Riemannian Manifolds with Structure Group G_2 (*).

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Summary. – *Riemannian manifolds with structure group G_2 are 7-dimensional and have a distinguished 3-form. In this paper such manifolds are treated as analogues of almost Hermitian manifolds. Thus S^7 has structure group G_2 just as S^6 is an almost Hermitian manifold. We study the covariant derivative of the fundamental 3-form as was done in* [GH] *for almost Hermitian manifolds.*

1. – Introduction.

The exceptional Lie group G_2 is one of the possible candidates for the holonomy group of an irreducible Riemannian manifold [BE]. Such a Riemannian manifold M must be 7-dimensional and have zero Ricci curvature [BO]. There is a representation of G_2 on each tangent space of M defined by means of a 2-fold vector cross product [CA], [GR 1-5], which is parallel. This vector cross product can be considered as a natural generalization of an almost complex structure [BG], [E 2], [GR 4], [GR 5]. Corresponding to the Kähler form, one has a fundamental 3-form φ which is parallel; thus if M is compact $H^3(M, \mathbf{R}) \neq 0$.

One approach to the study of a Riemannian manifold M whose holonomy group is contained in G_2 is to generalize the theory of Kähler manifolds. For example, one has a theory of harmonic forms on M which is a special case of a much more general treatment by CHERN [CH]. However, the authors have so far been unable to find examples of Riemannian manifolds whose holonomy group is actually equal to G_2.

The analogy between Kähler manifolds and Riemannian manifolds whose holonomy group is a subgroup of G_2 suggests the study of analogs of complex and symplectic manifolds, provided such analogs exist. More precisely, the situation is this: Consider the class $\mathcal{W}$ of 7-dimensional Riemannian manifolds M for which the structure group of the bundle of orthonormal frames can be reduced from $0(7)$ to G_2: This reduction can be described geometrically by saying that M has a 2-fold vector cross product P. The class $\mathcal{W}$ contains all parallelizable 7-dimensional manifolds and is analogous to the class of all almost Hermitian manifolds. Within the class $\mathcal{W}$ one can search for analogs of the classes of Hermitian and almost Kähler manifolds as well as analogs of other special types of almost Hermitian manifolds.

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This search, which we do in a systematic way using the method of [GH], is the principal subject of this paper. The idea is to study the representation of G_2 on the space W of tensors having the same symmetries as the covariant derivative $\nabla\varphi$ of the fundamental 3-form φ, and to decompose this representation into irreducible components. Corresponding to each invariant subspace of W there is a subclass of $\mathfrak{W}$.

In fact we shall show that the representation of G_2 on W has four irreducible components:

$$W = W_1 \oplus W_2 \oplus W_3 \oplus W_4 .$$

Thus, there are a total of 16 invariant subspaces of W, and hence 16 subclasses of $\mathfrak{W}$. Corresponding to $\{0\}$ are the manifolds with parallel vector cross products. The manifolds with $d\varphi = 0$ will correspond to W_2, and manifolds with locally conformally parallel vector cross products will correspond to W_4. On the sphere S^7 there is a naturally defined vector cross product, which will correspond to W_1 and is analogous to the canonical almost complex structure on S^6. Similar interpretations can be given to each of the 16 invariant subspaces of W. We explain all of this in section 5.

It should be remarked that there is one fundamental difference between 2-fold vector cross products and almost complex structures. Almost complex structures are defined without reference to a metric (although if a metric exists, a compatibility condition is required). In contrast to this, a 2-fold vector cross product has a unique (positivie definite) metric associated with it. We make this precise in section 2.

We discuss the algebra of a 2-fold vector cross product P in section 2, and extend the definition of P so that P operates on k-vectors and k-forms. Representations of the Lie group G_2 and the relevance of vector cross products to the study of these representations are studied in section 3.

The space W is defined in section 4, and the decomposition $W = W_1 \oplus W_2 \oplus \oplus W_3 \oplus W_4$ is established. It is also shown that the representation of G_2 on each W_i is irreducible. We define the 16 classes in section 5. Conformal relations between the 16 classes are studied in section 6. Each orientable hypersurface of $\boldsymbol{R}^8$ has a 2-fold vector cross product [GR 3]. These are studied in section 7. We define in section 8 the notion of complex vector cross product, and using it, we find nontrivial examples of manifolds in the class $\mathfrak{W}_2$. Finally we discuss the inclusion relations between the various classes in section 9.

2. – The algebra of 2-fold vector cross products.

A general definition of the notion of vector cross product has been given by ECKMANN [E 1]. See also [BG], [CA], [GR 1-5], [E 2], [YS], [Z]. There are four kinds. We shall be concerned exclusively with vector cross products of type (iii), the 2-fold vector cross products. Explicitly, we have

DEFINITION. – Let V be a finite dimensional vector space over $\boldsymbol{R}$ with a (positive definite) inner product $\langle\, ,\rangle$. A *2-fold vector cross product* on V is a bilinear map $P\colon V\times V\to V$ satisfying the axioms

$$\langle P(x, y), x\rangle = \langle P(x, y), y\rangle = 0\,, \tag{2.1}$$

$$\|P(x, y)\|^2 = \|x\|^2\|y\|^2 - \langle x, y\rangle^2\,, \tag{2.2}$$

for $x, y \in V$.

Let $\Lambda^k(V)$ denote the k-th Grassmann space over V (i.e., the space generated by the skew-symmetric products $v_1\wedge\ldots\wedge v_k$). It follows from (2.1) that $P(x, y) = -P(y, x)$. Thus we may extend P to a linear mapping $P\colon \Lambda^2(V)\to V$. For this reason we shall usually write $P(x\wedge y)$ instead of $P(x, y)$. Furthermore let the inner product $\langle\, ,\rangle$ be extended to $\Lambda^k(V)$ by the formula

$$\langle v_1\wedge\ldots\wedge v_k, w_1\wedge\ldots\wedge w_k\rangle = \det(\langle v_i, w_j\rangle)$$

for $v_1, \ldots, v_k, w_1, \ldots, w_k \in V$. Then (2.2) becomes

$$\|P(x\wedge y)\|^2 = \|x\wedge y\|^2\,. \tag{2.3}$$

Note that (2.3) does not say that P is an isometry, but only an isometry on decomposable vectors.

DEFINITION. – The fundamental 3-form φ of the 2-fold vector cross product P is given by

$$\varphi(x\wedge y\wedge z) = \langle P(x\wedge y), z\rangle$$

for $x, y, z\in V$. (From (2.1) it follows that φ is skew-symmetric.)

In [E 1], [BG] (for example) it is shown that if $(V, \langle\, ,\rangle)$ has a 2-fold vector cross product then necessarily $\dim V = 3$ or 7. When $\dim V = 3$ the vector cross product P is the « classical » vector cross product known to students of engineering. In this case a vector cross product determines a volume element (namely the fundamental 3-form φ) and vice versa. Thus, there is no advantage to studying vector cross products as such in this case. Therefore, we shall henceforth assume that $\dim V = 7$.

A simple explicit construction for P can be given via the Cayley numbers *Cay*. We view *Cay* as an 8-dimensional division algebra over $\boldsymbol{R}$ with identity 1 and orthonormal basis $\{1, e_0, \ldots, e_6\}$. The multiplication in *Cay* is then determined by

$$e_i^2 = -1\,, \quad e_ie_j = -e_je_i\,(i\neq j)\,, \quad e_ie_{i+1} = e_{i+3}\,,$$
$$e_{i+3}e_i = e_{i+1}\,, \quad e_{i+1}e_{i+3} = e_i\,(i\in Z_7)\,.$$

(This definition seems to be due to CARTAN [CN]. Many recent books, e.g. [BS], [FR], [GO], [J] have insisted on the multiplication in *Cay* by a complicated asymmetric table.) Let $V = \{1\}^\perp$ be the 7-dimensional space of pure imaginary Cayley numbers. On V the 2-fold vector cross product is then given by

$$P(x\wedge y) = xy + \langle x, y\rangle 1 \tag{2.4}$$

for $x, y \in V$. Here xy denotes the Cayley product of x and y.

DEFINITION. – A *Cayley basis* for V is an orthonormal basis $\{e_0, \dots, e_6\}$ such that $P(e_i, e_{i+1}) = e_{i+3}$ for $i \in Z_7$.

Some of our formulas are simpler when written using a Cayley basis. However, in general, we prefer to write the formulas using an arbitrary orthonormal basis $\{e_0, \dots, e_6\}$ of V.

First we write down without proof some elementary consequences of (2.1)-(2.3).

LEMMA 2.1. – For $x, y, z \in V$ we have

$$\langle P(x\wedge y), P(x\wedge z)\rangle = \langle x\wedge y, x\wedge z\rangle , \tag{2.5}$$

$$P(x\wedge P(x\wedge y)) = -\|x\|^2 y + \langle x, y\rangle x , \tag{2.6}$$

$$P(x\wedge P(y\wedge z)) + P(y\wedge P(x\wedge z)) = -2\langle x, y\rangle z + \langle x, z\rangle y + \langle y, z\rangle x . \tag{2.7}$$

COROLLARY 2.2. – The metric $\langle\, ,\rangle$ is determined by the vector cross product P.

PROOF. – From (2.6) we have

$$P\big(x\wedge P(x\wedge P(x\wedge y))\big) = -\|x\|^2 P(x\wedge y) . \tag{2.8}$$

Choose x and y linearly independent. Then $P(x\wedge y) \neq 0$. From (2.8) we can determine $\|x\|^2$ from P. By a standard polarization argument we get $\langle\, ,\rangle$.

As explained in [BG] a 1-fold vector cross product is nothing other than an almost complex structure. However, an almost complex structure does not determine a particular metric. (For example, complex manifolds can be defined without reference to a metric.) Thus because of corollary 2.2, 2-fold vector cross products are fundamentally different from almost complex structures.

We now introduce a mapping that will turn out to be the adjoint of P.

DEFINITION. – The linear mapping $p\colon V \to \Lambda^2(V)$ is given by

$$p(x) = -\tfrac{1}{2}\sum_{i=0}^{6} e_i\wedge P(e_i\wedge x) \tag{2.9}$$

for $x \in V$, where $\{e_0, \dots, e_6\}$ is any orthonormal basis of V.

