore so is λ.

Now we assume that X vanishes on the open set $\mathcal{O}$. For any vector fields Y, Z on M we have

$$(L_X g)(Y, Z) = X \cdot g(Y, Z) - g([X, Y], Z) - g(Y, [X, Z]) = 0 \text{ on } \mathcal{O} . \tag{3.5}$$

On the other hand,

$$(L_X g)(Y, Z) = 2\lambda g(Y, Z) ,$$

which shows that λ vanishes on $\mathcal{O}$. Hence there exists a point p on M such that $\lambda(p) < 0$ and $X_p \neq 0$. q.e.d.

Let $U(p)$ be a neighborhood of p, where λ is negative and X never vanishes. Then

$$X \cdot g(X, X) = (L_X g)(X, X) = 2\lambda g(X, X) , \tag{3.6}$$

which implies that $g(X, X)$ decreases along the integral curve of X on $U(p)$.

Lemma 3.3. *There exists a coordinate neighborhood U with local coordinate system $(x^1, \cdots, x^n)$ such that*

(1) *U is contained in $U(p)$,*

(2) *$x^i(p) = 0, i = 1, \cdots, n$,*

(3) *$|x^1| < a, |x^i| < b \ (i \geq 2)$ on U,*

(4) *the slice of U defined by the equation $x^1 = \xi$, where $|\xi| < a$, is an integral manifold of D,*

(5) *if we put $V \rightleftarrows \{q \in U \mid x^1(q) = 0\}$, then the set $\varphi_t(V)$ coincides with the set $\{q \in U \mid x^1(q) = t\}$.*

Proof. By Lemma 3.1. and Frobenius theorem (Chevalley [1]) we have a coordinate neighborhood U with a local coordinate system $(y^1, \cdots, y^n)$ which satisfies the conditions (1)–(4). Since V is an integral manifold of D and φ_t is a conformal transformation for a fixed t, $\varphi_t(V)$ is also an integral manifold, and X never vanishes on $U(p)$. So we can change y^i into x^i $(i = 1, \cdots, n)$ such that $x^1(\varphi_t(p)) = t$. Thus we have a desired coordinate system. q.e.d.

The value of $g(X, X)$ is constant on any integral manifold of D. In fact, for any $Z \in D$ we have

$$Z \cdot g(X, X) = 2g(\nabla_Z X, X) = 2\lambda g(Z, X) = 0 . \tag{3.7}$$

Let N be a unique maximal integral manifold of D containing the point p. Then $\varphi_t(N) \cap N = \emptyset$ for all $t, 0 < |t| < a$. By Lemma 3.3 and the above remark, the value of $g(X, X)$ on U is constant on each slice and decreases as the parameter t increases. This shows that $\varphi_t(V) \cap N = \emptyset$ and therefore $\varphi_t(N) \cap N = \emptyset$, for all $t, 0 < |t| < a$.

Lemma 3.4. *The above maximal integral manifold N is an* $(n-1)$*-dimensional compact manifold.*

Proof. We shall show that the closure $\bar{N}$ of N in M coincides with N. Let x be a point contained in $\bar{N}$, and $\{x_n\}$ be the sequence contained in N such that x_n converges to x in M as n tends to ∞. Since the value of $g(X, X)$ is a nonzero constant on N, $g_x(X, X)$ is equal to this value, and so there exists a neighborhood U_x of x in which the vector field X never vanishes. Now we take a coordinate neighborhood U' of x contained in U_x whose local coordinate system $(x'^1, \cdots, x'^n)$ has the same properties as in Lemma 3,3. If x'^1 is so taken that $|x'^1| < a' \leq a$, then it is clear from the above remark of this lemma that in U' there exists at most one of those slices contained in N. If there does not exist such a slice, we can not take the sequence $\{x_n\} \subset N$ such that $x_n \to x$ as $n \to \infty$. Therefore the slice passing through x is contained in N, so that $x \in N$. Moreover this shows that N has no boundary. q.e.d.

If $N \cap \varphi_t(N) \neq \emptyset$ for some t, then $N = \varphi_t(N)$, because N and $\varphi_t(N)$ are integral manifolds of D. Now we define the mapping F: $t \to \varphi_t(N)$. This mapping F is locally one-to-one. In fact, we have $\varphi_t(N) \neq \varphi_{t'}(N)$ for $t \neq t'$, $-a < t - t' < a$. Now we can consider the following two cases.