LEMMA 2.3. – The mapping p has the following properties:

(i) p is the adjoint of P; that is for $x \in V$, $\xi \in \Lambda^2(V)$;

$$\langle p(x), \xi\rangle = \langle x, P(\xi)\rangle ; \tag{2.10}$$

(ii) we have for $x \in V$

$$P(p(x)) = 3x ; \tag{2.11}$$

(iii) if $\{e_0, \ldots, e_6\}$ is a Cayley basis of V, then

$$p(e_i) = e_{i+1}\wedge e_{i+3} + e_{i+2}\wedge e_{i+6} + e_{i+4}\wedge e_{i+5} \qquad (i \in Z_7) . \tag{2.12}$$

PROOF. – (2.10) follows from (2.1), and (2.9). Also (2.11) is a consequence of (2.6) and (2.9). Finally, (2.12) is an easy calculation using (2.9) and the definition of Cayley basis.

Next we extend P and p to linear maps

$$P\colon \Lambda^{k+1}(V) \to \Lambda^k(V) \quad \text{and} \quad p\colon \Lambda^k(V) \to \Lambda^{k+1}(V) .$$

DEFINITION. – Let $v_1, \ldots, v_{k+1} \in V$. Then

$$P(v_1\wedge\ldots\wedge v_{k+1}) = \sum_{1\leqslant i<j\leqslant k+1} (-1)^{i+j+1} P(v_i\wedge v_j)\wedge v_1\wedge\ldots\wedge\hat{v}_i\wedge\ldots\wedge\hat{v}_j\wedge\ldots\wedge v_{k+1} ,$$

$$p(v_1\wedge\ldots\wedge v_k) = \sum_{i=1}^{k} (-1)^{i+1} p(v_i)\wedge v_1\wedge\ldots\wedge\hat{v}_i\wedge\ldots\wedge v_k .$$

We extend P and p linearly so that they become maps $P\colon \Lambda^{k+1}(V) \to \Lambda^k(V)$ and $p\colon \Lambda^k(V) \to \Lambda^{k+1}(V)$.

In particular we have

$$p(x\wedge y) = p(x)\wedge y - p(y)\wedge x ,$$

$$P(x\wedge y\wedge z) = \underset{xyz}{\mathfrak{S}}\, P(x\wedge y)\wedge z ,$$

for $x, y, z \in V$, where $\mathfrak{S}$ denotes the cyclic sum. Furthermore, we can compute the powers of p. In addition to (2.12), there are the following formulas:

$$p^2(e_i) = 3\{e_{i+1}\wedge e_{i+2}\wedge e_{i+5} - e_{i+1}\wedge e_{i+4}\wedge e_{i+6} - e_{i+2}\wedge e_{i+3}\wedge e_{i+4} + e_{i+3}\wedge e_{i+5}\wedge e_{i+6}\} ,$$

$$p^3(e_i) = 9e_i\wedge\{e_{i+1}\wedge e_{i+2}\wedge e_{i+4} + e_{i+2}\wedge e_{i+3}\wedge e_{i+5} + e_{i+3}\wedge e_{i+4}\wedge e_{i+6} + e_{i+5}\wedge e_{i+6}\wedge e_{i+1}\} ,$$

$$p^4(e_i) = 36e_i\wedge\{- e_{i+1}\wedge e_{i+2}\wedge e_{i+3}\wedge e_{i+6} + e_{i+1}\wedge e_{i+3}\wedge e_{i+4}\wedge e_{i+5} + e_{i+2}\wedge e_{i+4}\wedge e_{i+5}\wedge e_{i+6}\} ,$$

$$p^5(e_i) = -108e_{i+1}\wedge e_{i+2}\wedge e_{i+3}\wedge e_{i+4}\wedge e_{i+5}\wedge e_{i+6} ,$$

$$p^6(e_i) = 0 .$$

Let $\Lambda(V) = \bigoplus_{k=0}^{7} \Lambda^k(V)$. Recall that an antiderivation of $\Lambda(V)$ is a linear map $L: \Lambda(V) \to \Lambda(V)$ such that

$$L(\xi \wedge \eta) = L(\xi) \wedge \eta + (-1)^k \xi \wedge L(\eta)$$

for $\xi \in \Lambda^k(V)$ and $\eta \in \Lambda(V)$. Extend p to a map $p: \Lambda(V) \to \Lambda(V)$. From the definition of p we have:

LEMMA 2.4. – The map $p: \Lambda(V) \to \Lambda(V)$ is an antiderivation of $\Lambda(V)$.

Next let $*: \Lambda(V) \to \Lambda(V)$ be the Hodge star operator. Thus $*\big(\Lambda^k(V)\big) = \Lambda^{7-k}(V)$.

LEMMA 2.5. – $p = (-1)^{k+1} * P *: \Lambda^k(V) \to \Lambda^{k+1}(V)$.

PROOF. – This can be checked by choosing a Cayley basis for V and computing for each k the maps $* P *$ and p.

We also note

LEMMA 2.6. – For $x, y, z \in V$ we have

$$P^2(x \wedge y \wedge z) = \underset{xyz}{\mathfrak{S}} P\big(P(x \wedge y) \wedge z\big) = 3P\big(P(x \wedge y) \wedge z\big) - 3\langle x, z\rangle y + 3\langle y, z\rangle x\,, \tag{2.13}$$

$$\|P^2(x \wedge y \wedge z)\|^2 = 9\{\|x \wedge y \wedge z\|^2 - \varphi(x \wedge y \wedge z)^2\}\,. \tag{2.14}$$

PROOF. – These equations follow easily from the definitions and equations (2.3), (2.7).

We shall also need a formula for $* \varphi$. (Here the 4-form $* \varphi$ can be defined by $(* \varphi)(w \wedge x \wedge y \wedge z) = \varphi\big(*(w \wedge x \wedge y \wedge z)\big)$; this is equivalent to the usual definition.)

LEMMA 2.7. – Let $w, x, y, z \in V$. Then

$$(* \varphi)(w \wedge x \wedge y \wedge z) = \tfrac{1}{3}\langle w, \underset{xyz}{\mathfrak{S}} P\big(P(x \wedge y) \wedge z\big)\rangle = \langle w, P\big(P(x \wedge y) \wedge z\big)\rangle + \langle w \wedge z, x \wedge y\rangle\,. \tag{2.15}$$

PROOF. – Let ψ be the 4-form on V given by

$$\psi(w \wedge x \wedge y \wedge z) = \langle w, \underset{xyz}{\mathfrak{S}} P\big(P(x \wedge y) \wedge z\big)\rangle\,.$$

Then for a Cayley basis $\{e_0, \dots, e_6\}$ it is easy to verify that

$$\psi(e_i \wedge e_1 \wedge e_2 \wedge e_3) = -3\delta_{i6} = 3(* \varphi)(e_i \wedge e_1 \wedge e_2 \wedge e_3)\,. \tag{2.16}$$

Since (2.16) holds for any Cayley basis we get $\psi = 3 * \varphi$. This is the first part of equation (2.15). The second part is a consequence of (2.13).

3. – Some low dimensional representations of G_2.

The purpose of this section is to describe in a concrete way some representations of G_2 that we shall need in subsequent sections. Using Weyl's formula for the degrees of irreducible representations of simple Lie groups (see for example [SA, p. 130]), it can be verified that the degrees of the first five irreducible representations of G_2 are 1, 7, 14, 27, and 64. We shall need the first four of these. For convenience, all of the representations that we consider will be covariant.

The 7-dimensional representation of G_2 is best described by means of the vector cross product P:

$$G_2 = \{g \in 0(7) | P(gx \wedge gy) = gP(x \wedge y) \text{ for all } x, y \in V\} .$$

In fact, one purpose of the theory of vector cross products is to have this simple description of the 7-dimensional representation of G_2. The 14-dimensional representation of G_2 is the adjoint representation.

It will be convenient to consider covariant versions of P and p (which we shall denote by the same letters). Let V^* denote the dual space of V.

DEFINITIONS. – $P: \Lambda^k(V^*) \to \Lambda^{k+1}(V^*)$ and $p: \Lambda^k(V^*) \to \Lambda^{k-1}(V^*)$ are given by $P(\alpha) = \alpha \circ P$ and $p(\alpha) = \alpha \circ p$. For clarity we sometimes write P_k and p_k.

Thus we obtain the following (non-exact) sequences:

$$0 \to V^* \underset{p_1}{\overset{P_1}{\rightleftarrows}} \Lambda^2(V^*) \underset{p_2}{\overset{P_2}{\rightleftarrows}} \Lambda^3(V^*) \underset{p_3}{\overset{P_3}{\rightleftarrows}} \Lambda^4(V^*) \underset{p_4}{\overset{P_4}{\rightleftarrows}} \Lambda^5(V^*) \underset{p_5}{\overset{P_5}{\rightleftarrows}} \Lambda^6(V^*) \rightleftarrows 0 .$$

We shall determine the irreducible components of the induced representation of G_2 on each $\Lambda^k(V^*)$. The representation of G_2 on $\Lambda^k(V^*)$ and $\Lambda^{7-k}(V^*)$ are the same because the Hodge star operator $*: \Lambda^k(V^*) \to \Lambda^{7-k}(V^*)$ is an isometry. (Here we use the inner product $\langle \, , \rangle$ on $\Lambda^k(V^*)$ given by

$$\langle \alpha, \beta \rangle = \sum_{i_1 \ldots i_k = 0}^{6} \alpha(e_{i_1} \wedge \ldots \wedge e_{i_k}) \beta(e_{i_1} \wedge \ldots \wedge e_{i_k}) ,$$

where $\{e_0, \ldots, e_6\}$ is an arbitrary basis of V.) Therefore, it suffices to describe the representations of G_2 on V^*, $\Lambda^2(V^*)$, and $\Lambda^3(V^*)$. The representation of G_2 on V^* is the irreducible 7-dimensional representation; the representations of G_2 on $\Lambda^2(V^*)$ and $\Lambda^3(V^*)$ are both reducible. We now use the vector cross product P to describe in a geometrically useful way the irreducible summands.

First we define

$$\Lambda_1^2(V^*) = \{\alpha \in \Lambda^2(V^*) | p(\alpha) = 0\} ,$$

$$\Lambda_2^2(V^*) = \{\alpha \in \Lambda^2(V^*) | 3\alpha = Pp(\alpha)\} ,$$

$$\Lambda_1^3(V^*) = \{\varphi\} ,$$

$$\Lambda_2^3(V^*) = \left\{\alpha \in \Lambda^3(V^*) | p(\alpha) = 0 \quad \text{and} \quad \sum_{i,j=0}^{6} \alpha(e_i \wedge e_j \wedge P(e_i \wedge e_j)) = 0\right\} ,$$

$$\Lambda_3^3(V^*) = \{\alpha \in \Lambda^3(V^*) | \alpha(x \wedge y \wedge P(x \wedge y)) = 0 \text{ for all } x, y \in V\} .$$

LEMMA 3.1. – We have the following orthogonal direct sum:

$$\Lambda^2(V^*) = \Lambda_1^2(V^*) \oplus \Lambda_2^2(V^*) . \tag{3.1}$$

Also G_2 acts irreducibly on $\Lambda_i^2(V^*)$, and

$$\dim \Lambda_1^2(V^*) = 14 , \quad \dim \Lambda_2^2(V^*) = 7 .$$

PROOF. – Using the fact that $p_1 \circ P_1 = 3I_1$ (where I_k: $\Lambda^k(V^*) \to \Lambda^k(V^*)$ denotes the identity map), it is easy to verify that $\frac{1}{3} P_1 \circ p_1$ and $I_2 - \frac{1}{3} P_1 \circ p_1$ are projections onto $\Lambda_2^2(V^*)$ and $\Lambda_1^2(V^*)$ respectively. This proves (3.1).

The mapping P_1: $V^* \to \Lambda^2(V^*)$ is injective and Image $P_1 = \Lambda_2^2(V^*)$. Thus $\Lambda_1^2(V^*)$ and $\Lambda_2^2(V^*)$ have the stated dimensions.

To check that G_2 acts irreducibly on $\Lambda_i^2(V^*)$ one verifies that there are no non-trivial irreducible summands of dimension 1 or 7. This completes the proof.

LEMMA 3.2. – We have the following orthogonal direct sum:

$$\Lambda^3(V^*) = \Lambda_1^3(V^*) \oplus \Lambda_2^3(V^*) \oplus \Lambda_3^3(V^*) . \tag{3.2}$$

Also G_2 acts irreducibly on $\Lambda_i^3(V^*)$, and

$$\dim \Lambda_1^3(V^*) = 1 , \quad \dim \Lambda_2^3(V^*) = 27 , \quad \dim \Lambda_3^3(V^*) = 7 .$$

PROOF. – Consider the subspace A of $\Lambda^3(V)$ generated by elements of the form $\{x \wedge y \wedge P(x \wedge y) | x, y \in V\}$. By Corollary 2.8, A coincides with the kernel of $P^2 : \Lambda^3(V) \to V$. Since P^2 is onto it follows that $\dim A = 28$. Hence the set of forms vanishing of A, namely $\Lambda_3^3(V^*)$, has dimension 7. Similarly $P: \Lambda^3(V^*) \to V^*$ is onto so that its kernel has dimension 28. Also $\Lambda_1^3(V^*) \oplus \Lambda_2^3(V^*) = \text{kernel } P$. Hence all of the spaces have the stated dimensions. These representations of G_2 are irreducible and so they are orthogonal.

4. – The space of covariant derivatives of the fundamental 3-form.

In the next section we shall consider the covariant derivative $\nabla\varphi$ of the fundamental 3-form φ of a vector cross product on a 7-dimensional manifold M. The tensor field $\nabla\varphi$ has various symmetry properties. In the present section we define a finite dimensional vector space W that consists of those tensor fields having the same symmetries. Then we study the decomposition of W into irreducible components under the natural action of G_2.

The space W is given by

$$W = \{\alpha \in V^* \otimes \Lambda^3(V^*) | \alpha(x, y\wedge z\wedge P(y\wedge z)) = 0 \text{ for all } x, y, z \in V\} .$$

LEMMA 4.1. – $\dim W = 49$.

PROOF. – It is clear that W is naturally isomorphic to $V^* \otimes \Lambda^3_3(V^*)$. Since $\dim V^* = \dim \Lambda^3_3(V^*) = 7$, the result follows.

There is a natural inner product on W given by

$$\langle\alpha, \beta\rangle = \sum_{i,j,k,l=0}^{6} \alpha(e_i, e_j\wedge e_k\wedge e_l)\beta(e_i, e_j\wedge e_k\wedge e_l) .$$

Here $\{e_0, \dots, e_6\}$ is an arbitrary orthonormal basis of V. It will also be useful to consider linear maps $L_i\colon W \to \Lambda^i(V^*)$ for $i = 0, 1, 2$ given by

$$L_2(\alpha)(x\wedge y) = \sum_{i=0}^{6} \alpha(e_i, e_i\wedge x\wedge y) ,$$
$$L_1(\alpha)(x) = \sum_{i,j=0}^{6} \alpha(P(e_i\wedge e_j), e_i\wedge e_j\wedge x) ,$$
$$L_0(\alpha) = \sum_{i,j,k=0}^{6} \alpha\big(P(P(e_i\wedge e_j)\wedge e_k), e_i\wedge e_j\wedge e_k\big) ,$$

for $x, y \in V$, $\alpha \in W$.

LEMMA 4.2. – We have

$$L_0(\alpha) = \langle\alpha, *\varphi\rangle , \tag{4.1}$$

$$\begin{aligned} L_1(\alpha)(x) &= \sum_{i=0}^{6} \alpha\big(e_i, e_j\wedge P(e_i\wedge e_j)\wedge x\big) \\ &= -2pL_2(\alpha)(x) , \end{aligned} \tag{4.2}$$

for $x \in V$, $\alpha \in W$.

PROOF. – In the definition of L_1 we may replace e_j by $P(e_i \wedge e_j)$. Then using (2.6) we obtain

$$L_1(\alpha)(x) = \sum_{i,j=0}^{6} \alpha\big(P(e_i \wedge P(e_i \wedge e_j)),\, e_i \wedge P(e_i \wedge e_j) \wedge x\big) \tag{4.3}$$
$$= \sum_{i,j=0}^{6} \alpha(-e_j + \delta_{ij} e_i,\, e_i \wedge P(e_i \wedge e_j) \wedge x)$$
$$= -\sum_{i,j=0}^{6} \alpha(e_j,\, e_i \wedge P(e_i \wedge e_j) \wedge x)$$
$$= \sum_{i,j=0}^{6} \alpha(e_i,\, e_j \wedge P(e_i \wedge e_j) \wedge x)\,.$$

Furthermore, using (2.9) and (4.3) we see that

$$(pL_2)(\alpha)(x) = -\tfrac{1}{2} \sum_{i,j=0}^{6} \alpha(e_i,\, e_i \wedge e_j \wedge P(e_j \wedge x)) = -\tfrac{1}{2} L_1(\alpha)(x)\,.$$

This establishes (4.2). We get (4.1) from (2.15).

LEMMA 4.3. – Suppose there is a constant a such that

$$\alpha(P(x \wedge y),\, x \wedge y \wedge z) = a\{\alpha(x,\, P(x \wedge y) \wedge y \wedge z) - \alpha(y,\, P(x \wedge y) \wedge x \wedge z)\}\,, \tag{4.4}$$

for all $x, y, z \in V$. If $a \neq -\frac{1}{2}$ then $pL_2(\alpha) = 0$.

PROOF. – From lemma 4.2 and equation (4.4) we have

$$(pL_2)(\alpha)(x) = -\frac{1}{2} L_1(\alpha)(x) = -\frac{1}{2} \sum_{i,j=0}^{6} \alpha(P(e_i \wedge e_j),\, e_i \wedge e_j \wedge x)$$
$$= -\frac{a}{2} \sum_{i,j=0}^{6} \{\alpha(e_i,\, P(e_i \wedge e_j) \wedge e_j \wedge x) - \alpha(e_j,\, P(e_i \wedge e_j) \wedge e_i \wedge x)\}$$
$$= a \sum_{i,j=0}^{6} \alpha(e_i,\, e_j \wedge P(e_i \wedge e_j) \wedge x) = -2a(pL_2)(\alpha)(x)\,.$$

Hence the lemma follows.

We now define four subspaces of W:

$$W_1 = \{* \varphi\}\,,$$
$$W_2 = \{\alpha \in W | \alpha(w, x \wedge y \wedge z) - \alpha(x, w \wedge y \wedge z) + \alpha(y, w \wedge x \wedge z) - \alpha(z, w \wedge x \wedge y) = 0 \quad \text{for all } w, x, y, z \in V\}\,,$$
$$W_3 = \{\alpha \in W | L_2(\alpha) = L_0(\alpha) = 0\}\,,$$
$$W_4 = \{\alpha \in W | 12\alpha(w, x \wedge y \wedge z) = \mathop{\mathfrak{S}}_{xyz}\big(-(pL_2)(\alpha)(x)\varphi(w \wedge y \wedge z) + 3\langle w, x\rangle L_2(\alpha)(y \wedge z)\big) \quad \text{for all } w, x, y, z \in V\}\,.$$

Eventually we shall show that these are the four irreducible components of the representation of G_2 on W. First we need several alternate descriptions of these subspaces and their direct sums.

LEMMA 4.4. – $W_1 = \{\alpha \in W | \alpha = (1/168) L_0(\alpha) * \varphi\} = W \cap \Lambda^4(V^*)$.

PROOF. – Suppose $\alpha \in W_1$. Then $\alpha = a * \varphi$. Using (2.14), (2.15), and (2.2) we find that $a = (1/168) L_0(\alpha)$. Hence the lemma follows.

LEMMA 4.5. – W_1, W_2, W_3, are mutually orthogonal and

$$W_1 \oplus W_3 = \text{kernel } L_2 , \tag{4.5}$$

$$\begin{aligned} W_1 \oplus W_2 = \{\alpha \in W | \alpha(P(x\wedge y), x\wedge y\wedge z) \\ = \alpha(x, P(x\wedge y)\wedge y\wedge z) - \alpha(y, P(x\wedge y)\wedge x\wedge z) \quad \text{for all } x, y, z \in V\} . \end{aligned} \tag{4.6}$$

PROOF. – It is obvious from lemma 4.2 that W_1 and W_3 are orthogonal and that (4.5) holds. Also it can be verified by direct calculation using the definition of the inner product on W that W_2 is perpendicular to W_1. Furthermore,

$$\begin{aligned} W_1 \oplus W_2 \subseteq \{\alpha \in W | \alpha(P(x\wedge y), x\wedge y\wedge z) = \alpha(x, P(x\wedge y)\wedge y\wedge z) - \alpha(y, P(x\wedge y)\wedge x\wedge z) \\ \text{for all } x, y, z \in V\} . \end{aligned}$$

To establish the reverse inclusion we define $T\colon W \to \Lambda^4(V^*)$ by

$$T(\alpha)(w, x\wedge y\wedge z) = \tfrac{1}{4}\{\alpha(w, x\wedge y\wedge z) - \alpha(x, w\wedge y\wedge z) + \alpha(y, w\wedge x\wedge z) - \alpha(z, w\wedge x\wedge y)\}$$

for $w, x, y, z \in V$. Then one checks that if $\alpha \in W$ is such that

$$\alpha(P(x\wedge y), x\wedge y\wedge z) = \alpha(x, P(x\wedge y)\wedge y\wedge z) - \alpha(y, P(x\wedge y)\wedge x\wedge z)$$

for all $x, y, z \in V$, then $T(\alpha) \in W$. Furthermore, $T^2 = T$. Thus T is the projection of

$$\begin{aligned} \{\alpha \in W | \alpha(P(x\wedge y), x\wedge y\wedge z) = \alpha(x, P(x\wedge y)\wedge y\wedge z) - \alpha(y, P(x\wedge y)\wedge x\wedge z) \\ \text{for all } x, y, z \in V\} \end{aligned}$$

onto $W_1 = W \cap \Lambda^4(V^*)$. It is clear that the kernel of T is W_2. Thus we get (4.6).