(A) There exists $t \neq 0$ such that $N = \varphi_t(N)$.

(B) There does not exist $t \neq 0$ such that $N = \varphi_t(N)$.

Lemma 3.5. *In the case* (A), *M is diffeomorphic to a generalized twisted torus.*

Proof. Let t_0 be the minimum positive number such that $\varphi_{t_0}(N) = N$, and put

$$M'' \leftleftarrows \bigcup_{0 \leq t \leq t_0} \varphi_t(N) . \tag{3.8}$$

We shall show that M'' is an open and closed subset of M, so that $M = M''$. To this end we first show that M'' is open in M. For any point $q \in M''$, there exists s such that $0 \leq s \leq t_0$ and $q \in \varphi_s(N)$. We take a neighborhood V' of q in $\varphi_s(N)$ and a suitable positive number ε, so that the set $\bigcup_{-\varepsilon < t < \varepsilon} \varphi_t(V)$ is an open set of M which contains the point q.

Next we shall show that M'' is closed in M. For any point x of $\bar{M}''$, there exists a sequence $\{x_n\} \subset M''$ such that $x_n \to x$ as $n \to \infty$. Then we can write $x_n = \varphi_{t_n}(y_n)$, where $0 \leq t_n \leq t$ and $\{y_n\} \subset N$, and can choose the convergent subsequences of $\{y_n\}$ and $\{t_n\}$, so that we can assume that $y_n \to y, t_n \to s$ as $n \to \infty$, where $y \in N, 0 \leq s \leq t$. Now we estimate $d(x, \varphi_s(y))$, where d is the metric function on M:

$$
\begin{aligned}
d(x, \varphi_s(y)) &\leq d(x, \varphi_{t_n}(y_n)) + d(\varphi_{t_n}(y_n), \varphi_{t_n}(y)) + d(\varphi_{t_n}(y), \varphi_s(y)) \\
&\leq d(x, \varphi_{t_n}(y_n)) + \bar{d}_{t_n}(\varphi_{t_n}(y_n), \varphi_{t_n}(y)) + d(\varphi_{t_n}(y), \varphi_s(y)) \,,
\end{aligned} \tag{3.9}
$$

where $\bar{d}_{t_n}$ is the metric function on $\varphi_{t_n}(N)$. On the right hand side of (3.9), the first and third terms converge to 0 as $n \to \infty$. So we need only to estimate the second term. For any point $p \in N$,

$$
g_{\varphi t(p)}(X, X) = g_{\varphi t(p)}(\varphi_t X, \varphi_t X) = (\varphi_t{}^* g)_p(X, X) = f_t(p) \cdot g_p(X, X) \,. \tag{3.10}
$$

Since $g(X, X)$ is constant on $\varphi_t(N)$ for any $t, f_t(p)$ is independent of $p \in N$. $f_t(p), (p \in N)$, is a continuous function of t and satisfies $f_0(p) = 1, f_{t_0}(p) = 1$. So we have the maximum value C of $f_t(p)$ on $[0, t_0]$, and

$$
\bar{d}_{t_n}(\varphi_{t_n}(y_n), \varphi_{t_n}(y)) \leq C^{1/2} \bar{d}_0(y_n, y) \,. \tag{3.11}
$$

Since $\bar{d}_0(y_n, y) \to 0$ as $n \to \infty$, $\bar{d}_{t_n}(\varphi_{t_n}(y_n), \varphi_{t_n}(y)) \to 0$ as $n \to \infty$. This shows $d(x, \varphi_s(y)) = 0$, i.e., $x = \varphi_s(y)$. Therefore $\bar{M}'' = M''$, and hence M'' is closed in M.

Lemma 3.6. *In the case (B), M is homeomorphic to S^n.*

Proof. Since from (2.6) we have $\nabla_X X = \lambda X$, for any point $p \in N$ the curves τ and τ' defined by

$$
\tau \leftleftarrows \{\varphi_t(p) \mid t \in [0, \infty)\} \,, \tag{3.12}
$$

$$
\tau' \leftleftarrows \{\varphi_t(p) \mid t \in (-\infty, 0]\} \tag{3.13}
$$

are geodesics, and therefore their lengths $L(\tau)$ and $L(\tau')$ are independent of $p \in N$, due to the fact that $g(X, X)(\varphi_t(p))$ is independent of p for fixed t. Now we divide our discussion into the following four cases:

(a) $L(\tau) = \infty$ and $L(\tau') = \infty$.