LEMMA 4.6.

$$W_1 \oplus W_2 \oplus W_3 = \text{kernel}\, pL_2 . \tag{4.7}$$

PROOF. – One checks that $W_2 \cap \text{kernel}\, L_2 = \{0\}$. Furthermore, using lemma 4.3 and lemma 4.5, $pL_2(\alpha) = 0$ for $\alpha \in W_1 \oplus W_2 \oplus W_3$. Also $W_2 = \text{kernel}\, T$ so that $\dim W_2 \geqslant 14$. Then $W_1 \oplus W_2 \oplus W_3 = \text{kernel}\, pL_2$ because both spaces have the same dimension.

LEMMA 4.7. – Suppose $\alpha \in W$ with $pL_2(\alpha) \neq 0$ and that

$$\alpha(w, x\wedge y\wedge z) = \underset{xyz}{\mathfrak{S}} \{apL_2(\alpha)(x)\varphi(w\wedge y\wedge z) + b\langle w, x\rangle L_2(\alpha)(y\wedge z)\}, \tag{4.8}$$

for all $w, x, y, z \in V$. Then $a = -1/12$, $b = \frac{1}{4}$ and $PpL_2(\alpha) = 3L_2(\alpha)$.

PROOF. – In (4.8) we let $x = e_i$, $y = e_j$, $z = P(e_i \wedge e_j)$ and sum. Since $\alpha \in W$ we must have

$$\sum_{i,j=0}^{6} \alpha(w, e_i \wedge e_j \wedge P(e_i \wedge e_j)) = 0 .$$

This implies that

$$(18a + 6b)pL_2(\alpha)(w) = 0 . \tag{4.9}$$

On the other hand if we apply L_2 to both sides of (4.8) we obtain

$$(1 - 5b)L_2(\alpha) = aPpL_2(\alpha) . \tag{4.10}$$

Applying p to both sides of (4.10) we get

$$(1 - 5b - 3a)pL_2(\alpha) = 0 . \tag{4.11}$$

From (4.9) and (4.11) and the assumption that $pL_2(\alpha) \neq 0$ we obtain $a = -1/12$ $b = \frac{1}{4}$. Then substituting these values into (4.10) we find that $PpL_2(\alpha) = 3L_2(\alpha)$

LEMMA 4.8.

$$W_1 \oplus W_3 \oplus W_4 = \{\alpha \in W | PpL_2(\alpha) = 3L_2(\alpha)\} .$$

PROOF. – We have $W_1 \oplus W_3 \subseteq \{\alpha \in W | PpL_2(\alpha) = 3L_2(\alpha)\}$ because $L_2(\alpha) = 0$ for $\alpha \in W_1 \oplus W_3$. Also $W_4 \subseteq \{\alpha \in W | PpL_2(\alpha) = 3L_2(\alpha)\}$ by Lemma 4.7. Consider the mapping U: $V^* \to V^* \otimes \Lambda^3(V^*)$ given by

$$U(\gamma)(w, x\wedge y\wedge z) = -\frac{1}{12} \underset{xyz}{\mathfrak{S}} \{\gamma(x)\varphi(w\wedge y\wedge z) - \langle w, x\rangle \gamma(P(y\wedge z))\} .$$

It can be checked that U is injective and Image $U = W_4$. Thus dim $W_4 = 7$. Since dim $\{\alpha \in W | PpL_2(\alpha) = 3L_2(\alpha)\} = 35$ we must have $W_1 \oplus W_3 \oplus W_4 = \{\alpha \in W | PpL_2(\alpha) = 3L_2(\alpha)\}$ because both spaces have the same dimension.

The representation of G_2 on V induces a representation of G_2 on W. The next theorem describes the decomposition of this induced representation into irreducible components.

THEOREM 4.9. – We have $W = W_1 \oplus W_2 \oplus W_3 \oplus W_4$. This sum is direct and orthogonal, and it is preserved under the induced representation of G_2 on W. The induced representation of G_2 on W_i is irreducible. We have dim $W_1 = 1$, dim $W_2 = 14$, dim $W_3 = 27$, and dim $W_4 = 7$.

PROOF. – In the previous lemmas the dimensions of the W_i have been calculated. Also it is clear that $W_i \cap W_j = \{0\}$ for $i \neq j$. Since dim $W = 49$, it follows that $W_1 \oplus W_2 \oplus W_3 \oplus W_4 = W$ and the sum is direct and orthogonal.

That G_2 acts irreducibly on W_1 is obvious, and can be checked for W_2 and W_4. That G_2 acts irreducibly on W_3 follows because $W_3 = *\,(\Lambda_2^3(V^*))$. This completes the proof.

5. – Seven dimensional Riemannian manifolds with two fold vector cross products.

Let M be a C^∞ Riemannian manifold of dimension 7 with metric tensor field $\langle\, ,\rangle$. Denote by $\mathfrak{X}(M)$ the Lie algebra of C^∞ vector fields on M and by $\mathcal{F}(M)$ the algebra of C^∞ functions on M. For each $m \in M$ the tangent space at m will be denoted by M_m.

DEFINITION. – We say that $(M, \langle\, ,\rangle)$ has a 2-fold vector cross product P provided that each tangent space M_m has a 2-fold vector cross product $P_m\colon M_m \times M_m \to M_m$. We require that the mapping $m \to P_m$ be C^∞.

It is clear that P gives rise to a tensor field $P\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ of type (2,1) and that

$$\langle P(X, Y), X\rangle = \langle P(X, Y), Y\rangle = 0\,, \tag{5.1}$$

$$\|P(X, Y)\|^2 = \|X \wedge Y\|^2 = \det\begin{pmatrix} \|X^2\| & \langle X, Y\rangle \\ \langle X, Y\rangle & \|Y\|^2 \end{pmatrix}, \tag{5.2}$$

for $X, Y \in \mathfrak{X}(M)$. In fact, the linear algebra of the previous sections goes over in the obvious way to manifolds. The purpose of the present section is to study the way the vector cross product P changes from point to point, that is, the differential geometry of P. For this, it is important to study the covariant derivative of P.

We note that the fundamental 3-form φ becomes a differential 3-form on M. In special circumstances, (for example when P is parallel), it generates cohomology in dimension 3.

Let ∇ denote the Riemannian connection of $\langle\, ,\rangle$. The covariant derivatives ∇P and $\nabla\varphi$ are given by

$$\nabla_X(P)(Y, Z) = \nabla_X(P(Y, Z)) - P(\nabla_X Y, Z) - P(Y, \nabla_X Z)\,, \tag{5.3}$$

$$\nabla_W(\varphi)(X, Y, Z) = W\varphi(X, Y, Z) - \varphi(\nabla_W X, Y, Z) - \varphi(X, \nabla_W Y, Z) - \\ - \varphi(X, Y, \nabla_W Z)\,, \tag{5.4}$$

for $W, X, Y, Z \in \mathfrak{X}(M)$. Then we have from (5.3) and (5.4) that

$$\nabla_W(\varphi)(X, Y, Z) = \langle\nabla_W(P)(X, Y), Z\rangle\,, \tag{5.5}$$

and so the study of the covariant derivatives of P is equivalent to the study of the covariant derivatives of the fundamental 3-form φ. For convenience we shall do our calculations with $\nabla\varphi$.

LEMMA 5.1. – We have

$$\nabla_W(\varphi)(X, Y, Z) = -\nabla_W(\varphi)(Y, X, Z) = -\nabla_W(\varphi)(X, Z, Y)\,, \tag{5.6}$$

$$\nabla_W(\varphi)(X, Y, P(X, Y)) = 0\,, \tag{5.7}$$

for $W, X, Y, Z \in \mathfrak{X}(M)$.

PROOF. – (5.6) is easy to check directly from (5.4). Then (5.7) is proved by applying the vector field W to both sides of (5.2) and using (5.3) and (5.5).

We shall henceforth write $\nabla_W(\varphi)(X\wedge Y\wedge Z)$ for $\nabla_W(\varphi)(X, Y, Z)$, etc.

Consider the natural 7-dimensional representation of G_2 on each tangent space M_m. Put

$$W_m = \{\alpha \in M_m^* \otimes \Lambda^3(M_m^*) | \alpha(x, y\wedge z\wedge P(y\wedge z)) = 0 \text{ for all } x, y, z \in M_m\}\,.$$

Then the induced representation of G_2 on W_m has the four components W_{m1}, W_{m2}, W_{m3}, W_{m4} as described in the previous section. It is possible to form from these four a total of sixteen invariant subspaces of W_m (including $\{0\}$ and W_m).

DEFINITION. – Let U be one of the sixteen invariant subspaces of W. For a 7-dimensional Riemannian manifold M with 2-fold vector cross product and $m \in M$, let U_m denote the corresponding subspace of W_m. Then $\mathcal{U}$ will denote the class of all 7-dimensional Riemannian manifolds with a 2-fold vector cross product such that $(\nabla\varphi)_m \in U_m$ for all $m \in M$.

The class corresponding to W_i will be denoted by $\mathcal{W}_i$, and that corresponding to $W_i \oplus W_j$ by $\mathcal{W}_i \oplus \mathcal{W}_j$, etc. Also $\mathcal{P}$ will correspond to $\{0\}$ and $\mathcal{W}$ to W. Some, but not all, of these classes have been studied in [GR 3]. There are obvious analogies between some of these classes and corresponding ones for almost Hermitian manifolds [GH].

Let d and δ be the exterior differential and the coderivative of a manifold M. If η is a 3-form on M we have the following explicit formulas for $d\eta$ and $\delta\eta$:

$$(5.8)\qquad d\eta(W\wedge X\wedge Y\wedge Z) = \nabla_W(\eta)(X\wedge Y\wedge Z) - \nabla_X(\eta)(W\wedge Y\wedge Z) + \nabla_Y(\eta)(W\wedge X\wedge Z) - \nabla_Z(\eta)(W\wedge X\wedge Y)\,,$$

$$(5.9)\qquad \delta\eta(Y\wedge Z) = -\sum_{i=0}^{6}\nabla_{E_i}(\eta)(E_i\wedge Y\wedge Z)\,,$$

for $W, X, Y, Z\in\mathfrak{X}(M)$. Here $\{E_0, \ldots, E_6\}$ is an arbitrary local frame field.

Now assume that M is a 7-dimensional Riemannian manifold with a vector cross product P and fundamental 3-form φ. We note that

$$(5.10)\qquad \delta\varphi = -L_2(\nabla\varphi)\,.$$

Also, when written out for a manifold, the formulas for L_0 and L_1 become

$$(5.11)\qquad L_0(\nabla\varphi) = \sum_{i,j,k=0}^{6}\nabla_{P(P(E_i\wedge E_j)\wedge E_k)}(\varphi)(E_i\wedge E_j\wedge E_k)\,,$$

$$(5.12)\qquad L_1(\nabla\varphi)(X) = \sum_{i,j=0}^{6}\nabla_{P(E_i\wedge E_j)}(\varphi)(E_i\wedge E_j\wedge X)\,,$$

for $X\in\mathfrak{X}(M)$. Using lemma 4.2 we have

$$(5.13)\qquad L_0(\nabla\varphi) = \langle\nabla\varphi, *\varphi\rangle\,,$$

$$(5.14)\qquad L_1(\nabla\varphi) = 2p\,\delta\varphi\,.$$

THEOREM 5.2. – The defining relations for each of the 16 classes are given in table I below

TABLE I

Class	Defining relations
$\mathcal{T}$	$\nabla\varphi = 0$
$\mathcal{W}_1 = \mathcal{NT}$	$\nabla_X(\varphi)(X\wedge Y\wedge Z) = 0$ $\left(\text{or } d\varphi = 4\nabla\varphi, \text{ or } \nabla\varphi = \dfrac{1}{168}\langle\nabla\varphi, *\varphi\rangle *\varphi\right)$
$\mathcal{W}_2 = \mathcal{AT}$	$d\varphi = 0$
$\mathcal{W}_3$	$\delta\varphi = \langle\nabla\varphi, *\varphi\rangle = 0$

TABLE I (*continued*).

Class	Defining relations
$\mathcal{W}_4 = \mathcal{CT}$	$12\nabla_W(\varphi)(X\wedge Y\wedge Z) = \underset{XYZ}{\mathfrak{S}} \{p\,\delta\varphi(X)\,\varphi(W\wedge Y\wedge Z) - 3\langle W, X\rangle\,\delta\varphi(Y\wedge Z)\}$
$\mathcal{W}_1\oplus\mathcal{W}_2$	$\nabla_{P(X\wedge Y)}(\varphi)(X\wedge Y\wedge Z) = \nabla_X(\varphi)(P(X\wedge Y)\wedge Y\wedge Z) - \nabla_Y(\varphi)(P(X\wedge Y)\wedge X\wedge Z)$
$\mathcal{W}_1\oplus\mathcal{W}_3 = \mathcal{ST}$	$\delta\varphi = 0$
$\mathcal{W}_2\oplus\mathcal{W}_3$	$p\delta\varphi = \langle\nabla\varphi, *\varphi\rangle = 0$
$\mathcal{W}_1\oplus\mathcal{W}_4$	$\nabla_W(\varphi)(X\wedge Y\wedge Z) - \frac{1}{12}\underset{XYZ}{\mathfrak{S}}\{p\,\delta\varphi(X)\,\varphi(W\wedge Y\wedge Z) - 3\langle W, X\rangle\,\delta\varphi(Y\wedge Z)\} = \frac{1}{168}\langle\nabla\varphi, *\varphi\rangle *\varphi(W\wedge X\wedge Y\wedge Z)$
$\mathcal{W}_2\oplus\mathcal{W}_4$	$d\varphi = -\frac{1}{4}p\,\delta\varphi\wedge\varphi$
$\mathcal{W}_3\oplus\mathcal{W}_4$	$3\,\delta\varphi = Pp\,\delta\varphi$ and $\langle\nabla\varphi, *\varphi\rangle = 0$
$\mathcal{W}_1\oplus\mathcal{W}_2\oplus\mathcal{W}_3$	$p\,\delta\varphi = 0$
$\mathcal{W}_1\oplus\mathcal{W}_2\oplus\mathcal{W}_4$	$d\varphi = -\frac{1}{4}p\,\delta\varphi\wedge\varphi + \frac{1}{42}\langle\nabla\varphi, *\varphi\rangle *\varphi$
$\mathcal{W}_1\oplus\mathcal{W}_3\oplus\mathcal{W}_4$	$3\,\delta\varphi = Pp\,\delta\varphi$ or $12\nabla_X(\varphi)(X\wedge Y\wedge Z) = p\,\delta\varphi(X)\varphi(X\wedge Y\wedge Z) - 3\{\lVert X\rVert^2\delta\varphi(Y\wedge Z) - \langle X, Y\rangle\,\delta\varphi(X\wedge Z) + \langle X, Z\rangle\,\delta\varphi(X\wedge Y)\}$
$\mathcal{W}_2\oplus\mathcal{W}_3\oplus\mathcal{W}_4$	$\langle\nabla\varphi, *\varphi\rangle = 0$
$\mathcal{W}$	no relation

6. – Classes preserved under conformal changes of metric.

In this section, we determine which of the 16 classes are preserved under a conformal change of metric. Let M be a 7-dimensional manifold with metrics $\langle\,,\rangle$ and $\langle\,,\rangle^0$. We assume that these metrics are conformally related via

$$\langle\,,\rangle^0 = e^{2\sigma}\langle\,,\rangle\,, \tag{6.1}$$

where $\sigma\in\mathcal{F}(M)$. It is well known (see for example [GR 1]) that the connections ∇^0 of $\langle\,,\rangle^0$ and ∇ of $\langle\,,\rangle$ are related by

$$\nabla^0_X Y = \nabla_X Y + (X\sigma)\,Y + (Y\sigma)X - \langle X, Y\rangle\,\mathrm{grad}\,\sigma\,, \tag{6.2}$$

for X, $Y \in \mathfrak{X}(M)$. Here grad $\sigma \in \mathfrak{X}(M)$ is the vector field such that $\langle X, \operatorname{grad} \sigma\rangle = X\sigma$ for $X \in \mathfrak{X}(M)$.

What is a reasonable notion of a conformal relation between two vector cross products P and P^0 on M? It is clear that the corresponding metrics $\langle\,,\rangle$ and $\langle\,,\rangle^0$ should be conformally related and that $P^0 = fP$ for some $f \in \mathcal{F}(M)$. Assuming this, we calculate

$$\|P^0(X\wedge Y)\|^{02} = \|X\wedge Y\|^{02} = e^{4\sigma}\|X\wedge Y\|^2 = e^{4\sigma}\|P(X\wedge Y)\|^2 = e^{2\sigma}\|P(X\wedge Y)\|^{02},$$

for X, $Y \in \mathfrak{X}(M)$. Thus we must have $f^2 = e^{2\sigma}$. This leads us to the following

DEFINITION. – Let M be a 7-dimensional Riemannian manifold with metrics $\langle\,,\rangle$, $\langle\,,\rangle^0$ conformally related by (6.1). Let P and P^0 be vector cross products on M compatible with $\langle\,,\rangle$ and $\langle\,,\rangle^0$, respectively. We say that P and P^0 are conformally related provided

$$P^0 = e^{\sigma}P\,. \tag{6.3}$$

Let φ, φ^0 denote the fundamental 3-forms corresponding to P and P^0, and let p, p^0 be the corresponding adjoints. Also let δ, δ^0 denote the coderivatives of $\langle\,,\rangle$, $\langle\,,\rangle^0$, respectively.

LEMMA 6.1. – We have

$$\varphi^0 = e^{3\sigma}\varphi\,, \tag{6.4}$$

$$p^0 = e^{-\sigma}p\,, \tag{6.5}$$

$$\nabla^0_W(\varphi^0)(X\wedge Y\wedge Z) = e^{3\sigma}\{\nabla_W(\varphi)(X\wedge Y\wedge Z) - \underset{XYZ}{\mathfrak{S}}\big((X\sigma)\varphi(W\wedge Y\wedge Z) - \langle W, X\rangle P(Y\wedge Z)\sigma\big)\}\,, \tag{6.6}$$

for W, X, Y, $Z \in \mathfrak{X}(M)$,

$$\delta^0\varphi^0(Y\wedge Z) = e^{\sigma}\{\delta\varphi(Y\wedge Z) - 4P(Y\wedge Z)\sigma\}\,, \tag{6.7}$$

for Y, $Z \in \mathfrak{X}(M)$,

$$p^0\delta^0\varphi^0 = p\,\delta\varphi - 12d\sigma\,, \tag{6.8}$$

$$d\varphi^0 = e^{3\sigma}\{3d\sigma\wedge\varphi + d\varphi\}\,. \tag{6.9}$$

PROOF. – Equations (6.4) and (6.5) are obvious consequences of (6.1) and (6.3). Taking the exterior derivative of (6.4) we get (6.9). Equation (6.6) follows from (5.4), (6.2), and (6.4). Contracting (6.6) we obtain (6.7) and (6.8).

Next we introduce a tensor field ν that will turn out to be a conformal invariant for 2-fold vector cross products. A similar tensor field has been introduced in [GH] for almost Hermitian manifolds.

Definition. – Let M be a 7-dimensional manifold with metric tensor field $\langle\, ,\rangle$ and vector cross product P. Then ν is the covariant tensor field given by

$$(6.10)\qquad \nu(W, X, Y, Z) = \nabla_W(\varphi)(X\wedge Y\wedge Z) - \\ - \frac{1}{12} \underset{XYZ}{\mathfrak{S}} \{p\, \delta\varphi(X)\, \varphi(W\wedge Y\wedge Z) - 3\, \langle W, X\rangle\, \delta\varphi(Y\wedge Z)\},$$

for $W, X, Y, Z \in \mathfrak{X}(M)$.

Lemma 6.2. – Suppose $(P, \langle\, ,\rangle)$ and $(P^0, \langle\, ,\rangle^0)$ are conformally related. Then the corresponding tensor fields ν and ν^0 satisfy $\nu^0 = e^{3\sigma}\nu$.

Proof. – This follows from lemma 6.1 and equation (6.10).