(b) $L(\tau) = \infty$ and $L(\tau') < \infty$.

(c) $L(\tau) < \infty$ and $L(\tau') = \infty$.

(d) $L(\tau) < \infty$ and $L(\tau') < \infty$.

Case (a). Let c be the curve defined by $c = \{c(t) \mid c(t) \leftleftarrows \varphi_t(p), 0 \leq t < \infty, p \in N\}$. Since M and N are compact and c is perpendicular to N at p, we have the cut point $c(t_0)$ of N along c. If $t_1 > t_0$, then the shortest geodesic c' between $c(t_1)$ and N is different from the subarc $c \mid [0, t_1]$ of c, and the image of c' is integral curve of X because c' is perpendicular to N by construction. Hence the composite of $c \mid [0, t_1]$ and c' is an extension of $c \mid [0, t_1]$. This contradicts to our assumption (B), so Case (a) never happens.

Case (b). We first show $\varphi_t(N)$ converges to one point x as $t \to -\infty$. For any point $y \in N$, $\varphi_t(y)$ converges to a point y' as $n \to -\infty$. This implies $X_{y'} = 0$. Using the same argument as in (3.10), we have $f_t(p) \to 0$ as $t \to -\infty$. For any two points $y, z \in N$, let $x(s), 0 \leq s \leq 1$, be a curve in N joining y to z. Then for any fixed t, $\varphi_t(x(s)), 0 \leq s \leq 1$, is the curve in $\varphi_t(N)$ joining $\varphi_t(y)$ to $\varphi_t(z)$. Now we estimate the length of this curve in $\varphi_t(N)$.

$$\begin{aligned}(3.14)\qquad \int_0^1 g(\varphi_t\dot{x}(s), \varphi_t\dot{x}(s))^{1/2}ds &= \int_0^1 (\varphi_t{}^*g)(\dot{x}(s), \dot{x}(s))^{1/2}ds \\ &= \int_0^1 (f_t(p))^{1/2}(g(\dot{x}(s), \dot{x}(s)^{1/2}ds = (f_t(p))^{1/2}\int_0^1 g(\dot{x}(s), \dot{x}(s))^{1/2}ds \,.\end{aligned}$$

This shows $\int_0^1 g(\varphi_t\dot{x}(s), \varphi_t\dot{x}(s))^{1/2}ds \to 0$ as $t \to -\infty$, i.e., $d(y', z') = 0$, where $y' = \lim_{t\to-\infty} \varphi_t(y)$, $z' = \lim_{t\to-\infty} \varphi_t(z)$, and $\dot{x}(s)$ is the tangent vector at $x(s)$.

For any $s < 0$, the curve $\tau'(s) \underset{\leftarrow}{=} \{\varphi_t(p) \mid t \in [s, 0]\}$ is the shortest geodesic between $\varphi_s(p)$ and N. In fact, if the curve $\tau'(s)$ contains the cut point of N in its inner point, then we have a shortest geodesic τ_1' between $\varphi_s(p)$ and N, which is different from $\tau'(s)$. Since τ_1' is perpendicular to N, we can denote τ_1' by $\tau_1' = \{\varphi_t(q) \mid s < s' \leq t \leq 0\}$ or $\tau_1' = \{\varphi_t(q) \mid 0 \leq t \leq c_1\}$ for some $q \in N$. But we can easily show that these two cases do not happen. Hence $\tau'(s)$ is the shortest geodesic between $\varphi_s(p)$ and N.

For any $y \in N$, put $\tau'[y] \underset{\leftarrow}{=} \{\varphi_t(y) \mid -\infty < t \leq 0\}$. Then it has already been shown that $L(\tau'[y])$ is independent of $y \in N$ and $\bar{\tau}'[y] \underset{\leftarrow}{=} \tau'[y] \cup \{x\}$ is the shortest geodesic between x and N. This shows that for any $t \in (-\infty, 0]$, $\varphi_t(N)$ is a connected submanifold of $S_x(l) = \{z \in M \mid d(x, z) = l\}$, where $l = d(x, \varphi_t(p))$, $p \in N$. Since from its construction $\varphi_t(N)$ is an open and closed subset of $S_x(l)$, we have $S_x(l) = \varphi_t(N)$. For any $t \in \boldsymbol{R}$, put $\tau''(t) = \{\varphi_s(p) \mid -\infty < s \leq t\}$. Then it has already been shown that $\bar{\tau}''(t) = \tau''(t) \cup \{x\}$ is a geodesic joining x to $\varphi_t(p)$. By the same argument as above, $\bar{\tau}''(t)$ does not contain the cut point of x along $\bar{\tau}''(t)$. Since by the assumption $L(\bar{\tau}''(t)) \to \infty$, Case (b) never happens.