Let $\mathcal{U}$ be one of the sixteen classes given in table I. Then $\mathcal{U}^0$ will denote the class of all manifolds locally conformally related to manifolds in $\mathcal{U}$. In other words, $(M, P^0, \langle\, ,\rangle^0) \in \mathcal{U}^0$ if and only if for each $m \in M$ there exists an open neighborhood V of m such that $(V, P^0, \langle\, ,\rangle^0)$ is conformally related to $(V, P, \langle\, ,\rangle) \in \mathcal{U}$.

Next we prove

Theorem 6.3. – For any $\mathcal{U}$ given in table I we have $\mathcal{U}^0 \subseteq \mathcal{W}_4 \oplus \mathcal{U}$. Thus $\mathcal{U} = \mathcal{U}^0$ if and only if $\mathcal{W}_4 \subseteq \mathcal{U}$. Hence the conformally invariant classes are W_4, $\mathcal{W}_1 \oplus \mathcal{W}_4$, $\mathcal{W}_2 \oplus \mathcal{W}_4$, $\mathcal{W}_3 \oplus \mathcal{W}_4$, $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$, $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$, $\mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$, $\mathcal{W}$.

Proof. – We can rewrite the defining relations for each of the classes mentioned in the statement of the theorem in terms of ν. From table I we have

$M \in \mathcal{W}_4$ if and only if $\nu = 0$,

$M \in \mathcal{W}_1 \oplus \mathcal{W}_4$ if and only if $\nu = \frac{1}{168}\langle \nu, *\varphi\rangle * \varphi$,

$M \in \mathcal{W}_2 \oplus \mathcal{W}_4$ if and only if $\nu(W, X, Y, Z) - \nu(X, W, Y, Z) + \nu(Y, W, X, Z) - \nu(Z, W, X, Y) = 0$, for all $W, X, Y, Z \in \mathfrak{X}(M)$,

$M \in \mathcal{W}_3 \oplus \mathcal{W}_4$ if and only if $\langle \nu, *\varphi\rangle = \nu(X, X, Y, Z) = 0$, for all $X, Y, Z \in \mathfrak{X}(M)$,

$M \in \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$ if and only if $\nu(W, X, Y, Z) - \nu(X, W, Y, Z) + \nu(Y, W, X, Z) - \nu(Z, W, X, Y) = \frac{2}{147}\langle \nu, *\varphi\rangle (*\varphi)(W, X, Y, Z)$, for all $W, X, Y, Z \in \mathfrak{X}(M)$,

$M \in \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ if and only if $\nu(X, X, Y, Z) = 0$, for all $X, Y, Z \in \mathfrak{X}(M)$,

$M \in \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ if and only if $\langle \nu, *\varphi\rangle = 0$,

7. – Two-fold vector cross products on orientable hypersurfaces of $\boldsymbol{R}^8$.

In [GR 3] it is shown that every orientable hypersurface $M\subset \boldsymbol{R}^8$ has a 2-fold vector cross product. In this section we review this construction and introduce some improvements. Also, it will be shown that $M\in \mathcal{W}_1\oplus \mathcal{W}_3$ for any orientable hypersurface in $\boldsymbol{R}^8$, any necessary and sufficient conditions for M to belong to $\mathcal{W}_1$ or $\mathcal{W}_3$ will be given in terms of the second fundamental form of M.

First we recall

DEFINITION. – Let $\overline{V}$ be a finite dimensional vector space over $\boldsymbol{R}$ with (positive definite) inner product $\langle\, ,\rangle$. A *3-fold vector cross product* on $\overline{V}$ is a trilinear map $\overline{P}\colon \overline{V}\times\overline{V}\times\overline{V}\to\overline{V}$ satisfying the axioms

$$\langle\overline{P}(w,x,y),w\rangle=\langle\overline{P}(w,x,y),x\rangle=\langle\overline{P}(w,x,y),y\rangle=0\,, \tag{7.1}$$

$$\|\overline{P}(w,x,y)\|^2=\|w\wedge x\wedge y\|^2=\det\begin{pmatrix}\|w\|^2 & \langle w,x\rangle & \langle w,y\rangle\\ \langle w,x\rangle & \|x\|^2 & \langle x,y\rangle\\ \langle w,y\rangle & \langle x,y\rangle & \|y\|^2\end{pmatrix}, \tag{7.2}$$

for $w, x, y\in\overline{V}$.

Just as with 2-fold vector cross products, $\overline{P}(w,x,y)$ is antisymmetric in w, x, y Hence $\overline{P}$ may be extended to a linear mapping $\overline{P}\colon\Lambda^3(\overline{V})\to\overline{V}$. We write $\overline{P}(w\wedge x\wedge y)$ instead of $\overline{P}(w,x,y)$.

DEFINITION. – The fundamental 4-form $\bar{\varphi}$ of a 3-fold vector cross product $\overline{P}$ is given by

$$\bar{\varphi}(w\wedge x\wedge y\wedge z)=\langle\overline{P}(w\wedge x\wedge y),z\rangle$$

for $w, x, y, z\in\overline{V}$.

In [E 1], [W], or [BG] it is shown that if $\overline{V}$ has a 3-fold vector cross product then necessarily $\dim\overline{V}=4$ or 8. When $\dim\overline{V}=4$, the study $\overline{P}$ amounts to the study of the volume element of $\overline{V}$, namely $\bar{\varphi}$, so we restrict ourselves to the case $\dim\overline{V}=8$. In this case, it is shown in [BG] that there are two nonisomorphic 3-fold vector cross products $\overline{P}_+$ and $\overline{P}_-$. (The reason why there are two distinct 3-fold vector cross products is that the Cayley numbers are non-associative.) Furthermore, the automorphism groups of $\overline{P}_+$ and $\overline{P}_-$ are both isomorphic to Spin (7). We write $\bar{\varphi}_\pm$ for fundamental 4-form of $\overline{P}_\pm$.

There are explicit formulas expressing $\overline{P}_+$ and $\overline{P}_-$ in terms of the Cayley numbers. Let $x\to\bar{x}$ be the conjugation in *Cay* (taht is, $\bar{x}=-x+2\langle x,1\rangle 1$). Then we have [BG], [Z]:

$$\overline{P}_+(x\wedge y\wedge z) = -x(\bar{y}z) + \langle x, y\rangle z + \langle y, z\rangle x - \langle x, z\rangle y\ , \tag{7.3}$$

$$\overline{P}_-(x\wedge y\wedge z) = -(x\bar{y})z + \langle x, y\rangle z + \langle y, z\rangle x - \langle x, z\rangle y\ , \tag{7.4}$$

for $x, y, z \in \overline{V}$.

Let $V \subset \overline{V}$ be the pure imaginary Cayley numbers, that is, $V = \{1\}^\perp$. Then V has a 2-fold vector cross product P given by (2.4). We shall need to know the relations between P and $\overline{P}_\pm$.

LEMMA 7.1. – For $x, y, z \in V$ we have

$$P(x\wedge y) = \overline{P}_\pm(1\wedge x\wedge y)\ , \tag{7.5}$$

$$\begin{aligned} 3\overline{P}_\pm(x\wedge y\wedge z) &= \mp P^2(x\wedge y\wedge z) - 3\varphi(x\wedge y\wedge z)1 \\ &= \pm \underset{xyz}{\mathfrak{S}} P\big(x\wedge P(y\wedge z)\big) - 3\varphi(x\wedge y\wedge z)1\ , \end{aligned} \tag{7.6}$$

$$\bar{\varphi}_\pm = \pm * \varphi\ . \tag{7.7}$$

PROOF. – (7.5) is immediate from (7.3) and (7.4) and the fact that $\bar{x} = -x$ for $x \in V$. For (7.6) and (7.7), we do $\overline{P}_+$ and $\bar{\varphi}_+$. From (2.4) we have

$$\underset{xyz}{\mathfrak{S}} P\big(x\wedge P(y\wedge z)\big) = \underset{xyz}{\mathfrak{S}} \{x(yz) + \langle y, z\rangle x\} + 3\varphi(x\wedge y\wedge z)1\ , \tag{7.8}$$

for $x, y, z \in V$. On the other hand, from (7.3) we get

$$3\overline{P}_+(x\wedge y\wedge z) = \underset{xyz}{\mathfrak{S}} \overline{P}_+(x\wedge y\wedge z) = \underset{xyz}{\mathfrak{S}} \{x(yz) + \langle y, z\rangle x\}\ , \tag{7.9}$$

for $x, y, z \in V$. From (7.8) and (7.9) we obtain (7.6). Then (7.7) follows from (7.6) and lemma 2.7.

Let M be an orientable hypersurface of $\boldsymbol{R}^8$. Then there is a globally defined unit normal vector field N on M. In [GR 3], a 2-fold vector cross product on M is defined by means of the formula

$$P(A\wedge B) = \overline{P}_\pm(N\wedge A\wedge B)\ , \tag{7.10}$$

for $A, B \in \mathfrak{X}(M)$. In (7.10) we can take either of the vector cross products $\overline{P}_+$ or $\overline{P}_-$. They are defined by means of parallel translation on each tangent space of $\boldsymbol{R}^8$. Thus $\overline{\nabla}\overline{P}_\pm = 0$, where $\overline{\nabla}$ is the Riemannian connection of $\boldsymbol{R}^8$.

Let ∇ be the Riemannian connection of the hypersurface M, and let $S\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ be the shape operator (which is equivalent to the second fundamental form). Thus $SA = -\overline{\nabla}_A N$ for $A \in \mathfrak{X}(M)$.

LEMMA 7.2. – We have

$$\nabla_A(\varphi)(B\wedge C\wedge D) = \pm(*\varphi)(SA\wedge B\wedge C\wedge D)\,, \tag{7.11}$$

for $A, B, C, D \in \mathfrak{X}(M)$.

PROOF. – Using the fact that $\bar{\nabla}\bar{\varphi}_{\pm} = 0$ and (7.10), we calculate

$$\nabla_A(\varphi)(B\wedge C\wedge D) = -\,\bar{\varphi}_{\pm}(SA\wedge B\wedge C\wedge D)\,. \tag{7.12}$$

Then (7.11) follows from (7.7) and (7.12).

LEMMA 7.3. – For any hypersurface $M\subset \boldsymbol{R}^8$, $\delta\varphi = 0$.

PROOF. – From (7.11) we have for $A, B \in \mathfrak{X}(M)$

$$\delta\varphi(A\wedge B) = \pm\sum_{i=0}^{6}(*\varphi)(SE_i\wedge E_i\wedge A\wedge B)\,. \tag{7.13}$$

Choose the local frame field $\{E_0, \ldots, E_6\}$ so that S is diagonalized. Then the right hand side of (7.13) vanishes.