Case (c). This case can not happen in the same way as in Case (b).

Case (d). As we showed in Case (b), $\varphi_t(N)$ and $\varphi_{-t}(N)$ converge to x and x' respectively as $t \to +\infty$. For any $y \in N$, put $\tau'' \underset{\leftarrow}{=} \{\varphi_t(y) \mid -\infty < t < \infty\}$. Then $\bar{\tau}''$ is a shortest geodesic joining x to x', and $L(\bar{\tau}'')$ is independent of $y \in N$. As we showed in Case (b), $\varphi_t(N) = S_x(l) \underset{\leftarrow}{=} \{z \in M \mid d(x, z) = l, l = d(x, \varphi_t(p)), p \in N\}$.

Put $d(x, x') = r$. Let M_x be the tangent space of M at x, S^n an n-dimensional sphere of r/π in $\boldsymbol{R}^{n+1}$, and $\bar{x}'$ the antipodal point of $\bar{x} \in S^n$. Then construct the mapping $f\colon M \to S^n$ by

$$f \underset{\leftarrow}{=} \exp_{\bar{x}} \circ \iota \circ (\exp_x)^{-1} \qquad \text{on } M - \{x'\} \,,$$
$$f(x') = \bar{x}' \,,$$

where $\exp_x$ (resp. $\exp_{\bar{x}}$) is the exponential mapping at x (resp. $\bar{x}$) whose domain of definition is the open ball in M_x (resp. $S_{\bar{x}}$) of radius r/π and with the origin as its center, and $\iota\colon M_x \to S_{\bar{x}}$ is an isometric isomorphism. Then f is a homeomorphism of M onto S^n.

Lemma 3.7. *In the case (B), M is diffeomorphic to S^n.*

Proof. For any two points $y, z \in N$, put

$$\gamma \leftrightarrows \{\varphi_t(y) \mid -\infty < t < \infty\}\ , \qquad \bar{\gamma} \leftrightarrows \gamma \cup \{x\} \cup \{x'\}\ ,$$
$$\delta \leftrightarrows \{\varphi_t(z) \mid -\infty < t < \infty\}\ , \qquad \bar{\delta} \leftrightarrows \delta \cup \{x\} \cup \{x'\}\ .$$

Then the images of $\bar{\gamma}$ and $\bar{\delta}$ are two shortest geodesics joining x to x'. Let α (resp. α') be the angle between these two curves at x (resp. x'). Then we have

$$\alpha = \lim_{t\to\infty} \frac{\bar{d}_t(\varphi_t(y), \varphi_t(z))}{d(x, \varphi_t(y))}\ , \qquad \alpha = \lim_{t\to-\infty} \frac{\bar{d}_t(\varphi_t(y), \varphi_t(z))}{d(x', \varphi_t(y))}\ ,$$

where $\bar{d}_t(\varphi_t(y), \varphi_t(z))$ is the distance between $\varphi_t(y)$ and $\varphi_t(z)$ on $\varphi_t(N)$, which is the same set as $S_x(l) = \{w \in M \mid d(x, w) = l\}$ and $S_{x'}(l') = \{w \in M \mid d(x', w) = l'\}$, where $l = d(x, \varphi_t(p))$ and $l' = d(x', \varphi_t(p)), p \in N$. The proof of this is parallel to that of the lemma in Kobayashi-Nomizu [3, p. 170].