Then mean curvature H of a hypersurface is given by $\langle H, N\rangle = \sum_{i=0}^{6}\langle SE_i, E_i\rangle$.

LEMMA 7.4. – For any hypersurface $M\subset \boldsymbol{R}^8$,

$$\langle\nabla\varphi, *\varphi\rangle = \pm 8\langle H, N\rangle\,. \tag{7.14}$$

PROOF. – From (5.11), (5.13) and (7.11) we have

$$\langle\nabla\varphi, *\varphi\rangle = \pm\sum_{i,j,k=0}^{6}(*\varphi)\big(SP(P(E_i\wedge E_j)\wedge E_k)\wedge E_i\wedge E_j\wedge E_k\big)\,. \tag{7.15}$$

Then (7.14) follows from (7.15) and lemma 2.7.

We can now prove

THEOREM 7.5. – Let M be an orientable hypersurface of $\boldsymbol{R}^8$ with 2-fold vector cross product given by (7.10). Then

(i) $M \in \mathcal{W}_1 \oplus \mathcal{W}_3$;

(ii) $M \in \mathcal{P}$ if and only if M is totally geodesic;

(iii) $M \in \mathcal{W}_1$ if and only if M is totally umbilic (i.e., M is a part of a sphere);

(iv) $M \in \mathcal{W}_3$ if and only if M is a minimal variety (i.e., $\langle H, N\rangle = 0$).

PROOF. – (i) holds because $\delta\varphi = 0$ by lemma 7.3. Similarly, (ii) follows (7.11), (iii) follows from (7.11), and (iv) follows (7.14).

REMARK. – In [GR 3] it is shown that every hypersurface $M^7 \subset \boldsymbol{R}^8$ that is in $\mathcal{W}_2$ is automatically in $\mathcal{T}$ by a different method.

8. – Complex vector cross products.

There is a complex analogue of the notion of vector cross product. We shall use it, together with an auxiliary construction, to find nontrivial examples of manifolds in the class $\mathcal{W}_2$.

First we define the concept of a complex vector cross product on a vector space. Let V be a $2n$-dimensional real vector space equipped with an almost complex structure J and a (positive definite) inner product $\langle\, ,\rangle$. We assume that $\langle\, ,\rangle$ and J are compatible in the sense that $\langle Jx, Jy\rangle = \langle x, y\rangle$ for $x, y \in V$. We extend J to a complex linear map $J\colon V\otimes \boldsymbol{C} \to V\otimes \boldsymbol{C}$. Also let $(\, ,)$ be the (positive definite) hermitian inner product on $V\otimes \boldsymbol{C}$ correspond to $\langle\, ,\rangle$; thus

$$(x + \sqrt{-1}\, y, u + \sqrt{-1}\, v) = \langle x, u\rangle + \langle y, v\rangle + \sqrt{-1}\{-\langle x, v\rangle + \langle y, u\rangle\}\,,$$

for $x, y, u, v \in V$. Finally $\|\;\|^2$ will denote the norm corresponding to $(\, ,)$. (Then when restricted to V, $\|\;\|^2$ becomes the norm of $\langle\, ,\rangle$.)

DEFINITION. – An *r-fold complex vector cross product* on $(V, J, \langle\, ,\rangle)$ is a multilinear map

$$C\colon \overbrace{(V\otimes \boldsymbol{C})\times \ldots \times (V\otimes \boldsymbol{C})}^{r} \to V\otimes \boldsymbol{C}$$

such that the following conditions are satisfied for $x_1, \ldots, x_r \in V\otimes \boldsymbol{C}$:

(8.1) $\quad (C(x_1, \ldots, x_r), x_i) = 0\,, \qquad i = 1, \ldots, r\,,$

(8.2) $\quad C(Jx_1, x_2, \ldots, x_r) = -JC(x_1, \ldots, x_r)\,,$

(8.3) $\quad \|C(x_1 - \sqrt{-1}\, Jx_1, \ldots, x_r - \sqrt{-1}\, Jx_r)\|^2 = $
$$= 2^{r-1}\|(x_1 - \sqrt{-1}\, Jx_1)\wedge \ldots \wedge (x_r - \sqrt{-1}\, Jx_r)\|^2,$$

(8.4) $\quad C(x_1, \ldots, x_r) \in V \quad$ whenever $\quad x_1, \ldots, x_r \in V\,.$

Because of (8.1) we see that $C(x_1, \ldots, x_r)$ is antisymmetric in $x_1, \ldots, x_r$, and so we actually have a linear map $C\colon \Lambda^r(V) \to V$. Therefore, we shall write $C(x_1\wedge\ldots\wedge x_r)$ for $C(x_1, \ldots, x_r)$. Also there is a complex $(r+1)$-form ψ on V given by

$$\psi(x_1\wedge\ldots\wedge x_{r+1}) = \langle C(x_1\wedge\ldots\wedge x_r), x_{r+1}\rangle$$

for $x_1, \ldots, x_{r+1} \in V$. In fact, because of (8.2), ψ is the sum of forms of types $(r+1, 0)$ and $(0, r+1)$. Thus in case $r = n-1$, ψ is just a complex volume form.

Complex vector cross products enjoy many of the properties of ordinary vector cross products. Eckmann [E 2] has alluded to complex vector cross products, but it is not clear that his definition is the same as ours.

DEFINITION. – We say that an almost Hermitian manifold $(M, J, \langle\,,\rangle)$ has an r-fold complex vector cross product C provided each tangent space M_m has an r-fold complex vector cross product C_m. We require that the mapping $m \to C_m$ be C^∞.

We note that each r-fold complex vector cross product on $(M, J, \langle\,,\rangle)$ gives rise to a mapping (the vector cross product)

$$C\colon \overbrace{(\mathfrak{X}(M)\otimes \boldsymbol{C})\times \ldots \times (\mathfrak{X}(M)\otimes \boldsymbol{C})}^{r} \to \mathfrak{X}(M)\otimes \boldsymbol{C}$$

such that (8.1)-(8.4) are satisfied.

Complex vector cross products will be treated extensively in another paper. Perhaps, however, it will be useful to note a couple of examples arising from differential geometry.

EXAMPLE 1. – Let M be a $4n$-dimensional Riemannian manifold whose structure group is reducible to $Sp(n)$. It is well known that M admits almost complex structures I, J, and K, each compatible with the metric, such that $IJ = -JI = K$, etc. Then either I or K can be regarded as a 1-fold complex vector cross product with respect to the almost complex structure J.

EXAMPLE 2. – Let M be a $2n$-dimensional Riemannian manifold whose structure group is reducible to $U(n)$. Let $\langle\,,\rangle$ be the Riemannian metric and J the almost complex structure. Then the structure group can be further reduced to $SU(n)$ if and only if M has an $(n-1)$-fold complex vector cross product C. The associated n-form C is the complex volume form of M, and $\psi\Lambda * \psi$ is the ordinary Riemannian volume form. Note also that if F is the Kähler form associated to J then one has $\psi \wedge * \psi = n!\,F^n$.

EXAMPLE 3. – Let M be a 6-dimensional nearly Kähler manifold which is not Kählerian. Then it can be verified that there is a number a such that the function C defined by

$$C(X, Y) = a\nabla_X(J)\,Y\,,$$

is a 2-fold complex vector cross product.

For our purposes the case $n = 3$ in example 2 will be important. The linear algebra description of this complex vector cross product can be given entirely in terms of real vectors.

Lemma 8.1. – Let V be a 6-dimensional real vector space with positive definite inner product $\langle\,,\rangle$ and compatible almost complex structure J. Let $C\colon \Lambda^2(V) \to V$ be a linear map. Then the extension of C to a map from $\Lambda^2(V\otimes \mathbf{C}) \to V\otimes \mathbf{C}$ is a complex vector cross product if and only if C has the properties

$$C(Jx\wedge y) = -JC(x\wedge y)\,, \tag{8.5}$$

$$\langle C(x\wedge y), x\rangle = 0\,, \tag{8.6}$$

$$\|C(x\wedge y)\|^2 = \|x\|^2\|y\|^2 - \langle x, y\rangle^2 - \langle Jx, y\rangle^2\,, \qquad \text{for all } x, y \in V\,. \tag{8.7}$$

We shall now exploit this lemma to construct ordinary vector cross products on certain 7-dimensional manifolds.

Let $\hat{M}$ be a 6-dimensional Riemannian manifold whose structure group has been reduced to $SU(3)$, and let J be the almost complex structure and C the 2-fold vector cross product. Denote by F and ψ the 2-form and the 3-form associated with J and C respectively. Then one has the following formulas:

$$F(X, Y) = \langle JX, Y\rangle\,, \qquad \psi(X\wedge Y\wedge Z) = \langle C(X\wedge Y), Z\rangle\,,$$

$$\|C(X\wedge Y)\|^2 = \|X\|^2\|Y\|^2 - \langle X, Y\rangle^2 - \langle JX, Y\rangle^2\,,$$

$$C(JX\wedge Y) = -JC(X\wedge Y)\,,$$

$$\langle C(X\wedge Y), X\rangle = 0\,,$$

for $X, Y, Z \in \mathfrak{X}(M)$.

Theorem 8.2. – Let $\pi\colon E \to \hat{M}$ be a Riemannian fiber bundle with 1-dimensional fiber. We assume that E has a globally defined vertical vector field N with $\|N\| = 1$. Let ν be the 1-form dual to N. Then E has an ordinary 2-fold vector cross product P and the associated 3-form φ of P satisfies

$$\varphi = \pi^*(F)\wedge\nu + \pi^*(\psi)\,. \tag{8.8}$$

Proof. – We use (8.8) to define the 3-form φ on E, and then we put

$$\langle P(X\wedge Y), Z\rangle = \varphi(X\wedge Y\wedge Z)\,, \tag{8.9}$$

for $X, Y, Z \in \mathfrak{X}(E)$.

It is clear that P satisfies (5.1); therefore, it suffices to show that P satisfies (5.2) in order to conclude that P is a vector cross product.

Let $X, Y \in \mathfrak{X}(E)$. Without loss of generality, we may suppose that there are vector fields $\hat{X}, \hat{Y} \in \mathfrak{X}(\hat{M})$ such that $\pi_*(X) = \hat{X}$ and $\pi_*(Y) = \hat{Y}$. Let $\{E_1, \ldots, E_6, N\}$ be a local orthonormal frame on E such that $\{\hat{E}_1, \ldots, \hat{E}_6\}$ is a local orthonormal frame on $\hat{M}$, where $\hat{E}_a = \pi_*(E_a)$. We calculate as follows:

$$\begin{aligned}\|P(X\wedge Y)\|^2 &= \langle P(X\wedge Y), N\rangle^2 + \sum_{a=1}^{6}\langle P(X\wedge Y), E_a\rangle^2\\ &= \varphi(X\wedge Y\wedge N)^2 + \sum_{a=1}^{6}\varphi(X\wedge Y\wedge E_a)^2\\ &= (\pi^*(F)\wedge\nu)(X\wedge Y\wedge N)^2 + \sum_{a=1}^{6}\pi^*(\psi)(X\wedge Y\wedge E_a)^2\\ &= F(\hat{X}\wedge\hat{Y})^2 + \sum_{a=1}^{6}\psi(\hat{X}\wedge\hat{Y}\wedge\hat{E}_a)^2\\ &= \langle J\hat{X}, \hat{Y}\rangle^2 + \|C(\hat{X}\wedge\hat{Y})\|^2\\ &= \|X\wedge Y\|^2 .\end{aligned}$$

Hence P is indeed a 2-fold vector cross product on E.

Also we have

THEOREM 8.3. – Let M be any Riemannian manifold. Then the tangent bundle $T(M)$ has a complex vector cross product C. If ψ is the fundamental form associated with C then $\psi\wedge{}^*\psi$ is the volume element of $T(M)$. Also $d\psi = 0$.

PROOF. – Let $\sigma\colon T(M)\to M$ be the projection and let ω be the Riemannian volume element of M. Put $\psi = \sigma^*(\omega)$ and

$$\langle C(X_1\wedge\ldots\wedge X_{n-1}), X_n\rangle = \psi(X_1\wedge\ldots\wedge X_n)$$

for $X_1, \ldots, X_n \in \mathfrak{X}(M)$.

It is well known that $T(M)$ has an almost complex structure J that takes horizontal vectors into vertical vectors, and vice versa. This fact implies that C satisfies (8.2). It is obvious that (8.1) and (8.4) are satisfied. That (8.3) is satisfied follows from the fact that $\psi\wedge{}^*\psi$ is the volume element of $T(M)$. Finally $d\psi = \sigma^*(d\omega) = 0$.

We are now ready to give an example of a 7-dimensional manifold in $\mathcal{W}_2 - \mathcal{T}$.

THEOREM 8.4. – Let $\tilde{M}$ be any 3-dimensional nonflat Riemannian manifold and put

$$E = T(\tilde{M})\times \boldsymbol{R}\,.$$

Then $E\in\mathcal{W}_2-\mathcal{T}$.

PROOF. – From theorem 8.3 we know that $T(\tilde{M})$ has a complex vector cross product with $d\psi = 0$. (It can be checked that $\nabla\psi\neq 0$ because $\tilde{M}$ is not flat.) Since E is a fiber bundle over $T(\tilde{M})$ satisfying the requirements of theorem 8.2, it follows that E has a 2-fold vector cross product for which (8.8) is satisfied.

For $T(\tilde{M})$ it is well known that $dF = 0$, and we have just shown that $d\psi = 0$. Obviously $d\nu = 0$, and so from (8.8) it follows that $d\varphi = 0$. However, $\nabla F \neq 0$, so that $\nabla\varphi \neq 0$. Hence $E \in \mathcal{W}_2 - \mathcal{T}$.

9. – Inclusion relations.

In this section we establish the strictness of some of the inclusions among the sixteen classes.

THEOREM 9.1. – The following inclusion relations are strict: $\mathcal{T} \subset \mathcal{W}_1$, $\mathcal{T} \subset \mathcal{W}_2$, $\mathcal{T} \subset \mathcal{W}_3$, $\mathcal{T} \subset \mathcal{W}_4$, $\mathcal{W}_1 \cup \mathcal{W}_3 \subset \mathcal{W}_1 \oplus \mathcal{W}_3$, $\mathcal{W}_1 \cup \mathcal{W}_4 \subset \mathcal{W}_1 \oplus \mathcal{W}_4$, $\mathcal{W}_2 \cup \mathcal{W}_4 \subset \mathcal{W}_2 \oplus \mathcal{W}_4$, $\mathcal{W}_3 \cup \mathcal{W}_4 \subset \mathcal{W}_3 \oplus \mathcal{W}_4$, $\mathcal{W}_1 \cup \mathcal{W}_3 \cup \mathcal{W}_4 \subset \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$.

PROOF. – Let S^7 denote the 7-sphere, $M_H \subset \boldsymbol{R}^8$ a minimal hypersurface, and $M_1 \subset \boldsymbol{R}^8$ a hypersurface which is neither locally isometric to S^7 nor to a minimal hypersurface. Let $\tilde{M}$ be any non-flat 3-dimensional manifold. Also, let $\boldsymbol{R}^{7^0}$, S^{7^0}, M_H^0, M_1^0, and $(T(\tilde{M}) \times \boldsymbol{R})^0$ denote the manifolds $\boldsymbol{R}^7$, S^7, M_H, M_1, $T(\tilde{M}) \times \boldsymbol{R}$ with a nontrivial change of conformal metric. Then we have

$$S^7 \in \mathcal{W}_1 - \mathcal{T}$$

$$T(\tilde{M}) \times \boldsymbol{R} \in \mathcal{W}_2 - \mathcal{T}$$

$$M_H \in \mathcal{W}_3 - \mathcal{T}$$

$$\boldsymbol{R}^{7^0} \in \mathcal{W}_4 - \mathcal{T}$$

$$M_1 \in \mathcal{W}_1 \oplus \mathcal{W}_3 - \mathcal{W}_1 \cup \mathcal{W}_3$$

$$S^{7^0} \in \mathcal{W}_1 \oplus \mathcal{W}_4 - \mathcal{W}_1 \cup \mathcal{W}_4$$

$$(T(\tilde{M}) \times \boldsymbol{R})^0 \in \mathcal{W}_2 \oplus \mathcal{W}_4 - \mathcal{W}_2 \cup \mathcal{W}_4$$

$$M_H^0 \in \mathcal{W}_3 \oplus \mathcal{W}_4 - \mathcal{W}_3 \cup \mathcal{W}_4$$

$$M_1^0 \in \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 - \mathcal{W}_1 \cup \mathcal{W}_3 \cup \mathcal{W}_4 .$$

Thus we have shown that 9 of the inclusion relations among the 16 classes are strict.

REFERENCES

[AI] D. V. ALEKSEEVSKIJ, *Riemannian spaces with unusual holonomy groups*, Funkcional Anal. i Priloven, **2** (1968), pp. 1-10; translated in Functional Anal. Appl.

[BE] M. BERGER, *Sur les groupes d'holonomie homogène des variétés à connexion affine et des variétés riemanniennes*, Bull. Soc. Math. France, **83** (1955), pp. 279-330.

[BS] A. BESSE, *Manifolds all of whose geodesics are closed,* Ergebnisse der Mathematik, vol. 93, Springer-Verlag, 1978.

[BO] E. BONAN, *Sur les variétés riemanniennes à groupe d'holonomie G_2 ou Spin*(2), C. R. Acad. Sci. Paris, **262** (1966), pp. 127-129.

[BG] R. B. BROWN - A. GRAY, *Vector cross products,* Comment. Math. Helv., **42** (1967), pp. 222-236.

[CA] E. CALABI, *Construction and properties of some 6-dimensional almost complex manifolds,* Trans. Amer. Math. Soc., **87** (1958), pp. 407-438.

[CN] E. CARTAN, *Sur des familles remarquables d'hypersurfaces isoparametriques dans les sphériques,* Math. Z., **45** (1939), pp. 335-367.

[CH] S. S. CHERN, *On a generalization of Kähler geometry,* Lefschetz Jubilee Volume, Princeton University Press (1957), pp. 103-121.

[E 1] B. ECKMANN, *Systeme von Richtungsfeldern in Sphären und stetige Lösungen komplexer linearer Gleichungen,* Comment. Math. Helv., **15** (1943), pp. 1-26.

[E 2] B. ECKMANN, *Continuous solutions of linear equations. Some exceptional dimensions in topology,* Batelle Rencontres (1967); Lectures in Mathematics and Physics, W. A. Benjamin (1968), pp. 516-526.

[FR] H. FREUDENTHAL, *Oktaven, Ausnahmengruppen und Oktavengeometrie,* Utrecht, 1951.

[GO] S. GOLDBERG, *Curvature and Homology,* Academic Press, 1966.

[GR 1] A. GRAY, *Minimal varieties and almost Hermitian submanifolds,* Michigan Math. J., **12** (1965), pp. 273-279.

[GR 2] A. GRAY, *Some examples of almost Hermitian manifolds,* Ill. J. Math., **10** (1969), pp. 353-366.

[GR 3] A. GRAY, *Vector cross products on manifolds,* Trans. Amer. Math. Soc., **141** (1969), pp. 463-504. Correction, **148** (1970), p. 625.

[GR 4] A. GRAY, *Weak holonomy groups,* Math. Z., **123** (1971), pp. 290-300.

[GR 5] A. GRAY, *Vector cross products,* Rend. Sem. Mat. Univ. Politecn. Torino, **35** (1976-1977), pp. 69-75.

[GG] A. GRAY - P. GREEN, *Sphere transitive structures and the triality automorphism,* Pacific J. Math., **34** (1970), pp. 83-96.

[GH] A. GRAY - L. HERVELLA, *The sixteen classes of almost Hermitian manifolds and their linear invariant,* Annali di Mat. Pura Appl. (IV), **123** (1980), pp. 35-58.

[J] N. JACOBSON, *Lie Algebras,* John Wiley.

[L] H. C. LEE, *A kind of even-dimensional differential geometry and its application to exterior calculus,* Amer. J. Math., **65** (1943), pp. 433-438.

[SA] H. SAMELSON, *Lie Algebras,* Van Nostrand Reinhold Mathematical Studies, no. 23 (1969).

[SI] J. SIMONS, *On the transitivity of holonomy systems,* Ann. of Math., **76** (1967), pp. 213-234.

[V] I. VAISMAN, *On locally conformal almost Kähler manifolds,* Israel J. Math., **24** (1976), pp. 338-351.

[W] G. WHITEHEAD, *Note on cross-sections in Steifel manifolds,* Comment. Math. Helv., **37** (1962), pp. 239-240.

[YS] K. YANO - T. SUMITOMO, *Differential geometry on hypersurfaces in a Cayley space,* Proc. Roy. Soc. Edinburgh Sect. A, **66** (1962-64), pp. 216-231 (1965).

[Z] P. ZVENGROWSKI, *A 3-fold vector product in R^8,* Comment. Math. Helv., **40** (1966), pp. 149-152.