We have

$$\begin{aligned} \bar{d}_t(\varphi_t(y), \varphi_t(z)) &= f_t(y)^{1/2}\bar{d}_0(y, z)\ , \\ d(x', \varphi_t(y)) &= \int_{-\infty}^{t} g_{\varphi_u(y)}(X, X)^{1/2} du = \int_{-\infty}^{t} g_{\varphi_u(y)}(\varphi_u X, \varphi_u X)^{1/2} du \\ &= \int_{-\infty}^{t} (\varphi_u^* g)_y(X, X)^{1/2} du = g_y(X, X)^{1/2} \int_{-\infty}^{t} f_u(y)^{1/2} du\ , \end{aligned} \tag{3.15}$$

and therefore

$$\begin{aligned} \alpha' &= \lim_{t\to-\infty} \frac{f_t(y)^{1/2}\bar{d}_0(y, z)}{\left(\int_{-\infty}^{t} f_u(y)^{1/2} du\right) g_y(X, X)^{1/2}} \\ &= \lim_{t\to+\infty} \frac{f_{-t}(y)^{1/2}\bar{d}_0(y, z)}{\left(\int_{t}^{\infty} f_{-u}(y)^{1/2} du\right) \cdot g_y(X, X)^{1/2}}\ . \end{aligned}$$

Similarly,

$$\alpha = \lim_{t\to+\infty} \frac{f_t(y)^{1/2}\bar{d}_0(y, z)}{\left(\int_{t}^{\infty} f_u(y)^{1/2} du\right) \cdot g_y(X, X)^{1/2}}\ .$$

In order to prove $\alpha = \alpha'$, we estimate the ratio α'/α:

$$(3.16)\quad \frac{\alpha'}{\alpha} = \lim_{t\to\infty} \frac{f_{-t}(y)^{1/2}\cdot \bar{d}_0(y,z)}{\left(\int_t^\infty f_{-u}(y)^{1/2}du\right)\cdot g_y(X,X)} \cdot \frac{\left(\int_t^\infty f_u(y)^{1/2}du\right)\cdot g_y(X,X)^{1/2}}{f_t(y)^{1/2}\bar{d}_0(y,z)} ,$$

where

$$(3.17)\quad \begin{aligned} \lim_{t\to\infty}\frac{f_{-t}(y)^{1/2}}{f_t(y)^{1/2}} &= \lim_{t\to\infty}\frac{\exp\int_0^{-t}\lambda(\varphi_u(y))du}{\exp\int_0^t\lambda(\varphi_u(y))du} \\ &= \lim_{t\to\infty}\frac{\exp\left(-\int_t^0\lambda(\varphi_u(y))du\right)}{\exp\int_0^t\lambda(\varphi_u(y))du} = \lim_{t\to\infty}\frac{1}{\exp\int_{-t}^t\lambda(\varphi_u(y))du} . \end{aligned}$$

Since M is homeomorphic to S^n, M is orientable and N is also orientable by the construction, so that

$$(3.18)\quad \begin{aligned} 0 &= \int_M \lambda(x)dv = \int_N dv_1\int_{-\infty}^\infty\left\{\lambda(\varphi_u(x))\exp\left(u\int_0^u\lambda(\varphi_t(x))dt\right)\right\}du \\ &= \int_N\left[\frac{1}{u}\left(\exp\left(u\int_0^\infty\lambda(\varphi_t(x))dt - \exp\left(u\int_0^{-\infty}\lambda(\varphi_t(x))dt\right)\right)\right]dv_1 , \end{aligned}$$

where dv and dv_1 are volume elements on M and N respectively. Since the integrand of the right hand side of (3.18) is independent of x, we have

$$\exp\left(\int_0^\infty\lambda(\varphi_t(x))dt\right) = \exp\left(\int_0^\infty\lambda(\varphi_t(x))dt\right) .$$

Hence we have

$$(3.19)\quad \lim_{t\to\infty}\frac{f_{-t}(x)^{1/2}}{f_t(x)^{1/2}} = 1 .$$

Since the values of $d(x', \varphi_t(y))$ and $d(x, \varphi_t(y))$ are bounded, we obtain, in consequence (3.15),

$$(3.20)\quad \lim_{t\to\infty}\int_t^\infty f_u(y)^{1/2}du = 0 , \qquad \lim_{t\to\infty}\int_t^\infty e_{-u}(y)^{1/2}du = 0 ,$$

which together with (3.19) and l'Hospital's theorem implies

$$(3.21)\qquad \lim_{t\to\infty}\frac{\displaystyle\int_t^\infty f_u(y)^{1/2}du}{\displaystyle\int_t^\infty f_{-u}(y)^{1/2}du}=\lim_{t\to\infty}\frac{f_t(y)^{1/2}}{f_{-t}(y)^{1/2}}=1\ .$$

Hence by (3.19) and (3.21) we have

$$(3.22)\qquad \alpha=\alpha'\ .$$

Now we construct a diffeomorphism of M onto S^n. We put $d(x,x')=r$. Let M_x be the tangent space of M at x, S^n be an n-dimensional sphere of radius r/π in R^{n+1}, $\bar{x}'$ be the antipodal point $\bar{x}\in S^n$, $e_1,\cdots,e_n$ be an orthonormal basis for M_x, and e_i' $(i=1,\cdots,n)$ be the tangent vector at x', obtained by parallelly displacing e_i along the geodesic $\exp_x te_i$, $0\le t\le r$. By (3.22), $e_1',\cdots,e_n'$ is also an orthonormal basis for $M_{x'}$. Now we choose an orthonormal basis $\bar{e}_1,\cdots,\bar{e}_n$ for $S^n_{\bar{x}}$. Let $\bar{e}_i'$ $(i=1,2,\cdots,n)$ be the tangent vector at $\bar{x}'$, obtained by parallelly displacing $\bar{e}_i$ along the geodesic $\exp_{\bar{x}} t\bar{e}_i$, $0\le t\le r$. Then $\bar{e}_1',\cdots,\bar{e}_n'$ is also an orthonormal basis for $S^n_{\bar{x}}$. Let ι be the isometric isomorphism of M_x onto $S^n_{\bar{x}}$ such that $\iota(e_i)=\bar{e}_i$, $i=1,\cdots,n$, and ι' be the isometric isomorphism of $M_{\bar{x}}$ onto $S^n_{\bar{x}'}$ such that $\iota'(e_i')=\bar{e}_i'$, $i=1,\cdots,n$. Now define two mapping $f,f'\colon M\to S^n$ by:

$$\begin{aligned}
f &\;\overline{\overleftarrow{=}}\; \exp_{\bar{x}}\circ\iota\circ(\exp_x)^{-1} &&\text{on } M-\{x'\}\ ,\\
f(x') &= \bar{x}'\ ,\\
f' &\;\overline{\overleftarrow{=}}\; \exp_{\bar{x}'}\circ\iota'\circ(\exp_{x'})^{-1} &&\text{on } M-\{x\}\ ,\\
f'(x) &= \bar{x}\ .
\end{aligned}$$

By the construction, f is a diffeomorphism of $M-\{x'\}$ onto $S^n-\{\bar{x}'\}$, f' is a diffeomorphism of $M-\{x\}$ onto $S^n-\{\bar{x}\}$, and $f=f'$. Hence f is a diffeomorphism of M onto S^n.

3. Examples

In this section we give two examples of compact Riemannian minifolds admitting a closed conformal non-Killing vector field.

Example 1. In the (x,y)-plane, consider a curve $y=\sin x+a$, $0\le x\le 2\pi$, $a>1$. If we place this curve in the (x,y,z)-space and revolve it about the x-axis, then we obtain a smooth closed surface M' with boundary, on which we induce the natural Riemannian metric:

$$ds^2=dr^2+(\sin x(r)+a)^2d\theta^2\ ,$$

where we put

$$r = \int_0^x \sqrt{1 + \cos^2 t}\, dt \ .$$

Now we obtain a compact Riemannian manifold M by identifying a boundary, with two components, of M' by an isometry of two circles. Then M is diffeomorphic to a torus or a Klein's bottle, and $X = (\sin x(r) + a)\cdot\partial/\partial r$ is a closed conformal non-Killing vector field on M because it satisfies

$$L_X g = 2 \cos x(r) \frac{dx}{dr} g \ .$$

Example 2. In the (x, y)-plane, consider a smooth curve $y = f(x)$, $0 \leq x \leq l$, such that $f(0) = f(l) = 0$, $f(x) > 0$ on $(0, l)$ and $(dx/dy)_{x=0} = (dx/dy)_{x=l} = 0$. If we place this curve in the (x, y, z)-space and revolve it about the x-axis, then we obtain a smooth closed surface M on which we induce the natural Riemannian metric:

$$ds^2 = dr^2 + f(x(r))^2 d\theta^2 \ ,$$

where we put

$$r = \int_0^x \sqrt{1 + f'(t)^2}\, dt \ .$$

Thus M is diffeomorphic to a sphere S^2. If we set $f(x) = \sqrt{1 - \frac{4}{l^2}\left(x - \frac{l}{2}\right)^2}$, $X = f(x(r))\partial/\partial r$, then X is a closed conformal non-Killing vector field on M, because it satisfies

$$L_X g = 2 \frac{df}{dx} \frac{dx}{dr} g \ .$$

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